2000 VOL. XIII(XXI)

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ASYMPTOTICS OF THE WEIGHTED TRACE FOR THE FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATION

Abstract

In the present paper the asymptotic formula for the weighted trace of the 4-th order differential equation with operator coefficient is obtained.

Let H be a separable Hilbert space. In space $L_2[0,\infty;H]$ let's consider the differential expression

$$l(y)=y^{(IV)}+Q(x)y\,,\quad 0\le x<\infty \tag{1}$$

and the boundary conditions

$$\begin{aligned} y''(0)-Ay'(0)&=0,\\ y'''(0)-By(0)&=0\,. \end{aligned} \tag{2}$$

Let D' be totality of all four times continuously differentiable functions $y(x)$ with values from $D(Q)$, finitary at infinity and satisfying in zero the conditions (2) for which $Q(x)y(x)$ is continuous. Let D' be dense everywhere in $L_2(0,\infty;H)$. In D' let's determine operator L': $L'y=l(y), y\in D'$. Operator L' is symmetric and positive determined. We denote its closure by L. Under some assumptions, which we'll reduce below, operator L is self-conjugated and has pure discrete spectrum (see [3]).

Let $\lambda_1,\lambda_2,...,\lambda_n,...$ be eigen and $\varphi_1(x),\varphi_2(x)..\varphi_n(x)..$ be corresponding orthonormalized eigen functions of operator L. Let's introduce the function $N_s(\mu)$

$$N_s(\mu)=\sum_{\lambda_n<\mu}\mathbf{C}_n^{(s)}\,,$$

where $C_n^{(s)}=\int_0^\infty\left\|Q^s(x)\varphi_n(x)\right\|_H^2dx\,,\ s\le\frac{7}{8}$.

We'll call it the weighted trace of operator L. For $s=0$ it is converted to the number of eigen values of operator L less than the given number μ. Our main problem is to study the studying asymptotic behavior of the function $N_s(\mu)$ for $\mu\to\infty$.

We'll suppose that the function $Q(x)$ *and operators* A *and* B *satisfy the* following conditions:

1) Operators $Q(x)$ for almost all $x\in(0,\infty)$ are self-conjugated in H, moreover for almost all $x\in(0,\infty)$ they have general dense domain of determination $D(Q)$ in H for every $f\in D(Q)\,(Q(x)f,f)>(f,f)$.

2) Operators A and B are self-conjugated, non-bounded, moreover $D(Q)\subset D(A)$, $D(Q)\subset D(B)$. For every $f\in D(B)$ $(-Bf,f)>(f,f)$, and for $g\in D(A)\,(Ag,g)>(g,g)$, where $D(A)$ and $D(B)$ are domains of determination of operators A and B.

3) For all $x\in[0,\infty)\lim\limits_{\lambda\to\infty}\left\|K^{-3}B\right\|=\lim\limits_{\lambda\to\infty}\left\|A\cdot K^{-1}\right\|=\lim\limits_{\lambda\to\infty}\left\|K^{-2}A\cdot K\right\|=0$,

$$\left\|\left[(i\omega_j K)^2 - i\omega_j AK\right]^{-1}\left[(i\omega_l K)^2 + i\omega_l AK\right]\right\| < C_1, \quad j,l = 1,2$$

$$\left\|\left[(i\omega_j K) - i\omega_j AK\right]^{-1}\left[(i\omega_l K)^2 + i\omega_l AK\right]\right\| < C_2, \quad j,l = 1,2 \text{ and } j \neq l,$$

where $K = \{Q(x) + \lambda I\}^{1/4}$, I is a unit operator, λ is a positive parameter, ω_1, ω_2 are roots $\sqrt[4]{-1}$ arranging in the upper half-plane.

4) Series $\sum_{k=1}^{\infty} \beta_k^{2s-7/4}(x)$ converges for every $x \in [0,\infty)$ and its sum $F_s(x) \in L_1(0,\infty)$. Here $\beta_k(x)$ are eigen values of the operator $Q(x)$ in increase order.

5) For $|x-\eta| \leq 1$ the inequality

$$\left\|Q^s(x) \cdot Q^{-a}[Q(\xi) - Q(x)]Q^{-s}(\xi)\right\| < \mathbf{C}_3|x-\xi|, \quad 0 < a < \frac{5}{4}, \quad \mathbf{C} > 0$$

holds.

6) For $|x-\xi| > 1$,

$$\left\|Q(\xi) \cdot \exp\left(-\frac{Jm\omega_1}{2}|x-\xi|Q^{\frac{1}{4}}(x)\right)\right\| < B_1, \ \left\|Q^{-1}(x) \cdot Q(\xi)\right\| < B_2, \ B_1; B_2 > 0.$$

In order to obtain asymptotics of the function $N_s(\mu)$ first they study some properties of Green function of operator L. It was proved that Green function $G(x,\eta,\lambda)$ of operator L satisfies the following integral equation

$$G(x,\eta,\lambda) = G_0(x,\eta,\lambda) - \int_0^{\infty} G_0(x,\xi,\lambda)\{[Q(\xi) - Q(\eta)]G(\xi,\eta,\lambda)\}d\xi, \tag{3}$$

where

$$G_0(x,\eta,\xi,\lambda) = \frac{[q(\xi) + \lambda I]^{-3/4}}{4i} \cdot \sum_{k=1}^{2} \omega_k l^{i\omega_k [Q(\xi)+\lambda I]^{\frac{1}{4}}|x-\eta|}[1 + o(1)] \tag{4}$$

is a Green function of the problem (1)-(2) with the "frozen" in point ξ coefficient $Q(\xi)$.

Using conditions 5), 6) and integral equation (3) it is easy to show that for $\lambda \to \infty$ the correlation:

$$Q^s(x)G(x,\eta,\lambda) = Q^s(x)G_0(x,\eta,x,\lambda)[1 + o(1)] \tag{5}$$

holds.

It holds the following:

Theorem 1. *If conditions 1)-6) are fulfilled, then for $\lambda \to \infty$ the formula*

$$\sum_{n=1}^{\infty} \frac{C_n^{(s)}}{(\lambda_n + \lambda)^2} \sim \frac{7\sqrt{2}}{16} \sum_{i=1}^{\infty} \int_0^{\infty} \frac{\beta_i^{2s}(x)dx}{[\beta_i(x) + \lambda]^{7/4}}. \tag{6}$$

holds.

Proof. By the fact that $G(x,\eta,\lambda)$ is the kernel of operator $R_\lambda = (L + \lambda I)^{-1}$ we can write

$$\varphi_n(x) = (\lambda_n + \lambda)\int_0^{\infty} G(x,\eta,\lambda)\varphi_n(\eta)d\eta.$$

Applying the operator $Q^s(x)$ to both sides of this equality we obtain:

$$Q^s(x)\varphi_n(x)=(\lambda_n+\lambda)\int_0^\infty Q^s(x)G(x,\eta,\lambda)\varphi_n(\eta)d\eta .$$

Hence

$$\frac{Q^s(x)\varphi_n(x)}{\lambda_n+\lambda}=\int_0^\infty Q^s(x)G(x,\eta,\lambda)\varphi_n(\eta)d\eta .$$

Taking into account that $Q^s(x)G(x,\eta,\lambda)$ behaves asymptotically as $Q^s(x)G_0(x,\eta,\lambda)$, then for $\lambda\to\infty$ we obtain

$$\frac{Q^s(x)\varphi_n(x)}{\lambda_n+\lambda}\sim\int_0^\infty Q^s(x)G_0(x,\eta,\lambda)d\lambda .$$

Hence

$$\frac{\left\|Q^s(x)\varphi_n(x)\right\|^2}{(\lambda_n+\lambda)^2}\sim\left\|a_n\right\|^2, \tag{7}$$

where $a_n=\int_0^\infty Q^s(x)G(x,\eta,\lambda)\varphi_n(\eta)d\eta$.

Integrating the both sides of (7) on the half-axis, and then summing them we obtain:

$$\sum_{n=1}^\infty\frac{c_n^{(s)}}{(\lambda_n+\lambda)^2}\sim\sum_{n=1}^\infty\int_0^\infty\left\|a_n\right\|_H^2dx . \tag{8}$$

The expressions for a_n remember Fourier coefficients for operator-valued function $Q^{s(x)}G_0(x,\eta,\lambda)$ by orthonormalized system $\{\varphi_n(x)\}$. Then by Parseval's equality we'll have

$$\sum_{n=1}^\infty\left\|a_n\right\|_H^2=\int_0^\infty\sum_{m=1}^\infty r_{mm}^2(x,\eta,\lambda)d\eta , \tag{9}$$

where $r_{mm}(x,\eta,\lambda)$ are diagonal elements of the matrix corresponding to the operator $Q^{s(x)}G(x,\eta,\lambda)$ in the orthonormalized basis composed of eigen vectors $\beta_m(x)$ of the operator $Q(x)$, i.e.

$$r_{mm}(x,\eta,\lambda)=\frac{\beta_m^s(x)[\beta_m(x)+\lambda]^{-3/4}}{4i}\sum_{k=1}^2\omega_k e^{i\omega_k|x-\eta|\{\beta_k(x)+\lambda\}^{1/4}}(1+o(1)).$$

Then from (8)-(9) we obtain the correlation (6). Theorem has been proved.

Correlation (6) is main for obtaining the asymptotic formula for weighted trace $N_s(\mu)$.

With this purpose let's introduce the monotone functions

$$\sigma^{(i)}(\mu)=mes\{\beta_i(x)<\mu\}.$$

Assume

$$\varphi_s^{(i)}(\mu)=\int_0^\mu(\mu-v)^{1/4}v^{2s}d\sigma^{(i)}(v).$$

Then it holds the following:

$$\frac{7\sqrt{2}}{16}\int_0^\infty\frac{\beta_i^{2s}(x)dx}{\{\beta_i(x)+\lambda\}^{7/4}}=\frac{7\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)\cdot\Gamma\left(\frac{7}{4}\right)}\int_0^\infty\frac{d\varphi_s^{(i)}(\mu)}{(\mu+\lambda)^2}. \tag{10}$$

Indeed,

$$\int_0^\infty \frac{d\varphi_s^{(i)}(\mu)}{(\mu+\lambda)^2} = \int_0^\infty \frac{d\int_0^\mu (\mu-\nu)^{1/4}\nu^{2s}d\sigma^{(i)}(\nu)}{(\mu+\lambda)^2} = \frac{1}{4}\int_0^\infty \left[\int_0^\infty \frac{u^{-3/4}du}{(\nu+u+\lambda)^2}\right]\cdot \nu^{2s}d\sigma^{(i)}(\nu) =$$

$$= \frac{1}{4}\int_0^\infty \frac{\nu^{2s}d\sigma^{(i)}(\nu)}{(\nu+\lambda)^{7/4}}\cdot \int_0^\infty \frac{z^{-3/4}}{(1+z)^2}dz = \Gamma\left(\frac{5}{4}\right)\cdot\Gamma\left(\frac{7}{4}\right)\int_0^\infty \frac{\nu^{2s}d\sigma^{(i)}(\nu)}{(\nu+\lambda)^{7/4}}.$$

From (6), (10) it follows

$$\sum_{n=1}^\infty \frac{c_n^{(s)}}{(\lambda_n+\lambda)^2} = \int_0^\infty \frac{dN_s(\mu)}{(\mu+\lambda)^2} \sim \frac{7\sqrt{2}}{16\Gamma\left(\frac{5}{4}\right)\cdot\Gamma\left(\frac{7}{4}\right)}\int_0^\infty \frac{d\varphi_s(\mu)}{(\mu+\lambda)^2}. \tag{11}$$

Here $\varphi_s(\mu) = \sum_{i=1}^\infty \varphi_s^{(i)}(\mu)$.

In order to obtain from (11) the asymptotic formula for $N_s(\mu)$, with help of M.B. Keldysh is Tauber theorem [4] we must put on functions $\varphi_s(\mu)$ the following restrictions.

There exist such positive constants α and β that for sufficient large μ the inequalities:

$$\alpha\varphi_s(\mu) < \mu\varphi_s'(\mu) < \beta\varphi_s(\mu) \tag{12}$$

are fulfilled.

Thus, the following main theorem has been obtained.

Theorem 2. *Let the operator function $Q(x)$ and operators A and B satisfy conditions 1)-6). Suppose in addition the conditions (12) are fulfilled.*

Then for $\mu \to \infty$ the asymptotic formula

$$N_s(\mu) \sim \frac{7\sqrt{2}}{16\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{4}\right)}\sum_{i=1}^\infty \int_{\beta_i(x)<\mu} \beta_i^{2s}(x)[\mu-\beta_i(x)]^{1/4}dx$$

holds.

The authors thank to Prof. M. Bairamoglu for his useful advices.

References

[1]. Костюченко А.Г., Левитан Б.М. *Об асимптотическом поведении собственных значений операторной задачи Штурма-Лиувилля.* Функц.анализ и его прилож., 1967, вып.1, с.86-96.

[2]. Асланов Г.И. *Асимптотика числа собственных значений обыкновенных дифференциалных уравнений с операторными коэффициентами на полуоси.* Доклады АН Азерб.ССР, 1976, т.32, №3, с.3-7.

[3]. Исмаилов С.М. *Резольвента двучленного операторного уравнения четвертого порядка на полуоси с операторными граничными условиями.* Матер.VI Респуб.конф.мол.учен.по мат.и мех. (Баку, 6-8 мая 1985г.), Баку, 1985, с.86-89.

[4]. Келдыш М.В. *Об одной тауберовой теореме.* Труды мат.ин-та им. В.А.Стеклова АН СССР, 1951, 38, с.77-86.

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Received November 25, 1998; Revised February 14, 2000.
Translated by Soltanova S.M.