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ON THE UNIFORM CONVERGENCE OF THE GENERALIZED BIEBERBACH POLYNOMIALS IN REGIONS WITH K-QUASICONFORMAL BOUNDARY

Abstract

Let G be a finite domain in the complex plane with K-quasiconformal boundary, $z_\circ$ be an arbitrary fixed point in G and $p>0$. Let $\varphi(z)$ be the conformal mapping from G onto the disk with radius $r_\circ>0$ and centered at the origin 0, normalized by $\varphi(z_\circ)=0$ and $\varphi'(z_\circ)=1$. Let us set $\varphi_p(z):=\int_{z_\circ}^{z}[\varphi'(\zeta)]^{2/p}d\zeta$, and let $\pi_{n,p}(z)$ be the generalized Bieberbach polynomial of degree n for the pair $(G,z_\circ)$ that minimizes the integral $\iint_G|\varphi'_p(z)-P'_n(z)|^p d\sigma_z$ in the class $\tilde{A}_n$ of all polynomials of degree $\le n$ and satisfying the conditions $P_n(z_\circ)=0$ and $P'_n(z_\circ)=1$. In this work we prove the uniform convergence of the generalized Bieberbach polynomials $\pi_{n,p}(z)$ to $\varphi_p(z)$ on $\overline{G}$ in case of $p>2-\dfrac{K^2+1}{2K^4}$.

1.Introduction.

Let G be a finite domain in the complex plane bounded by a Jordan curve $L:=\partial G$, $\Omega:=ext\overline{G}$ and $z_\circ$ be an arbitrary fixed point in G. Let $w=\varphi(z)$ be the conformal mapping of G onto the disk $B(0,r_\circ):=\{w:|w|<r_\circ\}$ with $\varphi(z_\circ)=0$, $\varphi'(z_\circ)=1$; $r_\circ$ is called the conformal radius of G with respect to $z_\circ$. We denote ψ by the inverse of φ.

Let $p>0$. It is well known [24: p.435] that the function $\varphi_p(z):=\int_{z_\circ}^{z}[\varphi'(\zeta)]^{2/p}d\zeta$, $z\in G$, minimizes the integral

$$\|f\|_{L^1_p(G)}:=\|f'\|_{L_p(G)}:=\left(\iint_G|f'(z)|^p d\sigma_z\right)^{1/p}$$

in the class of all functions analytic in G and normalized $f(z_\circ)=0, f'(z_\circ)=1$.

Let $\tilde{A}_n$ be the class of all polynomials $P_n(z)$ of degree not exceeding n, and satisfying $P_n(z_\circ)=0, P'_n(z_\circ)=1$, and let $p>0$. Using a method similar to the one given in [11:p.137], it is seen that there exists a polynomial $\pi_{n,p}(z)$ furnishing a minimum to the integral $\|\varphi_p-P_n\|_{L^1_p(G)}$ in $\tilde{A}_n$, and if $p>1$ these polynomials $\pi_{n,p}(z)$ are determined uniquely[11:p.142]. We call such a polynomial $\pi_{n,p}(z)$ the generalized Bieberbach polynomial of degree n for the pair $(G,z_\circ)$ as it is in [17]. When $p=2$, let us emphasize that $\pi_{n,2}(z)$ is the same as the n-th Bieberbach polynomial for the pair (G, z_o), see ,for example [12].

If G is a Carathéodory region, then $\left\|\varphi_p - \pi_{n,p}\right\|_{L^1_p(G)} \to 0$ for $n \to \infty$ [28:p.63], and so the sequence $\{\pi_{n,p}(z)\}$ converges uniformly to $\varphi_p(z)$ on compact subsets of G. Therefore, the study of the uniform convergence of the sequence $\{\pi_{n,p}(z)\}$ in $\overline{G}$ and the estimation of the error $\left\|\varphi_p - \pi_{n,p}\right\|_{C(\overline{G})} := \max\left\{\left|\varphi_p(z) - \pi_{n,p}(z)\right| : z \in \overline{G}\right\}$ is closely related to the geometric properties of G.

In case of $p = 2$, M.V. Keldysh showed [18] that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ independent of n such that $\left\|\varphi - \pi_{n,2}\right\|_{C(\overline{G})} \le \frac{C(\varepsilon)}{n^{1-\varepsilon}}$ for every natural number n if the boundary L of G satisfies certain smoothness conditions. After that, the uniform convergence of the polynomials $\pi_{n,2}(z)$ to $\varphi(z)$ in the closed region $\overline{G}$ have been studied in [12],[13],[19],[20],[23],[25],[27]and [29] by weakening the conditions on the boundary L. If L is a K-quasiconformal curve, in [5] V.V. Andrievskii proved that there exist constants $C > 0$ and $\gamma > 0$ independent of n such that $\left\|\varphi - \pi_{n,2}\right\|_{C(\overline{G})} \le \frac{C}{n^{\gamma}}$ for all natural numbers n but he did not mention how to choose γ. In [21], M.Leclerc figured out γ in V.V. Andrievskii's work to be taken in $(0, \frac{1}{2K^2})$ arbitrarily. When L is piecewise quasiconformal , the problem of $\left\|\varphi - \pi_{n,2}\right\|_{C(\overline{G})} \to 0$ for $n \to \infty$ has been studied in [2],[4] and [6].

If $p > 2$ and L is a K-quasiconformal curve, D.M. Israfilov proved that there exist constants $C > 0$ and $\gamma > 0$ independent of n, not mentioning how to choose γ expilicity, such that $\left\|\varphi_p - \pi_{n,p}\right\|_{C(\overline{G})} \le \frac{C}{n^{\gamma}}$ [17].

In this work , we consider the case in which L is a K--quasiconformal curve and $p > 2 - \dfrac{K^2+1}{2K^4}$, and we prove the following theorem:

Main Theorem. *Let G be a finite domain in the complex plane with a K-quasiconformal boundary L and $p > 2 - \dfrac{K^2+1}{2K^4}$. Then there exists a constant $C > 0$ independent of n such that*

$$\left\|\varphi_p - \pi_{n,p}\right\|_{C(\overline{G})} \le C\, n^{-\gamma}$$

for all natural numbers $n \ge 2$ and every

$$\gamma \in \begin{cases} \left(0, \dfrac{1}{pK^2} - \dfrac{2K^2}{K^2+1}\left(\dfrac{2}{p} - 1\right)\right) & , \quad 2 - \frac{K^2+1}{2K^4} < p < 2, \\ \left(0, \dfrac{1}{pK^2}\right) & , \quad p \ge 2. \end{cases}$$

Remarks.

i. If $p = 2$, our theorem is the same one as in [21].

ii. If $p > 2$, the existence of $\gamma = \gamma(K)$ was proved in [17] .

iii. If $p \neq 2$, as it is in [5], it is possible to compute γ approximately for some simple regions as follows: Let ε be a sufficiently small positive number. Then;

a. If G is a square, $\gamma = \begin{cases} \frac{9p-16-\varepsilon}{6p}, & 2 \geq p > \frac{16}{9} \\ \frac{1-\varepsilon}{3p}, & p > 2. \end{cases}$

b. If G is a rectangle with the sides d and e, $d \geq e$,

$$\gamma = \begin{cases} \frac{(2p-3)(\pi\omega-1)+1-\varepsilon}{\pi p\omega(\pi\omega-1)}, & 2 \geq p > 2-\frac{\pi\omega}{2(\pi\omega-1)^2} \\ \frac{1-\varepsilon}{p(\pi\omega-1)}, & p > 2, \end{cases}$$

where $\omega := (arctg\,\frac{e}{d})^{-1}$.

c. If G is the L – shaped region, i.e.,

$G = \{z = x+iy : 0 < x < 2, 0 < y < 1\} \cup \{z = x+iy : 0 < x < 1, 1 \leq y < 2\}$,

$$\gamma = \begin{cases} \frac{1+3\rho-18\rho^2(2-p)-\varepsilon}{3\rho p(3\rho+1)}, & 2 > p > 2-\frac{1+3\rho}{18\rho^2} \\ \frac{1-\varepsilon}{3\rho p}, & p > 2, \end{cases} \qquad \text{where } \rho := \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)^2.$$

In order to prove the main theorem we need some auxiliary lemmas.

2. Some Auxiliary Lemmas.

Let a and b be nonnegative . From now on, we shall use the notations $a \asymp b$ and $a \preccurlyeq b$ to mean that $\frac{1}{c}b \leq a \leq cb$ and $a \leq cb$ for a positive constant c, independent of a and b.

Lemma 2.1. *Let G be a finite domain in the complex plane with a K – quasiconformal boundary L. If $p > 2(1-K^{-2})$, then the function $\varphi_p(z)$ can be extented to $\overline{G}$ continuously.*

Proof. It is clear that $\varphi_p(z)$ is uniformly continuous on every compact subset of G. We show that $\varphi_p(z)$ is uniformly continuous on G. Let z be any arbitrary point in G. From lemma 3 in [1], we know that

$$\left|\varphi_p'(z)\right| = \left|\varphi'(z)\right|^{\frac{2}{p}} \asymp \left[\frac{1-|\varphi(z)|}{d(z,L)}\right]^{\frac{2}{p}}. \tag{2.1}$$

On the other hand, since φ has a K^2 – quasiconformal extension to the whole plane [3:p.75] ,it satisfies K^{-2} -uniform Hölder condition in $\overline{G}$ [3:p.51], i.e.,

$$\left|\varphi(z)-\varphi(\zeta)\right| \preccurlyeq |z-\zeta|^{K^{-2}} \tag{2.2}$$

for every z, ζ in $\overline{G}$. So, we get

$$\left|\varphi_p'(z)\right| \preccurlyeq \left[d(z,L)\right]^{\frac{2(K^{-2}-1)}{p}}, \; z \in G. \tag{2.3}$$

Therefore, by (2.3) and the fact in [14] we have

$$\left|\varphi_p(z)-\varphi_p(\zeta)\right| \preccurlyeq \left|z-\zeta\right|^{\frac{2(K^{-2}-1)}{p}+1},\ z,\ \zeta \in G. \tag{2.4}$$

Since $p > 2(1-K^{-2})$, from (2.4) it follows that the function φ_p is continuous uniformly in G. Hence, it can be extented continuously to $\overline{G}$.

Let us mention that this lemma was proved in [17] in case of $p > 2$.

Lemma 2.2. *Let G be a finite domain in the complex plane with a K – quasiconformal boundary L and $p > 1$. Then*

$$\left\|\varphi_p - \pi_{n,p}\right\|_{L_p^1(G)} \preccurlyeq \frac{1}{n^{\gamma}},$$

for every natural number $n \geq 2$ and every $\gamma \in (0, \frac{1}{pK^2})$.

Proof. Since L is a K – quasiconformal curve, there exists a quasiconformal reflection $\alpha(z)$ across L [3:pp.77-80] such that it has bounded partial derivatives in a neighborhood of L and changes Euclidean lengths at most by a constant factor. Let us extend $\varphi_p(z)$ to the whole complex plane in the following way:

$$\widetilde{\varphi}_p(z) := \begin{cases} \varphi_p(z) & , \quad z \in \overline{G}, \\ \varphi_p(\alpha(z)) & , \quad z \in \Omega. \end{cases}$$

Then

$$\widetilde{\varphi}_{p,\bar{z}}(z) := \frac{\partial \widetilde{\varphi}_p(z)}{\partial \bar{z}} = \begin{cases} 0 & , \quad z \in G, \\ \varphi_p'(\alpha(z))\alpha_{\bar{z}}(z) & , \quad z \in \Omega. \end{cases} \tag{2.5}$$

It is obvious that $\varphi_p' \in L_1(G)$. Therefore $\varphi_p'(z)$ has the following integral representation [8] and [9] :

$$\varphi_p'(z) = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{\widetilde{\varphi}_{p,\bar{\zeta}}(\zeta)}{(\zeta - z)^2} d\sigma_{\zeta}, \quad z \in G. \tag{2.6}$$

The analog of this integral representation for an unbounded region G with boundary passing through ∞ is given in [10].

Now let us consider the conformal mapping $\phi(z)$ of Ω onto the exterior of the unit disc normalized by $\phi(\infty) = \infty$ and $\lim\limits_{z\to\infty} \frac{1}{z}\phi(z) > 0$. For $R > 1$, let us set $L_R := \{z : |\phi(z)| = R\}, G_R := \operatorname{int} L_R$ and $\Omega_R := extL_R$.

Let $0 < \varepsilon < 1$ be small enough and $R := 1 + n^{\varepsilon - 1}$. Since $\Omega = \overline{\Omega_R} \cup (G_R \setminus G)$ we get

$$\varphi_p'(z) = J_1(z) + J_2(z) \quad , \quad z \in G, \tag{2.7}$$

where

$$J_1(z) := -\frac{1}{\pi}\iint\limits_{\Omega_R} \frac{\widetilde{\varphi}_{p,\bar{\zeta}}(\zeta)}{(\zeta - z)^2} d\sigma_{\zeta}, \quad z \in G,$$

and

$$J_2(z) := -\frac{1}{\pi}\iint\limits_{G_R\setminus G}\frac{\widetilde{\varphi}_{p,\bar{\zeta}}(\zeta)}{(\zeta-z)^2}d\sigma_\zeta \;, \quad z\in G.$$

Since $J_1(z)$ is analytic in $\overline{G}$, there is a polynomial $Q_{n-1}(z)$ of degree $\leq n-1$ [26:p.142] such that $|J_1(z)-Q_{n-1}(z)| \preccurlyeq \frac{1}{n}$ for every $z\in\overline{G}$. Let us set $P_n(z) := \int\limits_{z_\circ}^{z} Q_{n-1}(\zeta)d\zeta$. Then $P_n(z_\circ)=0$ and from (2.7) we get

$$\|\varphi_p - P_n\|_{L_p^1(G)} \preccurlyeq \frac{1}{n} + \|J_2\|_{L_p(G)} \;, \tag{2.8}$$

Since the Hilbert transformation $T(f)(z) := -\frac{1}{\pi}\iint\frac{f(\zeta)}{(\zeta-z)^2}d\sigma_\zeta$ being defined as a Cauchy principle value is a bounded linear operator from L_p into itself for $p>1$, considering $|\widetilde{\varphi}_{p,\bar{\zeta}}(\zeta)|^p = |\varphi'(\alpha(\zeta))|^2 |\alpha_{\bar{\zeta}}(\zeta)|^p \preccurlyeq |\varphi'(\alpha(\zeta))|^2$ in $G_R\setminus G$, the Calderon-Zygmund inequality [3: p.89] shows that

$$\|J_2\|_{L_p(G)} \preccurlyeq \|\widetilde{\varphi}_{p,\bar{\zeta}}\|_{L_p(G)} \preccurlyeq \left(\iint\limits_{G_R\setminus G}|\varphi'(\alpha(\zeta))|^2 d\sigma_\zeta\right)^{1/p} \preccurlyeq \left(mes(\varphi(\alpha(G_R\setminus G)))\right)^{1/p}. \tag{2.9}$$

Let us take $R^* := 1+2(R-1)$. Let $L^* := \alpha(L_{R^*})$, and $\phi_{R^*}(z)$ be the conformal mapping of $extL_{R^*}$ onto the exterior of the unit disc with the properties $\phi_{R^*}(\infty)=\infty$ and $\lim\limits_{z\to\infty}\frac{1}{z}\phi_{R^*}(z)>0$. Since $\varphi(z)$ is K^2-quasiconformal in $\overline{G}$ and $\phi_{R^*}^{-1}(w)$ is conformal, using again Goldstein's theorem [15] we obtain

$$mes\varphi(\alpha(G_R\setminus G)) = mes\{(\varphi\circ\phi_{R^*}^{-1}\circ\phi_{R^*})(\alpha(G_R\setminus G))\} \preccurlyeq \{mes\phi_{R^*}(\alpha(G_R\setminus G))\}^{\delta}, \tag{2.10}$$

for $\delta\in(0,K^{-2})$.

On the other hand, according to Andrievskii's lemma [7: Lemma 2], there exists a $\widetilde{R}>1$ such that $\widetilde{R}-1 \preccurlyeq n^{\varepsilon-1}$ and if $S_{\widetilde{R}} := \{z : |\phi_{R^*}(z)| = \widetilde{R}\}$, then $mes\{\phi_{R^*}(\alpha(G_R\setminus G))\} \leq mes\{\phi_{R^*}(\operatorname{int}S_{\widetilde{R}}\setminus\overline{\operatorname{int}L^*}\} = \pi(\widetilde{R}^2-1) \preccurlyeq n^{\varepsilon-1}$. Thus, from (2.8) to (2.10) we get

$$\|\varphi_p - P_n\|_{L_p^1(G)} \preccurlyeq \left(\frac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}}. \tag{2.11}$$

Now let us consider the polynomial $\widetilde{P}_n(z) := P_n(z) + [1-P_n'(z_\circ)](z-z_\circ)$. It is clear that $\widetilde{P}_n(z)\in\check{A}_n$, and by means of (2.11) we obtain that

$$\|\varphi_p - \widetilde{P}_n\|_{L_p^1(G)} \preccurlyeq \left(\frac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}}.$$

So if we consider the extremal property of the polynomials $\pi_{n,p}(z)$ in $\check{A}_n$, the proof is completed.

Lemma 2.3. *Let G be a finite domain in the complex plane with a K – quasiconformal boundary L. Then, for every $z \in L$ there exists an arc $\beta(z_\circ, z)$ in G joining $z_\circ$ to z and with the following properties:*

i. $d(\zeta, L) \asymp |\zeta - z|$ for every $\zeta \in \beta(z_\circ, z)$,

ii. For every pair $\zeta_1, \zeta_2 \in \beta(z_\circ, z)$, if $\widetilde{\beta}(\zeta_1, \zeta_2)$ is the subarc of $\beta(z_\circ, z)$ joining ζ_1 to ζ_2 and $\ell(\widetilde{\beta}(\zeta_1, \zeta_2))$ is its length, then $\ell(\widetilde{\beta}(\zeta_1, \zeta_2)) \preccurlyeq |\zeta_1 - \zeta_2|$.

Proof. The proof of this lemma is similar to the one of lemma 4 in [6]. It follows that the arc $\beta(z_\circ, z) := \{\zeta : \zeta \in G, \arg\varphi(\zeta) = \arg\varphi(z)\}$ satisfies the required conditions by means of the modified version of lemma 3 in [6] for finite regions and lemmas 1,3 in [1].

Lemma 2.4. *Let G be a finite domain in the complex plane with a K – quasiconformal boundary L, and $P_n(z)$ be any polynomial of degree $\le n$ with $P_n(z_\circ) = 0$. Then*

$$\|P_n\|_{C(\overline{G})} \preccurlyeq \|P_n\|_{L_p^1(G)} \times \begin{cases} \sqrt{\log n} & , \quad p = 2, \\ 1 & , \quad p > 2, \\ n^{\frac{2K^2}{1+K^2}(\frac{2}{p}-1)} & , \quad 0 < p < 2. \end{cases}$$

Proof. When $p = 2$ and $p > 2$ the proofs are given in [5] and [17]. So we consider the case of $0 < p < 2$. Let z be an arbitrary point on L. For simplicity, let us set $s := \frac{2K^2}{1+K^2}$ and let $\beta(z_\circ, z)$ be the arc joining $z_\circ$ to z and satisfying the conditions of lemma 2.3. For an $\varepsilon > 0$ small enough, if $\beta_1 := \{\zeta : \zeta \in \beta(z_\circ, z), |\zeta - z| < \varepsilon n^{-s}\}$ and $\beta_2 := \beta(z_\circ, z) \setminus \beta_1$, we get

$$|P_n(z)| = \left| \int_{\beta(z_\circ, z)} P_n'(\zeta) d\zeta \right| \le \int_{\beta_1} |P_n'(\zeta)||d\zeta| + \int_{\beta_2} |P_n'(\zeta)||d\zeta|, \tag{2.12}$$

It is well known [16:Theorem 1] that there exists a $c_1 > 0$ such that $\|P_n'\|_{C(\overline{G})} \le c_1 n^s \|P_n\|_{C(\overline{G})}$. Therefore, since $\ell(\beta_1) \le c_2 \varepsilon n^{-s}$ for a $c_2 > 0$ which is independent of ε, and $|P_n'(\zeta)|^p \le \dfrac{1}{\pi d^2(\zeta, L)} \|P_n\|^p_{L_p^1(G)}$ [24:p.432], by lemma 2.3 we obtain

$$\begin{aligned} |P_n(z)| &\le c_1 n^s \|P_n\|_{C(\overline{G})} \int_{\beta_1} |d\zeta| + c_3 \|P_n\|_{L_p^1(G)} \int_{\beta_2} \frac{|d\zeta|}{|\zeta - z|^{2/p}}, \\ &\le \varepsilon \quad c_1 c_2 \|P_n\|_{C(\overline{G})} + c_3 \|P_n\|_{L_p^1(G)} \int_{c_2 \varepsilon n^{-s}}^{\ell(\beta)} \frac{dt}{t^{2/p}}, \\ &\le \varepsilon c_1 c_2 \|P_n\|_{C(\overline{G})} + c_4 \|P_n\|_{L_p^1(G)} n^{s(\frac{2}{p}-1)}. \end{aligned}$$

Using the maximum modulus principle and choosing ε such that $\varepsilon c_1 c_2 < 1$, the proof is completed.

3. Proof of the main theorem

We use the familiar method given in [5],[12] and [17]. Considering the case $2 > p > 2 - \frac{K^2+1}{2K^4}$, we proceed as follows:

Let be $\gamma \in (0, \frac{1}{pK^2} - \frac{2K^2}{K^2+1}(\frac{2}{p}-1))$. For n with $2^k \le n < 2^{k+1}$ and $\lambda = \gamma + \frac{2K^2}{K^2+1}(\frac{2}{p}-1)$ by Lemma 2.2, we have

$$\left\|\pi_{2^{k-1},p} - \pi_{n,p}\right\|_{L_p^1(G)} \preccurlyeq \frac{1}{n^\lambda} ;$$

and in particular,

$$\left\|\pi_{2^{j+1},p} - \pi_{2^j,p}\right\|_{L_p^1(G)} \preccurlyeq \frac{1}{2^{j\lambda}} \text{, for } j > k .$$

Now since

$$\varphi_p(z) - \pi_{n,p}(z) = \left[\pi_{2^{k+1},p}(z) - \pi_{n,p}(z)\right] + \sum_{j=k+1}^{\infty}\left[\pi_{2^{j+1},p}(z) - \pi_{2^j,p}(z)\right] \text{, for } z \in G ,$$

we have

$$\left\|\varphi_p - \pi_{n,p}\right\|_{C(\overline{G})} \le \left\|\pi_{2^{k+1},p} - \pi_{n,p}\right\|_{C(\overline{G})} + \sum_{j=k+1}^{\infty}\left\|\pi_{2^{j+1},p} - \pi_{2^j,p}\right\|_{C(\overline{G})} .$$

Now we put $n_j := (j+1)\frac{2K^2}{1+K^2}(\frac{2}{p}-1) - j\lambda$ and notice that since $\sum_{j=k+1}^{\infty} 2^{n_j} \preccurlyeq \frac{1}{2^{\gamma(k+1)}}$, using lemma 2.4 , we get

$$\left\|\varphi_p - \pi_{n,p}\right\|_{C(\overline{G})} \preccurlyeq \frac{1}{n^\gamma} .$$

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