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DIFFERENCE APPROXIMATION OF AN OPTIMAL CONTROL BY GOURSAT SYSTEMS WITH INTEGRAL BOUNDARY CONDITIONS**Abstract**

The convergence of difference approximations of optimal control problems by Goursat systems with integral boundary conditions is proved.

Optimization problems of the Goursat-Darboux system arise in describing the controllable sorption, desorption, drying and other processes [1].

Many problems (for example [2-5, 9]) have been devoted to the difference approximation of controllable systems, where the case of Goursat problem with ordinary boundary conditions has mainly been investigated. But in this paper an optimal control problem has been considered for Goursat systems with integral boundary conditions. At first the existence and uniqueness of the solution of the considered boundary value problem are proved. Then a discrete analogy on the given problem is constructed, the convergence of difference approximations on the basis of the method in [2, 6] is proved, one-sided convergence rate in the nonlinear case is indicated.

1. Problem statement. It is required to minimize the functional

$$J(u) = \sum_{i=1}^K \Phi(y(t_{\tau_i}, s_{r_i})) + \iint_Q F(t, s, y, y_t, y_s, u) dt ds \quad (1)$$

under restrictions

$$y_{ts} = f(t, s, y, y_t, y_s, u), \quad (t, s) \in Q \quad (2)$$

with integral boundary conditions

$$\int_0^T \mu(t) y(t, s) dt = \xi(s), \quad s \in [0, l]; \quad \int_0^l m(s) y(t, s) ds = \zeta(t), \quad t \in [0, T], \quad (3)$$

$$\left(\int_0^T \mu(t) \zeta(t) dt = \int_0^l m(s) \xi(s) ds = A = \text{const} \right),$$

$$u = u(t, s) \in U = \{u \in L_2(Q) / |u(t, s)| \leq R, \text{ a.e. } (t, s) \in Q\}, \quad (4)$$

where y is n -dimensional state vector; u is r -dimensional controlling function; $f(t, s, y, p, q, u)$, $\xi(s)$, $\zeta(t)$ are n -dimensional functions; the functions $\mu(t)$ and $m(s)$ are given $n \times n$ dimensional matrices; $\Phi(y)$, $F(t, s, y, p, q, u)$ are given functions; $Q = \{(t, s) / 0 \leq t \leq T; 0 \leq s \leq l\}$, $l, T > 0$ are given numbers; $\{(t_{\tau_i}, s_{r_i})\}$, $(\tau_i, r_i \in N, i = \overline{1, K})$ is an arbitrary set of points from Q ; $R > 0$ is a number.

Further, we shall assume that the functions $F(t, s, y, p, q, u)$, $f(t, s, y, p, q, u)$ and their partial derivatives

$$\begin{aligned} F_y &= (F_{y_1}, \dots, F_{y_n}), \quad F_p = (F_{p_1}, \dots, F_{p_n}), \quad F_q = (F_{q_1}, \dots, F_{q_n}), \\ F_u &= (F_{u_1}, \dots, F_{u_r}), \quad f_y = (\partial f^i / \partial y_j), \quad f_p = (\partial f^i / \partial p_j), \\ f_q &= (\partial f^i / \partial q_j), \quad f_u = (\partial f^i / \partial u_k), \quad (i, j = \overline{1, n}, k = \overline{1, r}) \end{aligned}$$

are measurable in (t, s) for all $(y, p, q, u) \in R^{3n+r}$ continuous in totality of $(y, p, q, u) \in R^{3n+r}$ a.e. $(t, s) \in Q$; the function $\Phi(y)$ has continuous partial derivatives

$\Phi_y(y) = (\Phi_{y_1}, \dots, \Phi_{y_n})$ for all $y \in R^n$; the matrices $\mu(t) \in L_\infty([0, T])$, $m(s) \in L_\infty([0, t])$ are permutational, that is $\mu(t)m(s) = m(s)\mu(t)$ a.e. $(t, s) \in Q$ and $\det \left[\int_0^T \mu(t) dt \right] \neq 0$,
 $\det \left[\int_0^t m(s) ds \right] \neq 0$; $\xi(s) \in H_n^1[0, t]$, $\zeta(t) \in H_n^1[0, T]$.

Definition. Under the solution of problem (2), (3) corresponding to the control $u = u(t, s) \in L_2(Q)$ we understand the vector-function $y = y(t, s; u) \in L_2^n(Q)$ having the generalized derivatives $y(t, s), y_s(t, s), y_{ss}(t, s) \in L_2^n(Q)$ and that satisfy equation (2) a.e. in Q , and conditions (3) in a classic sense.

According to [7] we can show that the continuous function $y(t, s)$ (more exactly, equivalent to the continuous on the function) is the solution of the boundary value problem (2), (3) if and only if it satisfies the integral equation

$$\begin{aligned} y(t, s) &= \tilde{m}^{-1}(l)\xi(s) + \tilde{n}^{-1}(T)\zeta(t) - \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)A - \tilde{n}^{-1}(T) \int_0^T \int_0^s n(t) \int_0^r f(\tau, y, y_t, y_s, u) d\tau dr dt - \\ &- \tilde{m}^{-1}(l) \int_0^l \int_0^s m(s) \int_0^r f(\tau, r, y, y_t, y_s, u) d\tau dr ds + \tilde{m}^{-1}(l)\tilde{n}^{-1}(T) \int_0^T \int_0^l m(s)n(t)* \\ &* \int_0^s \int_0^r f(\tau, r, y, y_t, y_s, u) d\tau dr dt ds + \int_0^s \int_0^r f(\tau, r, y, y_t, y_s, u) d\tau dr, \end{aligned} \quad (5)$$

where

$$\tilde{m}^{-1}(l) = \left[\int_0^l m(s) ds \right]^{-1}, \quad \tilde{n}^{-1}(T) = \left[\int_0^T \mu(t) dt \right]^{-1}.$$

Introduce the following conditions:

- I) $|f(t, s, 0, 0, 0, u)| \leq M_0$, $|f_y(t, s, y, p, q, u)| \leq M_1$, $|f_p(t, s, y, p, q, u)| \leq M_2$,
 $|f_q(t, s, y, p, q, u)| \leq M_3$, $|f_u(t, s, y, p, q, u)| \leq M_4$,
a.e. $(t, s) \in Q$ for any $(y, p, q, u) \in L_2(Q) \times L_2(Q) \times L_2(Q) \times U$;
- II) $|F_y(t, s, y, p, q, u)| \leq K_1$, $|F_p(t, s, y, p, q, u)| \leq K_2$, $|F_q(t, s, y, p, q, u)| \leq K_3$,
 $|F_u(t, s, y, p, q, u)| \leq K_4$ a.e. $(t, s) \in Q$ for any $(y, p, q, u) \in L_2(Q) \times L_2(Q) \times L_2(Q) \times U$,
 $|\Phi_y(y)|_{R^n} \leq N_1$ for any $y \in R^n$.

Let $R(\varepsilon)$ be a matrix with elements:

$$R_{11} = 12(1 + \varepsilon_1) \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{T^4 l^2}{3} + |\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \frac{T^2 l^4}{3} + |\tilde{m}^{-1}(l)|^2 |\tilde{n}^{-1}(T)|^2 \times \right.$$

$$\left. \times \|m\|_{L_\infty}^2 \frac{T^4 l^4}{9} + T^2 l^2 \right) M_1^2;$$

$$R_{12} = 12(1 + \varepsilon_1) \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{T^3 l^2}{2} + |\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \frac{T l^4}{3} + |\tilde{m}^{-1}(l)|^2 |\tilde{n}^{-1}(T)|^2 \times \right.$$

$$\left. \times \|\mu\|_{L_\infty}^2 \frac{T^3 l^4}{6} + T l^2 \right) M_2^2;$$

$$R_{13} = 12(1 + \varepsilon_1) \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{T^4 l}{3} + |\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \frac{T^2 l^3}{2} + |\tilde{m}^{-1}(l)|^2 |\tilde{\mu}^{-1}(T)|^2 \times \right. \\ \left. \times \|\mu\|_{L_\infty}^2 \|m\|_{L_\infty}^2 \frac{T^4 l^3}{6} + T^2 l \right) M_3^2;$$

$$R_{21} = 6(1 + \varepsilon_2) \left(|\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \frac{Tl^4}{3} + Tl^2 \right) M_1^2;$$

$$R_{22} = 6(1 + \varepsilon_2) \left(|\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \frac{l^4}{3} + l^2 \right) M_2^2;$$

$$R_{23} = 6(1 + \varepsilon_2) \left(|\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \frac{Tl^3}{3} + Tl \right) M_3^2;$$

$$R_{31} = 6(1 + \varepsilon_3) \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{T^4 l}{3} + T^2 l \right) M_1^2;$$

$$R_{32} = 6(1 + \varepsilon_3) \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{T^3 l}{2} + Tl \right) M_2^2;$$

$$R_{33} = 6(1 + \varepsilon_3) \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{T^4}{3} + T^2 \right) M_3^2;$$

for any $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_i > 0$, ($i = \overline{1, 3}$).

By using the method from [8] we can prove the following

Theorem 1. Let the above shown condition I) be fulfilled. Besides, for some ε a spectrum of the matrix $R(\varepsilon)$ lies in a unit circle. Then the solution of problem (2)-(3) exists for any $u \in U$, it is unique and has the following properties

$$\begin{aligned} \max_Q |y|^2 &\leq C_1(\varepsilon), \quad \max_{[0, l]} \int_0^T |y_t|^2 dt \leq C_2(\varepsilon), \quad \max_{[0, T]} \int_0^l |y_s|^2 ds \leq C_3(\varepsilon), \\ \max_Q |y(t, s) - y(\tau, r)| &\leq C_4(|t - \tau| + |s - r|), \quad \max_{[0, T]} \int_0^l |y_s(t, s) - y_s(\tau, s)| ds \leq C_6 |t - \tau|, \\ \max_{[0, l]} \int_0^T |y_t(t, s) - y_t(t, r)| dt &\leq C_5 |s - r|, \quad \max_Q |y(t, s; u + \bar{u}) - y(t, s; u)|^2 \leq C_7(\varepsilon) \|\bar{u}\|_{L_2}^2, \\ \max_{[0, l]} \int_0^T |y_t(t, s; u + \bar{u}) - y_t(t, s; u)|^2 dt &\leq C_8(\varepsilon) \|\bar{u}\|_{L_2}^2, \quad \max_{[0, T]} \int_0^l |y_s(t, s; u + \bar{u}) - y_s(t, s; u)|^2 ds \leq \\ &\leq C_9(\varepsilon) \|\bar{u}\|_{L_2}^2. \end{aligned} \tag{6}$$

if along with I), condition II) is fulfilled, then

$$|J(u + \bar{u}) - J(u)| \leq C_{10}(\varepsilon) \|\bar{u}\|_{L_2}^2,$$

where C_i , ($i = \overline{1, 10}$) are constants that are independent of $\tau, t \in [0, T]$, $s, r \in [0, l]$ and controls $u(t, s)$; $u + \bar{u}$, $u \in U$.

2. Difference approximation of the problem. For approximate solution of problem (1)-(4) we introduce on Q the consequence of rectangular grids $\{(t_i, s_j)\}_x$ that

$(t_i, s_{r_i}), i = \overline{1, K}$ are contained in these grids, $\chi = 1, 2, \dots$, where $0 = t_{0,\chi} < \dots < t_{p_{\chi},\chi} = T$, $0 = s_{0,\chi} < \dots < s_{L_{\chi},\chi} = l$ and denote by

$$p = p_\chi, L = L_\chi, t = t_{i,\chi}, s = s_{j,\chi}, \Delta t_i = t_{i+1} - t_i, \Delta s_j = s_{j+1} - s_j,$$

$$T_i = [t_i, t_{i+1}), S_j = [s_j, s_{j+1}), Q_{ij} = T_i \times S_j, i = \overline{0, p-1}, j = \overline{0, L-1}.$$

$$\Delta_\chi t = \max_{i=1,p} \Delta t_i, \Delta_\chi s = \max_{i=1,L} \Delta s_j, \delta_\chi t = \min_{i=1,p} \Delta t_i, \delta_\chi s = \min_{i=1,L} \Delta s_j,$$

$$(y_y)_i = (y_{i+1,j} - y_{i,j})/\Delta t_i, (y_y)_s = (y_{i,j+1} - y_{i,j})/\Delta s_j, (y_y)_{is} = (y_{i,j+1})_i - (y_{i,j})_i/\Delta s_j.$$

For each natural χ consider a problem on the minimization of a discrete functional

$$I_\chi[u] = \sum_{i=1}^K \Phi(y_{r_i,r_i}) + \sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \iint F(t, s, y_{ij}, (y_y)_i, (y_y)_s, u_{ij}) dt ds \quad (7)$$

in solutions $y_y = y_y[u]$ of the following difference scheme:

$$\begin{aligned} y_{i+1,j+1} &= \tilde{m}^{-1}(l)\zeta(t_{i+1}) + \tilde{n}^{-1}(T)\xi(s_{j+1}) - \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)A_\chi - \tilde{n}^{-1}(T)\sum_{i=0}^{p-1} \int_{T_i} n(t) dt \times \\ &\quad \times \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds - \tilde{m}^{-1}(l)\sum_{j=0}^{L-1} \int_{S_j} m(s) ds \times \\ &\quad \times \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds + \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)\sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \int_{T_i} n(t) dt \int_{S_j} m(s) ds \times \\ &\quad \times \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds + \\ &\quad \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds, i = \overline{0, p-1}, j = \overline{0, L-1}, \\ y_{i+1,0} &= \tilde{m}^{-1}(l)\zeta(t_{i+1}) + \tilde{n}^{-1}(T)\xi(0) - \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)A_\chi - \tilde{m}^{-1}(l)\sum_{j=0}^{L-1} \int_{S_j} m(s) ds \times \\ &\quad \times \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds + \tilde{m}^{-1}(l)\tilde{n}^{-1}(T) \times \quad (8) \\ &\quad \times \sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \int_{T_i} n(t) dt \int_{S_j} m(s) ds \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds, i = \overline{0, p-1}, \\ y_{0,j+1} &= \tilde{m}^{-1}(l)\zeta(0) + \tilde{n}^{-1}(T)\xi(s_{j+1}) - \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)A_\chi - \tilde{n}^{-1}(T)\sum_{i=0}^{p-1} \int_{T_i} n(t) dt \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, \\ &\quad y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds + \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)\sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \int_{T_i} n(t) dt \int_{S_j} m(s) ds \times \\ &\quad \times \sum_{i=0}^l \sum_{j=0}^L \iint f(t, s, y_{ij}, (y_{ij})_i, (y_{ij})_s, u_{ij}) dt ds, j = \overline{0, L-1}, \\ y_{0,0} &= \tilde{m}^{-1}(l)\zeta(0) + \tilde{n}^{-1}(T)\xi(0) - \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)A_\chi + \tilde{m}^{-1}(l)\tilde{n}^{-1}(T)\sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \int_{T_i} n(t) dt \times \end{aligned}$$

$$\times \sum_{s_j} \int_0^T \sum_{i=0}^{j-1} \sum_{l=0}^L \iint f(t, s, y_{i,j}, (y_{i,j})_t, (y_{i,j})_s, u_{i,j}) dt ds .$$

We choose the discrete controls [4] from the set

$$U_\chi = \left\{ [u_y] \in L_{2,\chi}^r \mid |u_y| \leq R, i = \overline{0, p-1}, j = \overline{0, L-1} \right\}, \quad (9)$$

where $L_{2,\chi}^r$ is a space of discrete functions with a scalar product

$$\langle [u], [\vartheta] \rangle_{L_{2,\chi}^r} = \sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \langle u_{ij}, \vartheta_{ij} \rangle \Delta t_i \Delta s_j .$$

Denote in problems (1)-(4) and (7)-(9)

$$J^* = \inf_{u \in U} J(u), \quad I_n^* = \inf_{[u] \in U_n} I_n[u], \quad n = 1, 2, \dots$$

2. Convergence in functional. Let the following conditions be fulfilled:

III) the functions f, ξ_s, ζ_t are equipotentially continuous in the norm L_2 , and more exactly in the following sense

$$\begin{aligned} & \iint_Q |f(t+\tau, s+r, y, y_t, y_s, u) - f(t, s, y, y_t, y_s, u)|^2 dt ds + \int_0^T |\zeta_t(t+\tau) - \zeta_t(t)|^2 dt + \\ & + \int_0^T |\xi_s(s+r) - \xi_s(s)|^2 ds = O(\tau^2, r^2) \rightarrow 0, \quad (t+\tau, s+r) \in Q, \end{aligned}$$

for $(\tau, r) \rightarrow 0$ uniformly in controls $u \in U$;

IV) the sequence $\{(t_i, s_j)\}$ is such that $\lim_{\chi \rightarrow \infty} p_\chi = \lim_{\chi \rightarrow \infty} L_\chi = +\infty$, $p_\chi \Delta_\chi t \leq (1 + \alpha_\chi) T$,

$L_\chi \Delta_\chi s \leq (1 + \beta_\chi)$ and there will be found $C_1^0, C_2^0 > 0$, not depending on

χ , $\Delta_\chi t \leq C_1^0 \delta_\chi t$, $\Delta_\chi s \leq C_2^0 \delta_\chi s$, where $\alpha_\chi, \beta_\chi \rightarrow 0$, $\chi \rightarrow \infty$.

We can show that problem (8), (9) is equivalent to the following difference equations:

(11)

Below in notations [4] we shall identify the grid function with its piecewise-constant complement.

Lemma. Let the conditions of theorem I and conditions III), IV) be fulfilled, besides, there are arbitrary sequences of functional and discrete controls $u_\chi \in U$, $[u_\chi] \in U_\chi$, $\chi = 1, 2, \dots$. Then there will be found such a $\chi_0 > 0$, ($\chi_0 \in N$), that for each $\chi > \chi_0$ for corresponding solutions $y_\chi(t, s) = y(t, s; u_\chi)$ and $y_{\chi_0} = y_{\chi_0}[u_\chi]$ of problem (2), (3) and (8) it is valid the estimate

$$\begin{aligned} & \max_{i,j} |y_\chi(t_i, s_j) - y_{\chi_0}|^2 \leq C_{10}(\varepsilon) (\Delta_\chi^2 t + \Delta_\chi^2 s + \|u_\chi - [u_\chi]\|_{L_2}^2), \\ & \max_j \sum_{i=0}^{p-1} \left| \int_{t_i}^{t_{i+1}} (y_{\chi_0}(t, s_j) - (y_{\chi_0})_t) dt \right|^2 \leq C_{11}(\varepsilon) (\Delta_\chi^2 t + \Delta_\chi^2 s + O(\Delta_\chi^2 t; \Delta_\chi^2 s) + \|u_\chi - [u_\chi]\|_{L_2}^2), \\ & \max_i \sum_{j=0}^{L-1} \left| \int_{s_j}^{s_{j+1}} (y_{\chi_0}(t_i, s) - (y_{\chi_0})_s) ds \right|^2 \leq C_{12}(\varepsilon) (\Delta_\chi^2 t + \Delta_\chi^2 s + O(\Delta_\chi^2 t; \Delta_\chi^2 s) + \|u_\chi - [u_\chi]\|_{L_2}^2), \end{aligned} \quad (10)$$

where the constants C_{10}, C_{11}, C_{12} are independent of χ , and $O(\Delta_\chi^2 t; \Delta_\chi^2 s)$ is the quantity $O(\tau^2, r^2)$ from condition III) for $\tau = \Delta_\chi t$, $r = \Delta_\chi s$. And if in addition to the condition of Lemma, condition II) is fulfilled, then

$$\|J(u_\chi) - I_\chi[u_\chi]\| \leq C_{13}(\varepsilon) \sqrt{\Delta_\chi^2 t + \Delta_\chi^2} + O(\Delta_\chi^2 t; \Delta_\chi^2) + \|u_\chi - [u_\chi]\|_{L_2}^2,$$

where $C_{13} > 0$ are constants not depending on χ .

Proof. From equalities (5), (8) follow the following equalities:

$$\begin{aligned} y_t &= \tilde{m}^{-1}(l)\zeta(t) - \tilde{m}^{-1}(l) \int_0^t m(s) \int_0^s f(t, r, y, y_t, y_s, u) dr ds + \int_0^t f(t, r, y, y_t, y_s, u) dr \\ y_s &= \tilde{\mu}^{-1}(T)\xi_s(s) - \tilde{\mu}^{-1}(T) \int_0^T \mu(t) \int_0^t f(\tau, s, y, y_t, y_s, u) d\tau dt + \int_0^t f(\tau, s, y, y_t, y_s, u) d\tau, \end{aligned} \quad (11)$$

a.e. $(t, s) \in Q$

and

$$\begin{aligned} (y_{i,j+1}) &= \tilde{m}^{-1}(l)(\zeta_i)_t - \tilde{m}^{-1}(l) \frac{1}{\Delta t_i} \sum_{j=0}^{L-1} \int_0^t m(s) ds \sum_{j_l=0}^j \iint f(t, s, y_{ij_l}, (y_{ij_l})_t, (y_{ij_l})_s, u_{ij_l}) dt ds + \\ &+ \frac{1}{\Delta t_i} \sum_{j_l=0}^j \iint f(t, s, y_{ij_l}, (y_{ij_l})_t, (y_{ij_l})_s, u_{ij_l}) dt ds, \quad i = \overline{0, p-1}, j = \overline{0, L-1}, \\ (y_{i,0})_t &= \tilde{m}^{-1}(l)(\zeta_i)_t, \quad i = \overline{0, p-1}, \\ (y_{i+1,j})_s &= \tilde{\mu}^{-1}(T)(\xi_j)_s - \tilde{\mu}^{-1}(T) \frac{1}{\Delta s_j} \sum_{i=0}^{p-1} \int_0^t \mu(t) dt \sum_{i_l=0}^i \iint f(t, s, y_{i_l j}, (y_{i_l j})_t, (y_{i_l j})_s, u_{i_l j}) dt ds + \\ &+ \frac{1}{\Delta s_j} \sum_{i_l=0}^i \iint f(t, s, y_{i_l j}, (y_{i_l j})_t, (y_{i_l j})_s, u_{i_l j}) dt ds, \quad i = \overline{0, p-1}, j = \overline{0, L-1}, \\ (y_{0,j})_s &= \tilde{\mu}^{-1}(T)(\xi_j)_s, \quad j = \overline{0, p-1}. \end{aligned} \quad (12)$$

Using formulas (5), (9), (11), (12) condition 1) Lagrange's formula, relation (6) from theorem 1, Hölder's inequality, and the known " ε " inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, $\varepsilon > 0$

we have:

$$\begin{aligned} |y_\chi(t_i, s_j) - y_{\chi ij}|^2 &\leq \left(\|\tilde{\mu}^{-1}(T)\| \|\tilde{m}^{-1}(l)\| \|A - A_\chi\| + \left| \tilde{\mu}^{-1}(T) \sum_{i=0}^{p-1} \int_0^t \mu(t) dt \right| \times \right. \\ &\times \sum_{i_l=0}^i \sum_{j_l=0}^{L-1} \iint f(t, s, y, y_t, y_s, u_\chi) - f(t, s, y_{i_l j_l}, (y_{i_l j_l})_t, (y_{i_l j_l})_s, u_{i_l j_l}) dt ds + \left| \tilde{m}^{-1}(l) \right| \times \\ &\times \sum_{j=0}^{L-1} \int_0^t m(s) ds \cdot \sum_{i_l=0}^i \sum_{j_l=0}^{L-1} \iint f(t, s, y, y_t, y_s, u_\chi) - f(t, s, y_{i_l j_l}, (y_{i_l j_l})_t, (y_{i_l j_l})_s, u_{i_l j_l}) dt ds + \\ &+ \left| \tilde{\mu}^{-1}(T) \right| \left| \tilde{m}^{-1}(s) \right| \sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \iint \mu(t) \|m(s)\| dt ds \times \sum_{i_l=0}^i \sum_{j_l=0}^{L-1} \iint f(t, s, y, y_t, y_s, u_\chi) - \\ &- f(t, s, y_{i_l j_l}, (y_{i_l j_l})_t, (y_{i_l j_l})_s, u_{i_l j_l}) dt ds \right|^2 \leq C_{10}^1 \left(\Delta_\chi^2 t + \Delta_\chi^2 s + \|u_\chi - [u_\chi]\|_{L_2}^2 \right) + \\ &+ R_{11}^\chi \max_{i,j} |y_\chi(t_i, s_j) - y_{\chi ij}|^2 + R_{12}^\chi \max_j \sum_{i=0}^{p-1} \left| \int_0^t y_\chi(t, s_j) - (y_{\chi j})_t \right|^2 dt + \\ &+ R_{13}^\chi \max_i \sum_{j=0}^{L-1} \left| \int_0^s y_\chi(t_i, s) - (y_{\chi j})_s \right|^2 ds, \end{aligned}$$

$$\begin{aligned}
& \sum_{i=0}^{p-1} \int_{T_i}^{\tau} \left| y_{x_i}(t, s_j) - (y_{x_i})_t \right|^2 dt \leq \sum_{i=0}^{p-1} \int_{T_i}^{\tau} \left(|\tilde{m}^{-1}(l)| |\zeta_i(t) - (\zeta_i)_t| + |\tilde{m}^{-1}(l)| \sum_{j=0}^{L-1} \int_{S_j} m(s) ds \times \right. \\
& \times \sum_{j=0}^L \int_{S_j} \left| f(t, s, y, y_t, y_s, u_x) - \frac{1}{\Delta t_i} \int_{T_i}^{\tau} f(t, s, y_{ij}, (y_{ij})_t, (y_{ij})_s, u_{xij}) dt \right| ds + \\
& \left. + \sum_{j=0}^L \left| f(t, s, y, y_t, y_s, u_x) - \frac{1}{\Delta t_i} \int_{T_i}^{\tau} f(t, s, y_{ij}, (y_{ij})_t, (y_{ij})_s, u_{xij}) dt \right|^2 \right)^2 dt \leq \\
& \leq C_{11}^1 \left(\Delta_x^2 t + \Delta_x^2 s + O(\Delta_x^2 t, \Delta_x^2 s) + \|u_x - [u_x]\|_{L_2}^2 \right) + R_{21}^x \max_y |y_x(t, s_j) - y_{xy}|^2 + \\
& + R_{22}^x \max_y \sum_{i=0}^{p-1} \int_{T_i}^{\tau} \left| y_{x_i}(t, s_j) - (y_{xy})_t \right|^2 dt + R_{23}^x \max_y \sum_{i=0}^{p-1} \int_{S_j} \left| y_{x_i}(t, s) - (y_{xy})_s \right|^2 ds.
\end{aligned} \tag{13}$$

In a similar way

$$\begin{aligned}
& \sum_{j=0}^{L-1} \int_{S_j} \left| y_{x_i}(t_i, s) - (y_{xy})_s \right|^2 ds \leq C_{12}^1 \left(\Delta_x^2 t + \Delta_x^2 s + O(\Delta_x^2 t, \Delta_x^2 s) + \|u_x - [u_x]\|_{L_2}^2 \right) + \\
& + R_{31}^x \max_y |y_x(t_i, s_j) - y_{xy}|^2 + R_{32}^x \max_y \sum_{i=0}^{p-1} \int_{T_i}^{\tau} \left| y_{x_i}(t, s_j) - (y_{xy})_t \right|^2 ds + \\
& + R_{33}^x \max_y \sum_{i=0}^{p-1} \int_{S_j} \left| y_{x_i}(t_i, s) - (y_{xy})_s \right|^2 ds.
\end{aligned}$$

Here $C_{10}^1, C_{11}^1, C_{12}^1 > 0$ are constants

$$\begin{aligned}
R_{11}^x = 12(1 + \varepsilon_1) & \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{\Delta_x^4 p^2 (p+1)(2p+1) \Delta_x^2 s L^2}{6} + |\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \times \right. \\
& \times \frac{\Delta_x^2 p^2 \Delta_x^4 s L^2 (L+1)(2L+1)}{6} + |\tilde{m}^{-1}(l)|^2 |\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \|m\|_{L_\infty}^2 \times \\
& \left. \times \frac{\Delta_x^4 t p^2 (p+1)(2p+1) \Delta_x^4 s L^2 (L+1)(2L+1)}{36} + \Delta_x^2 t p^2 \Delta_x^2 s L^2 \right) M_1^2;
\end{aligned}$$

$$\begin{aligned}
R_{12}^x = 12(1 + \varepsilon_1) & \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{\Delta_x^3 p^2 (p+1) \Delta_x^2 s L^2}{2} + |\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \times \right. \\
& \times \frac{\Delta_x p \Delta_x^4 s L^2 (L+1)(2L+1)}{6} + |\tilde{m}^{-1}(l)|^2 |\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \|m\|_{L_\infty}^2 \times \\
& \left. \times \frac{\Delta_x^3 t p^2 (p+1) \Delta_x^4 s L^2 (L+1)(2L+1)}{12} + \Delta_x t p^2 \Delta_x^2 s L^2 \right) M_2^2;
\end{aligned}$$

$$\begin{aligned}
R_{13}^x = (12 + \varepsilon_1) & \left(|\tilde{\mu}^{-1}(T)|^2 \|\mu\|_{L_\infty}^2 \frac{\Delta_x^4 p^2 (p+1)(2p+1) \Delta_x s L}{6} + |\tilde{m}^{-1}(l)|^2 \|m\|_{L_\infty}^2 \times \right. \\
& \times \frac{\Delta_x^2 t p^2 \Delta_x^3 s L^2 (L+1)}{2} + |\tilde{m}^{-1}(T)|^2 |\tilde{m}^{-1}(l)|^2 \|\mu\|_{L_\infty}^2 \|m\|_{L_\infty}^2 \times
\end{aligned}$$

$$\begin{aligned}
& \times \frac{\Delta_x^4 t p^2 (p+1)(2p+1) \Delta_x^3 s L^2 (L+1)}{12} + \Delta_x^2 t p^2 \Delta_x^2 s L \Big) M_3^2; \\
R_{21}^\chi &= 6(1+\varepsilon_2) \left(\|\tilde{m}^{-1}(l)\|^2 \|m\|_{L_\infty}^2 \frac{\Delta_x t p \cdot \Delta_x^4 s L^2 (L+1)(2L+1)}{6} + \Delta_x t p \Delta_x^2 s L^2 \right) M_1^2; \\
R_{22}^\chi &= 6(1+\varepsilon_2) \left(\|\tilde{m}^{-1}(l)\|^2 \|m\|_{L_\infty}^2 \frac{\Delta_x^4 s L^2 (L+1)(2L+1)}{6} + \Delta_x^2 s L^2 \right) M_2^2; \\
R_{23}^\chi &= 6(1+\varepsilon_2) \left(\|\tilde{m}^{-1}(l)\|^2 \|m\|_{L_\infty}^2 \frac{\Delta_x t p \cdot \Delta_x^3 s L^2 (L+1)(2L+1)}{2} + \Delta_x t p \Delta_x s L \right) M_3^2; \\
R_{31}^\chi &= 6(1+\varepsilon_3) \left(\|\tilde{\mu}^{-1}(T)\|^2 \|\mu\|_{L_\infty}^2 \frac{\Delta_x^4 t p^2 (p+1)(2p+1) \Delta_x s L}{6} + \Delta_x^2 t p^2 \Delta_x s L \right) M_1^2; \\
R_{32}^\chi &= 6(1+\varepsilon_3) \left(\|\tilde{\mu}^{-1}(T)\|^2 \|\mu\|_{L_\infty}^2 \frac{\Delta_x^3 t p^2 (p+1) \cdot \Delta_x s L}{2} + \Delta_x t p \Delta_x s L \right) M_2^2; \\
R_{33}^\chi &= 6(1+\varepsilon_3) \left(\|\tilde{\mu}^{-1}(T)\|^2 \|\mu\|_{L_\infty}^2 \frac{\Delta_x^4 t p^2 (p+1)(2p+1)}{6} + \Delta_x^2 t p^2 \right) M_3^2.
\end{aligned}$$

By R_χ denote a matrix with elements R_{ij}^χ , $i, j = \overline{1, 3}$. We can easily see that the elements of the matrix R_χ may be estimated as follows: $R_{ij}^\chi \leq R_{ij} + \delta_{ij}^\chi$, $\delta_{ij}^\chi \rightarrow 0$, $(i, j = \overline{1, 3})$ for $\chi \rightarrow \infty$.

Hence it follows that there exists such $\chi_0 > 0$ that for $\chi > \chi_0$ a spectrum of the matrix R_χ lies in a unit circle. Then for any $\chi > \chi_0$ from (13) it follows the following relation:

$$\begin{cases}
\max_{i,j} |y_\chi(t_i, s_j) - y_{\chi\theta}|^2 \\
\max_j \sum_{i=0}^{p-1} \int |y_{\chi i}(t, s_j) - (y_{\chi\theta})_i|^2 dt \\
\max_i \sum_{j=0, S_j}^{L-1} \int |y_{\chi i}(t_i, s) - (y_{\chi\theta})_s|^2 ds
\end{cases} \leq (E - R_\chi)^{-1} \begin{cases}
C_{10}^1 (\Delta_x^2 t + \Delta_x^2 s + \|u_\chi - [u_\chi]\|_{L_2'}^2) \\
C_{11}^1 (\Delta_x^2 t + \Delta_x^2 s + O(\Delta_x^2 t, \Delta_x^2 s) \|u_\chi - [u_\chi]\|_{L_2'}^2) \\
C_{12}^1 (\Delta_x^2 t + \Delta_x^2 s + O(\Delta_x^2 t, \Delta_x^2 s) \|u_\chi - [u_\chi]\|_{L_2'}^2)
\end{cases}$$

where E is 3×3 -dimensional unit matrix, from which the validity of inequalities (10) follow.

Now, we will prove the last inequality of lemma.

Using formulas (1), (7), we estimate $|J(u_\chi) - I_n[u_\chi]|$:

$$\begin{aligned}
|J(u_\chi) - I_n[u_\chi]|^2 &\leq \left(\sum_{i=1}^K |\Phi(y(t_{\tau_i}, s_{\tau_i})) - \Phi(y_{\tau_i, \tau_i})| + \right. \\
&\quad \left. + \sum_{i=0}^{p-1} \sum_{j=0}^{L-1} \int \int |F(t, s, y, y_i, y_s, u_\chi) - F(t, s, y_\theta, (y_\theta)_i, (y_\theta)_s, u_{\chi\theta})| dt ds \right)^2,
\end{aligned}$$

taking into account inequalities (6), (10), condition II), Lagrange's formula, Hölder's inequality, we get:

$$|J(u_\chi) - I_\chi[u_\chi]| \leq C_{13} \sqrt{\Delta_\chi^2 t + \Delta_\chi^2 s + O(\Delta_\chi^2 t, \Delta_\chi^2 s)} + \|u_\chi - [u_\chi]\|_{L_2}^2.$$

The proof of the lemma is completed.

Introduce the mappings

$$A_\chi : U \rightarrow U_\chi, \quad B_\chi : U_\chi \rightarrow U$$

functioning by the rule:

$$A_\chi(u)_y = \frac{1}{\Delta t_i \Delta s_j} \iint_{Q_{ij}} u(t, s) dt ds,$$

$$B_\chi[u](t, s) = u_{i,j}, \quad (t, s) \in Q_{ij}.$$

Further, by following [2] and using analogous arguments based on the above obtained results we can cite the following

Corollary. Let under the conditions of the lemma the discrete controls $[u_\chi] \in U_\chi$ be arbitrary, and $u_\chi = B_\chi[u_\chi]$. Then the left sides of the inequalities of the lemma converge to zero and for $\chi \rightarrow \infty$, and the convergence rate is estimated by the quantity

$$C(\varepsilon) \sqrt{\Delta_\chi^2 t + \Delta_\chi^2 s + (\Delta_\chi^2 t, \Delta_\chi^2 s)} \quad (C > 0 \text{ are constants}).$$

If under the conditions of the control lemma $u_\chi \equiv u$, and $[u] = [A_\chi(u)]$ is a fixed element from U , then the left sides of the lemma inequalities converge to zero for $\chi \rightarrow \infty$. But in this case the convergence rate estimate is absent.

Theorem 2. Let all the conditions of the lemma be fulfilled. Then a sequence of difference extremal problems (7)-(9) will approximate in functional the initial minimization problem (1)-(4) in the following sense:

$$\lim_{\chi \rightarrow \infty} I_\chi^* = J^*. \quad (14)$$

Remark 1. In the convergence (14) we can indicate the one-sided estimate of the convergence rate

$$J^* \leq I_\chi^* + C(\varepsilon) \sqrt{\Delta_\chi^2 t + \Delta_\chi^2 s + O(\Delta_\chi^2 t, \Delta_\chi^2 s)}.$$

Remark 2. In some classes of controllable systems (for example, in a linear case), we can show the convergence rate estimate. In these case we are to modify somehow the proof method of the lemma.

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