BALAYEV M.K.

ON THE SPECTRAL PROPERTIES OF THE OPERATOR, GENERATED BY IRREGULAR BOUNDARY CONDITIONS AND ORDINARY DIFFERENTIAL EXPRESSIONS

Abstract

In the paper it is proved, that the resolvent of the operator, generated by the general irregular boundary conditions and the ordinary differential expressions of the second order, have not the minimal growth.

As it is known irregular in Birkoff-Tamarkin sense boundary-value conditions for ordinary differential operator of the second order have the form [1, p.433]

$$L_1 u = \gamma u'(0) + \delta u'(1) + \alpha u(0) + \beta u(1) = 0,$$
$$L_2 u = \gamma u(0) - \delta u(1) = 0.$$

In 1927 Stone [2] investigated Green's function of the operator L generated by such boundary value conditions and proved expansion of functions in series by eigen and joined functions.

In S.Y. Yakubov and K.S. Mamedov's work [3] it was shown that in the regular case the minimal growth of the resolvent is stipulated for any order of the ordinary differential equation, but for an odd order the class of irregular boundary value conditions was chosen for which minimal growth exists. But if the boundary value conditions are not regular then the resolvent can not have maximum order of decreasing and even can increase for $|\lambda| \to \infty$. So, if $L_0 u = -u''(x), L_1 u = u(0), \; L_2 u = \int_0^1 u(x) dx$, then

$$\left\| (L + \lambda I)^{-1} \right\|_{B(L_p(0,1))} < C |\lambda|^{-\frac{1}{2} - \frac{1}{2p}}.$$

But if $L_1 u = u(0), L_2 u = u'(0)$, then

$$\left\| (L + \lambda I)^{-1} \right\|_{B(L_p(0,1))} = C \left| \lambda^{-1} e^{\sqrt{\lambda}} \right|.$$

In [4] it was shown that for the class of irregular boundary value conditions if the quantity $\theta_0 = \alpha\delta + \beta\gamma \neq 0$, then minimal growth of resolvent doesn't exist.

In the given paper some classes of irregular boundary value conditions have been selected for which $\theta_0 = 0$ and the resolvent can not have maximal order of decreasing and can increase for $|\lambda| \to \infty$.

Let's consider in space $L_2(0,1)$ the operator L determined by the equalities

$$D(L) = W_2^2((0,1)); \; L_\nu u = 0, \; \nu = 1,2), \; Lu = u''(x).$$

Let's consider the equation:

$$Lu - \lambda u = f. \tag{1}$$

Theorem 1. *Let $\theta_0 = \alpha\delta + \beta\gamma = 0, \theta_1 = \delta^2 - \gamma^2 \neq 0$. Then $\forall \varepsilon > 0 \; \exists R_\varepsilon$ such that all points of the complex plane, for which $|\arg \lambda| < \pi - \varepsilon$, $|\lambda| > R_\varepsilon$ are regular for L and for solution (1) it holds the estimation:*

$$e^{-\operatorname{Re}\sqrt{\lambda}}\|u''\| + |\lambda|^{\frac{1}{2}}e^{-\operatorname{Re}\sqrt{\lambda}}\|u'\| + |\lambda|e^{-\operatorname{Re}\sqrt{\lambda}}\|u\| \le C_{\varepsilon}\|f\|. \tag{2}$$

Proof. Let us denote $\lambda = \rho^2$, where $|\arg\rho| < \dfrac{\pi-\varepsilon}{2}$. Then equation (1) is equal to the problem:

$$u''(x) - \rho^2 u(x) = f(x), \quad L_1 u = 0, \quad L_2 u = 0.$$

The solution of this problem is represented in the form of sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is contraction on $[0,1]$ of the solution of the equation $u_1''(x) - \rho^2 u_1(x) = \tilde{f}$ on the whole axis, where

$$\tilde{f}(x) = \begin{cases} f(x), & x \in [0,1] \\ 0, & x \overline{\in} [0,1] \end{cases}$$

and $u_2(x)$ is the solution of the problem:

$$u_2''(x) - \rho^2 u(x) = 0, \quad L_1 u_2 = -L_\nu u_1, \quad \nu = 1,2. \tag{3}$$

By help of Fourier transformation it is easy to prove the estimation

$$\|u_1^{(k)}\| \le c|\rho|^{-(2-k)}\|f\|, \quad k = 0,1,2. \tag{4}$$

Now let's estimate the solution of problem (3). The general solution $u_2''(x) - \rho^2 u_2(x) = 0$ has the form:

$$u_2(x) = c_1 e^{-\rho x} + c_2 e^{\rho(x-1)}. \tag{5}$$

Putting (5) in the boundary conditions of problem (3) we obtain a system for determination of c_1 and c_2

$$\begin{cases} \left(-\gamma\rho - \delta\rho e^{-\rho} + \alpha + \beta e^{-\rho}\right)c_1 + \left(\gamma\rho e^{-\rho} + \delta\rho + \alpha e^{-\rho} + \beta\right)c_2 = -L_1 u_1 \\ \left(\gamma - \delta e^{-\rho}\right)c_1 + \left(\gamma e^{-\rho} - \delta\right)c_2 = -L_2 u_1. \end{cases} \tag{6}$$

The determinant of system (6) has the form $D(\rho) = \theta_1 R(\rho)$ where $R(\rho) = 2\rho e^{-\rho}$.

Then in sector $|\arg\rho| < \dfrac{\pi-\varepsilon}{2}$ for $|\rho| \to \infty$ we have

$$\|u_2\| \le c\left(\int_0^1 \left(\left|e^{-\rho x}\right|\frac{|1+e^{\rho}|}{|\rho|}|L_1 u_1| + \left|e^{\rho(x-1)}\right|\left(\left|c_1 + \frac{1}{\rho}c_2\right|\cdot\left|e^{\rho}+1\right|\right)\cdot|L_2 u_2|\right)^2 dx\right)^{\frac{1}{2}} \le ce^{\operatorname{Re}\rho}.$$

$$\cdot\left[|\rho|^{-\frac{3}{2}}|L_1 u_1| + \left(|\rho|^{-\frac{1}{2}} + |\rho|^{-\frac{3}{2}}\right)|L_2 u_1|\right]. \tag{7}$$

In order to estimate $|L_\nu u_1|$, let us use the formula in [5, p.145]

$$\sup_{x\in[0,1]}\left|u^{(k)}(x)\right| \le c\left(h^{1-\chi}\|u''\| + h^{-\chi}\|u\|\right),$$

where $0 \le k \le 2$, $0 < h \le h_0$, $\chi = \dfrac{k+\dfrac{1}{2}}{2}$. Let $h = |\rho|^{-2}$. Denote the order of the operator L_ν by k_ν. Then $k_1 = 1$, $k_2 = 0$ and by virtue of (4)

$$|L_\nu u_1| \le C\|u_1\|_{C^{k_\nu}[0,1]} \le C\sum_{j=0}^{k_\nu}\left(|\rho|^{-2\left(1-\frac{j+\frac{1}{2}}{2}\right)}\|u_1''\| + |\rho|^{2\cdot\frac{j+\frac{1}{2}}{2}}\|u_1\|\right) \le C|\rho|^{k_\nu-\frac{3}{2}}\|f\|.$$

Putting these estimations in (7), we obtain in sector $\left|\arg\rho\right| < \dfrac{\pi-\varepsilon}{2}$ for sufficient large $|\rho|$

$$\|u_2\| \le C_\varepsilon e^{\operatorname{Re}\rho}\left(|\rho|^{-\frac{3}{2}}|\rho|^{1-\frac{3}{2}} + \left(|\rho|^{-\frac{1}{2}} + |\rho|^{-\frac{3}{2}}\right)\cdot|\rho|^{-\frac{3}{2}}\right)\cdot\|f\| \le$$

$$\le C_\varepsilon e^{\operatorname{Re}\rho}\left(|\rho|^{-2} + |\rho|^{-3}\right)\cdot\|f\| \le C_\varepsilon e^{\operatorname{Re}\rho}|\rho|^{-2}\|f\|. \tag{8}$$

From (4) and (8) we get the estimation

$$\|u\| \le \|u_1\| + \|u_2\| \le C_\varepsilon\left(|\rho|^{-2}\|f\| + e^{\operatorname{Re}\rho}|\rho|^{-2}\|f\|\right) \le$$

$$\le C_\varepsilon|\rho|^{-2}\left(e^{\operatorname{Re}\rho} + 1\right)\|f\| \le C_\varepsilon|\lambda|^{-1}e^{\operatorname{Re}\sqrt{\lambda}}\|f\|. \tag{9}$$

in sector $\left|\arg\lambda\right| < \pi - \varepsilon$ for sufficient large $|\lambda|$. Further, since $u''(x) = \lambda u(x) + f(x)$, then

$$\|u''(x)\| \le |\lambda|\|u\| + \|f\| \le C_\varepsilon e^{\operatorname{Re}\sqrt{\lambda}}\|f\|, \tag{10}$$

and finally, using the estimation

$$\|u'(x)\| \le C\left(\|u''\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + \|u\|\right),$$

cited for example, in [5, p.237] we get

$$\|u'\| \le C_\varepsilon\left(e^{\frac{1}{2}\operatorname{Re}\sqrt{\lambda}}\|f\|^{\frac{1}{2}}|\lambda|^{-\frac{1}{2}}e^{\frac{1}{2}\operatorname{Re}\sqrt{\lambda}}\|f\|^{\frac{1}{2}} + |\lambda|^{-1}e^{\operatorname{Re}\sqrt{\lambda}}(f)\right) \le$$

$$\le C_3|\lambda|^{-\frac{1}{2}}e^{\operatorname{Re}\sqrt{\lambda}}\|f\|. \tag{11}$$

Then on the base of (9), (10), (11) we obtain the required estimation (2).

Theorem 2. *Let* $\theta_0 = \alpha\delta + \beta\gamma = 0$, $\theta_1 = \delta^2 - \gamma^2 \ne 0$. *Then* $\forall\varepsilon > 0$, $\exists R_\varepsilon > 0$ *such that*

$$\|R(\lambda, L)\| \le C_\varepsilon|\lambda|^{-1}e^{\operatorname{Re}\sqrt{\lambda}},$$

for $\left|\arg\lambda\right| < \pi - \varepsilon$, $|\lambda| > R_\varepsilon$.

The proof follows from theorem 1 and estimation (9). From theorem 2 as a corollary it is easy to get the more general theorem.

Let's consider on the space $L_2(0,1)$ the operator determined by equalities $D(L_a) = W_2^2\left((0,1), L_\nu u = 0, \nu = 1,2\right)$, $L_a u = au''(x)$.

Theorem 3. *Let* $\theta_0 = \alpha\delta + \gamma\beta = 0$, $\theta_1 = \delta^2 - \gamma^2 \ne 0$. *Then* $\forall\varepsilon > 0$ $\exists R_\varepsilon$, *such that*

$$\|R(\lambda, L_a)\| \le C_\varepsilon|\lambda|^{-1}e^{\operatorname{Re}\sqrt{\lambda a^{-1}}}.$$

for $\left|\arg\lambda - \arg a\right| < \pi - \varepsilon$, $|\lambda| > R_\varepsilon$.

Proof. Since $L_a - \lambda I = aL - \lambda I = a\left(L - \lambda a^{-1}I\right)$, then for $\left|\arg\lambda a^{-1}\right| < \pi - \varepsilon, \left|\lambda a^{-1}\right| > R'_\varepsilon$, on the base of theorem 2 we get

$$\left\|\left(L - \lambda a^{-1}I\right)^{-1}\right\| \le C'_\varepsilon\left|\lambda a^{-1}\right|^{-1}e^{\operatorname{Re}\sqrt{\lambda a^{-1}}}$$

so for

$$\left|\arg\lambda - \arg a\right| < \pi - \varepsilon, \ |\lambda| > R_\varepsilon, \quad \left\|\left(L_a - \lambda I\right)^{-1}\right\| = |a|\left\|\left(L_a - \lambda a^{-1}\right)^{-1}\right\| \le c_\varepsilon|\lambda|^{-1}e^{\operatorname{Re}\sqrt{a^{-1}\lambda}}.$$

Q.E.D.

References

[1]. Данфорд Н., Шварц Дж.Т. *Линейные операторы.* Т.3., М., 1974, 661 с.

[2]. Stone M.N. *Irregular differential systems of the second order and related expansion problems.* Trans. Amer. Math. Soc., 29, 1927, 23-53.

[3]. Якубов С.Я. Мамедов К.С. *Полнота собственных и присоединенных функций некоторых нерегулярных краевых задач для обыкновенных дифференциальных уравнений.* Функциональный анализ и его приложения.1980, 14, вып. 4, с. 93-94.

[4]. Якубов С.Я. *Линейные дифференциальные операторные уравнения и их приложения.* Баку, Элм, 1985, 220 с.

[5]. Бесов О.В., Ильин В.П., Никольский С.М. *Интегральные представления функций и теоремы вложения.* М., 1975, 480 с.

Balayev M.K.

Institute of Mathematics & Mechanics of Azerbaijan AS.

9, F. Agayev str., 370141, Baku, Azerbaijan.