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MATHEMATICS

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ON EXISTENCE AND PROPERTIES OF LIMIT VALUES OF CAUCHY TYPE INTEGRAL BY NON-SMOOTH INFINITE CURVE

Abstract

In the work Sokhotsky's formulas are established for Cauchy type integral by the non-smooth infinite curve and the properties of limit values are studied in the scale of Hölder's spaces of functions determined on the closed functions at infinity which satisfy the condition $\Theta(\delta) \approx \delta$.

In the work Sokhotsky's formulas are established for the integral of type $F(z)$ determined in (2) and the properties of limiting values are studied in the scale of space H_ω.

This problem is solved by following way: with help of fractional linear mapping (1) the investigated case is reduced to the case of Cauchy type integral and singular integral by closed Jordan curve and then the results of [4], [5], [6] are used. The results of the present work are deposited partially in ([1]).

On realization of this approach the necessity appears to study the connection between $\Theta_\Gamma(\delta)$ and $\Theta_\gamma(\delta)$ of curve Γ and its prototype γ for fractional linear mapping (1).

Let's remark that such approach to study of Cauchy type integral and singular integral (in the case of sufficient smooth curves) was realized in [4].

Let Γ be infinity orientated Jordan rectified (in its each finite part) curve, Γ^+ be one of connected components $C \setminus \Gamma$ which is remained from the left for positive circuit Γ, $\Gamma^- = C \setminus \left(\Gamma \cup \Gamma^+\right)$.

Everywhere further we will suppose that $z_0 \overline{\in} \Gamma$ is the fixed point.

Let's denote by γ_Γ the curve, which the curve Γ turns into for fractional-linear mapping

$$F : Z \to \frac{1}{z - z_0}. \tag{1}$$

It is obvious that then curve γ_Γ passes through point 0 and the positive direction on Γ corresponds to the positive direction on γ_Γ.

Domain Γ^+ is mapped into domain γ^+ when that does not reduce to missunder standing.

Let's denote

$$\Theta_\Gamma(z,\delta) = mes\{\xi \in \Gamma : |\xi - z| \le \delta\},\ z \in \Gamma,\ \delta > 0,$$
$$\Theta_\Gamma(z_0,\delta) = mes\{\xi \in \Gamma : |\xi - z_0| \le \delta\},$$
$$\Theta_\gamma(\tau,\delta) = mes\{\xi \in \gamma : |\xi - \tau| \le \delta\},\ \tau \in \gamma,\ \delta > 0,$$

where *mes* denotes Lebeg's line measure.

Suppose

$$\Theta_\Gamma(\delta)=\max\left\{\sup_{z\in\Gamma}\Theta_\Gamma(z,\delta),\,\Theta_\Gamma(z_0,\delta)\right\},$$
$$\Theta_\gamma(\delta)=\sup_{\tau\in\gamma}\Theta_\gamma(\tau,\delta).$$

The connection between metric characteristics $\Theta_\gamma(\delta)$ of curve Γ and its prototype γ for fractional linear mapping is given by following theorem.

Theorem 1. $\Theta_\gamma(\delta)\approx\delta$ *then and only then when*

$$\Theta_\gamma(\delta)\approx\delta .$$

Let's note that $\varphi(\delta)\approx\psi(\delta)$ *means, that*

$$\exists C_1,C_2>0,\ \ \forall\delta>0\ ,\ C_1\varphi(\delta)\le\psi(\delta)\le C_2\varphi(\delta).$$

Everywhere further Γ is the closed in infinity orientiated Jordan rectified (in its each finite part) curve for which

$$\Theta_\Gamma(\delta)\le k\delta,\quad \delta>0,\quad k=Const .$$

Let's note $\Gamma^{|}=\Gamma\cup\{\infty\}$. Let $C_{\Gamma^|}-(B)$ be a space of continuous on compact $\Gamma^{|}$ functions

$$\|f\|_{C_{\Gamma^|}}=\sup_{t\in\Gamma^|}|f(t)|.$$

By definition $f\in C_{\Gamma^|}$ means, that $f:\Gamma\to C$ is continuous and has a finite limit at infinity.

Further we will consider that $\omega:(0,+\infty)\to R^+$ is an arbitrary function which has the properties: ω increases, $\omega(\delta)/\delta$ decreases, $\lim\limits_{\delta\to 0}\omega(\delta)=0$.

By definition $\omega\in J_0$ if $\int\limits_0\omega(t)/t\,dt<\infty$.

Let's denote by H_ω the class of functions $f\in C_\Gamma$ which have the properties: there exists constant C_f such that for any $z_1,z_2\in\Gamma$.

$$|F(z_1)-F(z_2)|\le C_f O(\varepsilon_{z_0}(z_1,z_2)),$$

where

$$\varepsilon_{z_0}(z_1,z_2)=\left|\frac{1}{z_1-z_0}-\frac{1}{z_2-z_0}\right|.$$

Let $f\in H_\omega$. Let's consider Cauchy type integral

$$F(z)=\int_\Gamma\frac{f(\xi)}{\xi-z}d\xi\overset{def}{=}\lim_{R\to\infty}\int_{\{\xi\in\Gamma,|\xi|\le R\}}\frac{f(\xi)}{\xi-z}d\xi,\ \ z\in C\setminus\Gamma. \tag{2}$$

Let's denote by R the totality of curves Γ for which there exists the finite limit

$$\lim_{R\to\infty}\int_{\{\xi\in\Gamma,|\xi|\le R\}}\frac{d\xi}{\xi-z_0} \tag{3}$$

for some $z_0\in C\setminus\Gamma$.

Theorem 2. *Let* $\Gamma\in R$, $f\in H_\omega$, $\omega\in J_0$. *Then for every* $z_0\in C\setminus\Gamma$ *there exists the finite limit in right-hand side of (2).*

First let's prove the following lemma.

Lemma 1. *Let* $f\in H_\omega$, $\omega\in J_0$. *Then* *a) for every* $z_0 \in C \setminus \Gamma$

$$\int \frac{(\zeta) - f(\infty)}{|\zeta - z|} \Big/ d\zeta < +\infty ;$$

b) for any $z, z_1 \in C \setminus \Gamma$

$$\int_{\Gamma} \frac{|f(z)|}{|\zeta - z||\zeta - z_1|} |d\zeta| < +\infty .$$

Proof. Let's prove a). It's sufficient to check that

$$A \overset{def}{=} \int_{\{\zeta \in \Gamma : |\zeta| \geq r\}} \frac{|f(\zeta) - f(\infty)|}{|\zeta - z|} |d\zeta| < +\infty ,$$

where $r = 2\max\{|z|, |z_0|\}$.

For $|\zeta| \geq r$, $|\zeta - z| \geq \frac{1}{2}|\zeta|$ and $|\zeta - z_0| \geq \frac{1}{2}(\zeta)$.

Then $|f(\zeta) - f(\infty)| \leq C\omega\left(\frac{1}{|\zeta - z_0|}\right) \leq 2C\omega\left(\frac{1}{|\zeta|}\right)$ and so

$$A \leq 4c \int_{\{\zeta \in \Gamma : |\zeta| \geq r\}} \frac{\omega\left(\frac{1}{|\zeta|}\right)}{|\zeta|} |d\zeta| = 4c \sum_{n=0}^{\infty} A_n , \tag{4}$$

where $A_n = \int_{\{\zeta \in \Gamma : 2r \leq |\zeta| < 2^{n+1} r\}} \frac{\omega\left(\frac{1}{|\zeta|}\right)}{|\zeta|} |d\zeta|$.

Taking into account the properties ω and condition (4) we have:

$$A_1 \leq \frac{\omega\left(2^{\frac{1}{n}} \cdot r\right)}{2^n \cdot r} \int_{\{2^n r \leq |\zeta| \leq 2^{n+1} r\}} |d\zeta| \leq \frac{\omega\left(\frac{1}{2^n \cdot r}\right)}{2^n \cdot r} \cdot k \cdot 2^{n+1} \cdot r = 2kr\omega\left(\frac{1}{2^n \cdot r}\right) =$$

$$= 4kr \int_{\frac{1}{2^{n+1} \cdot r}}^{\frac{1}{2^n \cdot r}} \frac{\omega\left(\frac{1}{2^n \cdot r}\right)}{\frac{1}{2^n \cdot r}} dx \leq 4kr \int_{1/2^{n+1} \cdot r}^{1/2^n \cdot r} \frac{\omega(t)}{t} dx .$$

From the last taking (4) into account we have:

$$A \leq 16ckr \int_0^{1/r} \frac{\omega(t)}{t} dx .$$

Point a) has been proved. Point b) is proved by analogy taking into account boundedness of f. Using these results Sokhotsky's formulas are proved.

Let's denote by $d(d')$ the diameter of prototype Γ for mapping $z \to \frac{1}{z - z_0}\left(z \to \frac{1}{z - z_0'}\right)$ for $z_0 \in \Gamma^-\left(z_0^1 \in \Gamma^+\right)$ and also by $F^+(W)\left(F^-(W)\right)$ let's denote limit values of Cauchy type integral $F(z)$ for $z \in \Gamma^+ \to W \in \Gamma$ (for $z \in \Gamma^+ \to W \in \Gamma$).

Theorem 3. *Let* $\Gamma \in R$, $f \in H_\omega$, $\omega \in I_0$. *Then*

$$F^+(W)=\frac{\omega-z_0}{2\pi i}\int_\Gamma\frac{f(\zeta)-f(W)}{(\zeta-z)(\zeta-z_0)}d\xi+F(z_0),\, W\in\Gamma,\ W\neq\infty;$$

$$F^+(\infty)=-\frac{1}{2\pi i}\int_\Gamma\frac{f(\zeta)-f(\infty)}{\zeta-z_0}d\zeta+f(\infty)+F(z_0).$$

Further, for any $z_1,z_2\in\Gamma^-$

$$|F(z_1)-F(z_2)|\le C_{f,\Gamma,z}\left(\int_0^\varepsilon\frac{\omega(t)}{t}dt+\varepsilon\int_\varepsilon^d\frac{\omega(t)}{t^2}dt\right)$$

$\varepsilon=\varepsilon_{z_0}(z_1,z_2)$, $z_0\in F^-$ *the fixed point.*

Theorem 4. *Let* $f\in H_\omega$, $\omega\in J_0$

$$F^-(W)=\frac{\omega-\omega_0}{2\pi i}\int_\Gamma\frac{f(\xi)-f(W)}{(\xi-\omega)(\xi-z_0)}d\xi+F(z_0),\, W\in\Gamma,\ W\neq\infty,$$

then

$$F^-(\infty)=-\frac{1}{2\pi i}\int_\Gamma\frac{f(\xi)-f(\infty)}{\xi-z_0}d\xi+F(z_0).$$

Further, for any $z_1,z_2\in\Gamma^-$

$$|F(z_1)-F(z_2)|\le C_{f,\Gamma,z_0}\left(\int^\varepsilon\frac{\omega(t)}{t}dx+\varepsilon\int_\varepsilon^{\alpha''}\frac{\omega(t)}{t^2}dx\right),$$

where $\varepsilon=\varepsilon_{z_0'}(z_1,z_2)$, $z_{0'}\in\Gamma^-$ *is the fixed point.*

From $f\in H_\omega$ and from Theorem 3 it follows that the estimation of Theorem 3 is saved also for singular theorem

$$\bar f(z)=\frac{z-z_0}{2\pi i}\int_\Gamma\frac{f(\xi)-f(z)}{(\xi-z)(\xi-z_0)}d\xi,\ z\in\Gamma$$

and that is the analogue of Zigmund's estimation for the singular integral by the closed curve. Similar problems in the case when Γ is the real straight line or the smooth curve with one asymptotic direction at infinity were considered in [2], [3].

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