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NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM WITH THE INTEGRAL CONDITION

Abstract

In this paper the boundary value problems with the integral condition are investigated numerically.

In the work some boundary-value problems with the integral condition are numerically solved by Riccati transformation [1] method.

1. First let's consider the boundary-value problem with the integral condition for a linear differential equation of the first order

$$y' = a(x)y + b(x) \quad (0 \le x \le T), \tag{1}$$

$$\int_0^T \alpha(s)y(s)ds = \gamma_0 .$$

Let's consider the function

$$z(x) = \int_0^x \alpha(s)y(s)ds .$$

Then boundary-value problem (1) is equal to the following problem

$$\begin{gathered} y' = a(x)y + b(x), \\ z'(x) = \alpha(x)y(x), \quad 0 \le x \le T, \\ z(0) = 0, \quad z(T) = \gamma_0 . \end{gathered} \tag{2}$$

Let's divide segment $[0,T]$ into N equal parts with step $h = \frac{T}{N}$; $0 = x_0 < x_1 < ... < x_N = b$ and consider the difference scheme [2]:

$$\begin{gathered} y_{i+1} = (1 + ha_i)y_i + hb_i , \\ z_{i+1} = z_i + h\alpha_i y_i \quad (i = 0,1,...,N-1), \\ z_0 = 0,\ z_N = \gamma_0 , \end{gathered} \tag{3}$$

Here $a_i = a(x_i)$, $b_i = b(x_i)$, $\alpha_i = \alpha(x_i)$. We'll search the solution of difference scheme (3) in the form

$$z_i = R_i y_i + S_i . \tag{4}$$

Then $z_{i+1} = R_{i+1}y_{i+1} + S_{i+1}$.

Taking into account relation (3) we have

$$\begin{gathered} R_i y_i + S_i + h\alpha_i y_i = R_{i+1}(1 + h\alpha_i)y_i + R_{i+1}hb_i + S_{i+1}, \\ R_i = R_{i+1}(1 + ha_i) - h\alpha_i ,\ S_i = S_{i+1} + hR_{i+1}b_i \quad (i = 0,1,...,N-1) . \end{gathered} \tag{5}$$

Let's take $i = N$ in equality (4), then taking into account relation $z_N = \gamma_0$ we have

$$R_N = 0,\ S_N = \gamma_0 .$$

By these values from equality (5) we find $R_{N-1}, S_{N-1},...,R_0,S_0$. By the known values R_0S_0 and from the equality $z_0 = 0$ we determine y_0. Further, the values of the solution at other points of the net are found.

2. Now let us consider the boundary-value problem for the equations of the second order:

$$
\begin{aligned}
&y'' = a(x)y' + b(x)y + c(x) \qquad 0 \le x \le T, \\
&y'(0) = \gamma_0, \\
&\int_0^T \alpha(s)y(s)ds = \gamma_1 .
\end{aligned} \tag{6}
$$

Introducing the denotations

$$y' = V, \ z(t) = \int_0^t \alpha(s)y(s)ds$$

for the given problem we have the difference scheme:

$$
\begin{aligned}
&y_{i+1} = y_i + hV_i, \\
&V_{i+1} = (1 + ha_i)V_i + hb_i y_i + hc_i, \\
&z_{i+1} = z_i + h\alpha_i y_i, \quad (i = 0,1,\ldots,N-1) \\
&V_0 = \gamma_0, \ z_0 = 0, \ z_N = \gamma_1 .
\end{aligned} \tag{7}
$$

The solution of this problem is sought in the form

$$z_i = R_i y_i + p_i V_i + S_i .$$

Then

$$z_{i+1} = R_{i+1} y_{i+1} + p_{i+1} V_{i+1} + S_{i+1} .$$

Taking into account relation (7) we obtain

$$
\begin{aligned}
R_i y_i + p_i V_i + S_i + h\alpha_i y_i = R_{i+1} y_i + hR_{i+1} V_i + \\
+ p_{i+1} V_i + ha_i p_{i+1} V_i + hb_i p_{i+1} y_i + hp_{i+1} c_i + S_{i+1}, \\
R_i = R_{i+1} + hb_i p_{i+1} - h\alpha_i, \ \ p_i = hR_{i+1} + p_{i+1} + ha_i p_{i+1}, \\
S_i = S_{i+1} + hp_{i+1} c_i, \qquad (i = 0,1,\ldots,N-1).
\end{aligned}
$$

Further from the equality

$$R_N y_N + p_N V_N + S_N = \gamma_1 ,$$

we have

$$R_N = 0, \ p_N = 0, \ S_N = \gamma .$$

By the known values R_N, p_N, S_N we find $R_N, p_N, S_N, \ldots, R_0, p_0, S_0$.

The unknown value y_0 is found by the values R_0, p_0, S_0 found from the equality $z_i = R_i y_i + p_i y_i + S_i$ for $i = 0$. The values of the solution and its derivative of the first order at the nodes of the net are determined with help of equality (7).

3. Finally let us consider the analogous boundary-value problem for the equations of the third order

$$
\begin{aligned}
&y''' = a(x)y'' + b(x)y' + c(x)y + f(x) \quad 0 \le x \le T, \\
&y(0) = \gamma_0, \\
&y'(0) = \gamma_1, \\
&\int_0^T \alpha(s)y(s)ds = \gamma_2 .
\end{aligned} \tag{8}
$$

Introducing the denotations

$$y' = V, \ V' = u, \ z(x) = \int_0^x \alpha(s)y(s)ds,$$

the boundary-value problem (8) will be equal to the boundary-value problem

$$
\begin{gathered}
u' = a(x)u + b(x)V + c(x)y + f(x) \quad 0 \le x \le T, \\
y' = V, \\
V' = u, \\
z'(x) = \alpha(x)y(x), \\
y(0) = \gamma_0, \;\; y'(0) = \gamma_1, \\
z = 0, \;\; z(T) = \gamma_2 .
\end{gathered} \tag{9}
$$

The difference scheme for this problem is such:

$$
\begin{aligned}
& u_{i+1} = (1 + ha_i)u_i + hb_iV_i + hc_iy_i + hf_i, \\
& y_{i+1} = y_i + hV_i, \quad (i = 1,2,\dots,N-1), \\
& V_{i+1} = V_i + hu_i, \quad y_0 = \gamma_0, \; V_0 = \gamma_1, \\
& z_{i+1} = z_i + h\alpha_i y_i \quad z_0 = 0, \; z_N = \gamma_2 .
\end{aligned} \tag{10}
$$

The solution of this scheme we seek in the form

$$
z_i = R_iu_i + P_iy_i + \Omega_iV_i + S_i . \tag{11}
$$

Then

$$
z_{i+1} = R_{i+1}u_{i+1} + P_{i+1}y_{i+1} + \Omega_{i+1}V_{i+1} + S_{i+1} .
$$

Here taking into account the relation (10) we'll obtain

$$
\begin{gathered}
R_iu_i + P_iy_i + \Omega_iV_i + S_i + h\alpha_iy_i = R_{i+1}(1 + ha_i)u_i + R_{i+1}hb_iV_i + \\
+ hR_{i+1}c_iy_i + hR_{i+1}f_i + P_{i+1}y_i + hP_{i+1}V_i + \Omega_{i+1}V_i + h\Omega_{i+1}u_i + S_{i+1}; \\
(R_i - R_{i+1}(1 + ha_i) - h\Omega_{i+1})u_i + (P_i + h\alpha_i - hR_{i+1}c_i - P_{i+1})y_i + \\
+ (\Omega_i - R_{i+1}hb_i - hP_{i+1} - \Omega_{i+1})V_i = hR_{i+1}f_i + S_{i+1} - S_i .
\end{gathered}
$$

Hence we have

$$
\begin{gathered}
R_i = R_{i+1}(1 + ha_i) + h\Omega_{i+1}, \\
P_i = P_{i+1} - h\alpha_i + hR_{i+1}c_i, \\
\Omega_i = R_{i+1}hb_i + hP_{i+1} + \Omega_{i+1}, \\
S_i = S_{i+1} + hR_{i+1}f_i ,
\end{gathered} \tag{12}
$$

Let's assume $i = N$ in the equality (11), then we have

$$
R_Nu_N + P_Ny_N + \Omega_NV_N + S_N = \gamma_2 .
$$

Hence we have

$$
R_N = P_N = \Omega_N = 0, \;\; S_N = \gamma_2 .
$$

By these values the values $R_{N-1}, P_{N-1}, \Omega_{N-1}, S_{N-1}, R_0, P_0, \Omega_0, S_0$ are found from (12).

By the found values R_0, P_0, Ω_0, S_0 and by the given values y_0, V_0 for $i = 0$ u_0 is found from the equality (11). Further the solutions of the boundary-value problem and their derivatives of the first and second order at the nodes of the net are found from equalities (10).

Thus it holds the following

Theorem. *Let boundary value problem (6) has a unique solution. Then the approximate solution obtained by formulas (10) converges to this solution with velocity* h.

References

[1]. Беллман Р., Энджел Э. *Динамическое программирование и уравнения в частных производных.* Мир, 1974.

[2]. Самарский А.А. *Введение в теорию разностных схем.* Наука, Москва.

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