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THE INVESTIGATION OF STABILITY OF A DIFFERENCE PROBLEM FOR A SYSTEM OF DIFFERENTIAL EQUATIONS OF SIMULTANEOUS PROPAGATION OF SOUND AND HEAT

Abstract

Application of a finite difference method to the solution of mixed problems for system of differential equations describing sound and heat propagation in gas is considered in the paper. The stability of a difference problem approximating the initial value problem to within $O(h+\tau)$ is proved by energetic inequalities method.

It is known that sound propagation in gas with heatconductivity σ, isothermic velocity of sound C and adiabatic constant $\gamma > 1$ is described by a system of differential equations [1]

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\frac{\partial w}{\partial x} - c(\gamma - 1)\frac{\partial e}{\partial x}, \\ \frac{\partial w}{\partial t} &= c\frac{\partial u}{\partial x}, \\ \frac{\partial e}{\partial t} &= \sigma\frac{\partial^2 e}{\partial x^2} - c\frac{\partial u}{\partial x}. \end{aligned} \tag{1}$$

Here $u = u(x,t)$ is sound propagation in gas, $w = w(x,t)$ and $e = e(x,t)$ are perturbations of volume and inner energy of gas.

In the paper we'll consider the following mixed problem for the system (1):

we are to find the functions $u = u(x,t)$, $w = w(x,t)$ and $e = e(x,t)$, $0 \le x \le l$, $0 \le t \le T$, satisfying for $0 < x < l$, $0 < t \le T$ the system (1) and additional conditions

$$u(x,o) = u_0(x),\ w(x,0) = w_0(x),\ e(x,0) = e_0(x),\ 0 \le x \le l, \tag{2}$$

$$u(0,t) = \mu_1(t),\ w(l,t) = \mu_2(t),\ e(0,t) = \nu_1(t),\ e(l,t) = \nu_2(t),\ 0 \le t \le T. \tag{3}$$

§1. Difference problem.

In a closed domain $\overline{D} = \{0 \le x \le l,\ 0 \le t \le T\}$ we define a grid domain $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$, where $\overline{\omega}_h = \left\{x_j = jh,\ j = 0,1,\dots,N,\ h = \frac{l}{N}, x_0 = 0,\ x_N = l\right\}$, $\overline{\omega}_\tau = \left\{t_n = n\tau,\ n = 0,1,\dots,M,\quad \tau = \frac{T}{M}, t_0 = 0,\ t_M = T\right\}$. The value of a grid function $y = y(x,t)$ in the node (x_j, t_n) of the grid $\overline{\omega}_{h\tau}$ is denoted by $y_j^n : y_j^n = y(x_j, t_n)$.

Associate the problem (1)-(3) with the following difference problem:

$$\frac{u_j^{n+1} - u_j^n}{\tau} = c\frac{w_j^n - w_{j-1}^n}{h} - c(\gamma - 1)\frac{e_j^n - e_{j-1}^n}{h},\ j = 1,2,\dots,N,$$

$$\frac{w_j^{n+1} - w_j^n}{\tau} = c\frac{u_{j+1}^{n+1} - u_j^{n+1}}{h},\ j = 0,1,\dots,N-1,$$

$$\frac{e_j^{n+1}-e_j^n}{\tau}=\frac{\sigma}{2}\left(\frac{e_{j-1}^{n+1}-2e_j^{n+1}+e_{j+1}^{n+1}}{h^2}+\frac{e_{j-1}^n-2e_j^n+e_{j+1}^n}{h^2}\right)-\tag{4}$$

$$-c\frac{u_{j+1}^{n+1}-u_j^{n+1}}{h},\ j=1,2,...,N-1, n=0,1,...,M-1,$$

$$u_j^0=u_0(x_j),\ w_j^0=w_0(x_j),\ e_j^0=e_0(x_j),\ j=0,1,...,N,\tag{5}$$

$$u_0^n=\mu_1(t_n),\ w_N^n=\mu_2(t_n),\ e_0^n=\nu_1(t_n),\ e_N^n=\nu_2(t_n),\ n=0,1,...,M.\tag{6}$$

We are to note that this difference problem involving partial derivatives of functions $u(x,t)$, $w(x,t)$ and $e(x,t)$ up to needed orders in variables x and t, approximates the initial value problem (1)-(3) to within $O(h+\tau)$.

We can easily verify that the difference problem (4)-(6) admits under known u_j^n, w_j^n and $e_j^n, j=0,1,..,N$ to find u_j^{n+1}, w_j^{n+1} and e_j^{n+1}, $j=0,1,..,N$.

Let $\alpha=\frac{c\tau}{h}$, $\beta=\frac{\sigma\tau}{2h^2}$. Further we shall use the following denotations adopted in [1] and [2]

$$\Delta_+y_j^n=y_{j+1}^n-y_j^n,\ \ \Delta_-y_j^n=y_j^n-y_{j-1}^n,\ \Delta_-\Delta_+y_j^n=y_{j+1}^n-2y_j^n+y_{j-1}^n,$$

$$(y^n,z^n)=\sum_{j=1}^{N-1}y_j^nz_j^nh,\ \ (y^n,z^n]=\sum_{j=1}^{N}y_j^nz_j^nh,\ \ [y^n,z^n)=\sum_{j=0}^{N-1}y_j^nz_j^nh,$$

$$[y^n,z^n]=\sum_{j=0}^{N}y_j^nz_j^nh,\ \ \|y^n\|=\sqrt{(y^n,y^n)},\ \ \|y^n]|=\sqrt{(y^n,y^n]},$$

$$|[y^n\|=\sqrt{[y^n,y^n)},\ \ |[y^n]|=\sqrt{[y^n,y^n]}.$$

Using these denotations we rewrite the difference equations (4) in the following way:

$$\begin{cases}u_j^{n+1}-u_j^n=\alpha\Delta_-w_j^n-\alpha(\gamma-1)\Delta_-e_j^n,\ j=1,2,..,N,\\ w_j^{n+1}-w_j^n=\alpha\Delta_+u_j^{n+1},\ j=0,1,..,N-1,\\ e_j^{n+1}-e_j^n=\beta\Delta_-\Delta_+\left(e_j^{n+1}+e_j^n\right)-\alpha\Delta_+u_j^{n+1},\\ j=1,2,...,N-1,\ \ n=0,1,..M-1.\end{cases}\tag{7}$$

§2. The investigation of stability of a difference problem.

Consider the difference equations (7) with homogeneous boundary conditions

$$u_0^n=0,\ w_N^n=0,\ e_0^n=0,\ e_N^n=0,\ n=0,1,...,M,\tag{8}$$

and initial conditions (5). We shall study the stability of the difference problem (7), (8), (5) by energetic inequalities method.

Multiply both sides of difference equations in (7) by $\left(u_j^{n+1}+u_j^n\right)h$, $\left(w_j^{n+1}+w_j^n\right)h$, $\left(e_j^{n+1}+e_j^n\right)h$ respectively, and sum the obtained equalities in all j:

$$\|u^{n+1}]|^2-\|u^n]|^2=\alpha\sum_{j=1}^{N}\Delta_-w_j^n\left(u_j^{n+1}+u_j^n\right)h-$$

$$-\alpha(\gamma-1)\sum_{j=1}^{N}\Delta_-e_j^n\left(u_j^{n+1}+u_j^n\right)h,$$

$$\left|\left[w^{n+1}\right|\right|^2-\left|\left[w^n\right|\right|^2=\alpha\sum_{j=0}^{N-1}\Delta_+u_j^{n+1}\left(w_j^{n+1}+w_j^n\right)h,$$

$$\left\|e^{n+1}\right\|^2-\left\|e^n\right\|^2=\beta\sum_{j=1}^{N-1}\Delta_-\Delta_+\left(e_j^{n+1}+e_j^n\right)\left(e_j^{n+1}+e_j^n\right)h-$$

$$-\alpha\sum_{j=1}^{N-1}\Delta_+u_j^{n+1}\left(e_j^{n+1}+e_j^n\right)h.$$

By virtue of boundary conditions of (8) after some transformations we get:

$$\sum_{j=0}^{N-1}\Delta_+u_j^{n+1}\left(w_j^{n+1}+w_j^n\right)h=-\left(u^{n+1},\Delta_-\left(w^{n+1}+w^n\right)\right],$$

$$\sum_{j=1}^{N-1}\Delta_+u_j^{n+1}\left(e_j^{n+1}+e_j^n\right)h=-\left(u^{n+1},\Delta_-\left(e^{n+1}+e^n\right)\right],$$

$$\sum_{j=1}^{N-1}\Delta_-\Delta_+\left(e_j^{n+1}+e_j^n\right)\left(e_j^{n+1}+e_j^n\right)h=-\left\|\Delta_+\left(e^{n+1}+e^n\right)\right\|^2-\left(e_1^{n+1}+e_1^n\right)^2h.$$

With regard to these three equalities we can write the last three equalities in the form

$$\left\|u^{n+1}\right]\right|^2-\left\|u^n\right]\right|^2=\alpha\left(u^{n+1}+u^n,\Delta_-w^n\right]-$$

$$-\alpha(\gamma-1)\left(u^{n+1}+u^n,\Delta_-e^n\right],$$

$$\left|\left[w^{n+1}\right\|^2-\left|\left[w^n\right\|^2=-\alpha\left(u^{n+1},\Delta_-\left(w^{n+1}+w^n\right)\right],$$

$$\left\|e^{n+1}\right\|^2-\left\|e^n\right\|^2=-\beta\left\|\Delta_+\left(e^{n+1}+e^n\right)\right\|^2-\beta\left(e_1^{n+1}+e_1^n\right)^2h+$$

$$+\alpha\left(u^{n+1},\Delta_-\left(e^{n+1}+e^n\right)\right].$$

The validity of the following equality follows from these three equalities

$$S_{n+1}-S_n=-\beta(\gamma-1)\left[\left\|\Delta_+\left(e^{n+1}+e^n\right)\right\|^2+\left(e_1^{n+1}+e_1^n\right)^2h\right], \tag{9}$$

where

$$S_n=\left|\left[u^n\right]\right|^2+\left|\left[w^n\right]\right|^2+(\gamma-1)\left|\left[e^n\right]\right|^2+\alpha\left(u^n,\Delta_-\left(w^n-(\gamma-1)e^n\right)\right]. \tag{10}$$

From (9) by virtue of inequalities $\beta>0,\gamma>1$ we have

$$S_{n+1}\le S_n,\ n=0,1,\dots,M-1. \tag{11}$$

Now prove that by fulfilling some conditions S_n determines the norm.

Let's consider $\left|\left(u^n,\Delta_-w^n\right]\right|$:

$$\begin{gathered}\left|\left(u^n,\Delta_-w^n\right]\right|\le\left\|u^n\right]\right|\cdot\left\|\Delta_-w^n\right]\right|\le\left\|u^n\right]\right|\times\\ \times\sqrt{\sum_{j=1}^{N}\left(\Delta_-w_j^n\right)^2h}\le\left|\left[u^n\right]\right|\sqrt{4\sum_{j=0}^{N}\left(w_j^n\right)^2h}\le\\ \le2\left|\left[u^n\right]\right|\left|\left[w^n\right]\right|\le\left|\left[u^n\right]\right|^2+\left|\left[w^n\right]\right|^2.\end{gathered} \tag{12}$$

In a similar way we have

$$\left|\left(u^n,\Delta_-e^n\right]\right|\le\left\|u^n\right]\right|\cdot\left\|\Delta_-e^n\right]\right|\le\left|\left[u^n\right]\right|^2+\left|\left[e^n\right]\right|^2. \tag{13}$$

With regard to the last two equalities for S_n we have:

$$S_n \geq (1-\alpha\gamma)\left\|\left[u^n\right]\right\|^2 + (1-\alpha)\left\|\left[w^n\right]\right\|^2 + (\gamma-1)(1-\alpha)\left\|\left[e^n\right]\right\|^2 . \tag{14}$$

Hence, by virtue of $\gamma > 1$, it follows that if $1-\alpha\gamma > 0$ or the same $\tau < h/c\gamma$, then $S_n \geq 0$.

So, if it the condition

$$\tau < \frac{h}{c\gamma} \tag{15}$$

is fulfilled. Then S_n determines the norm. On the other hand by virtue of the inequality (11) we have:

$$S_n \leq S_0, \quad n = 1,2,...,M . \tag{16}$$

Estimate S_0 from above. With regard to inequalities (12) and (13) we have:

$$\begin{aligned} S_0 &\leq (1+\alpha\gamma)\left\|[u^0]\right\|^2 + (1+\alpha)\left\|[w^0]\right\|^2 + (\gamma-1)(1+\alpha)\left\|[e^0]\right\|^2 \leq \\ &\leq K\left(\left\|[u^0]\right\|^2 + \left\|[w^0]\right\|^2 + \left\|[e^0]\right\|^2\right), \end{aligned} \tag{17}$$

where

$$K = \max\{1+\alpha\gamma, (\gamma-1)(1+\alpha)\}. \tag{18}$$

Thus, by virtue of inequalities (16) and (17) the next theorem is valid

Theorem 1. *For solving a difference problem (7), (8), (5) by fulfilling the condition (15) the estimation*

$$S_n \leq K\left(\left\|[u_0]\right\|^2 + \left\|[w_0]\right\|^2 + \left\|[e_0]\right\|^2\right), \quad n = 1,2,...,M,$$

is valid. Where S_n and K are respectively determined by equalities (10) and (18).

Let $0 < \varepsilon < 1$ be an arbitrary number. Instead of the condition (15) we require the fulfillment of the condition

$$\tau \leq \frac{h(1-\varepsilon)}{c\gamma} \tag{19}$$

By fulfilling this inequality we have

$$1-\alpha\gamma \geq \varepsilon, \quad 1-\alpha \geq \frac{\varepsilon}{\gamma}.$$

With regard to these inequalities, by virtue of the inequality (14) we get:

$$S_n \geq \varepsilon \cdot L\left(\left|[u^n]\right|^2 + \left|[w^n]\right|^2 + \left|[e^n]\right|^2\right),$$

where

$$L = \min\left(\frac{1}{\gamma}, \frac{\gamma-1}{\gamma}\right). \tag{20}$$

Hence and from the inequalities (16) and (17) validity of the inequality

$$\left\|[u^n]\right\|^2 + \left\|[w^n]\right\|^2 + \left\|[e^n]\right\|^2 \leq \frac{K}{\varepsilon L}\left(\left\|[u_0]\right\|^2 + \left\|[w_0]\right\|^2 + \left\|[e_0]\right\|^2\right) \tag{21}$$

follows.

Theorem 2. *By fulfilling condition (19) the difference problem (7), (8), (5) is stable and for its solution the estimation (21) is valid, where $0 < \varepsilon < 1$ is an arbitrary number, K and L are positive constants that are defined by the equalities (18) and (20).*

References

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