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DOSIEV A.A.

ON FREDHOLM PAIRS OF OPERATORS

Abstract

In this paper the ultrapower technique is applied to study Fredholm pairs of operators.

1. Introduction. The concept of Fredholm complexes (or Fredholm operator families) has appeared in several different contexts. The significance of this concept for general spectral theory in several variables is reflected in works [1], [2, chapter 2], [3, chapter 3]. In this note we apply the ultrapower techniques to study of Fredholm complexes. As in [2, chapter 2.2] we introduce more simple and in some meaning more general notion than Fredholm complexes. It is the Fredholm pair of operators (see below definition 1). For pairs of operators we give necessary and sufficient conditions in terms of ultrapowers to be Fredholm pair. These conditions are similar to the assertions founded in [4] and actually generalize these ones.

2. Preliminaries. All vector spaces are considered to be defined on the field of complex numbers $\mathbf{C}$. For Banach spaces X and Y, the Banach space of all bounded linear operators (with operator norm) from X into Y is denoted by $B(X,Y)$. We set $B(X)=B(X,X)$. The kernel and the image of an operator $T\in B(X,Y)$ are denoted by $N(T)$ and $R(T)$, respectively. For a closed subspace $E\subseteq X$, the restriction of $T\in B(X)$ on E is denoted by $T|_E$ and the induced quotient operator is denoted by $T^{\sim}\in B(X/E)$ (which defined as $T^{\sim}(x^{\sim}\bmod E)=(Tx^{\sim})(\bmod E)$) if E is invariant under T. As usually $\mathbf{N}$ is the set of all positive integers.

Let us recall the ultrapowers of Banach spaces and operators. Let I be a infinite set and $\mathcal{U}$ be a nontrivial ultrafilter (i.e. $\bigcap\{M:M\in\mathcal{U}\}=\varnothing$) on I. The ultrafilter $\mathcal{U}$ is said be $\aleph_0$-incomplete if there exists a countable partition $\{I_n:n\in\mathbf{N}\}$ of I such that $I_n\notin\mathcal{U}$ for each $n\in\mathbf{N}$. All non-trivial ultrafilters on $\mathbf{N}$ are $\aleph_0$-incomplete (take $I_n=\{n\}$). It was indicated in [4, lemma 1], that on any infinite set, there exists $\aleph_0$-incomplete ultrafilters. In the sequel by an ultrafilter we shall mean a non-trivial $\aleph_0$-incomplete ultrafilter. Let X be a Banach space and let $\ell_\infty(I,K)$ be the Banach space of all bounded families $(x_i)_{i\in I}$ of X with sup-norm. For an ultrafilter $\mathcal{U}$ on I, let $N_{\mathcal{U}}(X)$ be the closed subspace in $\ell_\infty(I,X)$ which consists of all families $(x_i)_{i\in I}$ such that $\lim_{\mathcal{U}} x_i=0$. The ultrapower of X following $\mathcal{U}$ is called the quotient space $X_{\mathcal{U}}=\ell_\infty(I,X)/N_{\mathcal{U}}(X)$. The element of $X_{\mathcal{U}}$ including as a representative the family $(x_i)_{i\in I}\in\ell_\infty(I,X)$ is denoted by $[x_i]$. One easily check that the norm $\|[x_i]\|$ is $\lim_{\mathcal{U}}\|x_i\|$. The space X is contained in $X_{\mathcal{U}}$ as a subspace generated by the constant families of $\ell_\infty(I,X)$, and $X_{\mathcal{U}}=X$ iff X is finite dimensional space (see [4, proposition 7]). An operator $T\in B(X,Y)$ is extended to $T_{\mathcal{U}}\in B(X_{\mathcal{U}},Y_{\mathcal{U}})$, $T_{\mathcal{U}}[x_i]=[Tx_i]$ and $\|T_{\mathcal{U}}\|=\|T\|$. The following two assertions are proved in [4, proposition 15, 16, 20, 22].

Proposition 1. *Let* $T \in B(X,Y)$. *Then* $N(T)_{\mathcal{U}} \subseteq N(T_{\mathcal{U}}), R(T_{\mathcal{U}}) \subseteq R(T)_{\mathcal{U}}$. *Moreover the following statements are equivalent:*

(i) $R(T)$ *is closed;*

(ii) $R(T_{\mathcal{U}})$ *is closed;*

(iii) $N(T)_{\mathcal{U}} = N(T_{\mathcal{U}})$;

(iv) $R(T_{\mathcal{U}}) = R(T)_{\mathcal{U}}$.

Let us recall that an operator $T \in B(X,Y)$ is said to be lower semi-Fredholm if its cokernel $Y/R(T)$ is finite dimensional. An operator $T \in B(X,Y)$ is said to be upper semi-Fredholm if its kernel $N(T)$ is finite dimensional and $R(T)$ is closed. We denote the class of all lower (upper) semi-Fredholm operators acting from X to Y by $\mathsf{F}_-(X,Y)$ $(\mathsf{F}_+(X,Y))$.

Proposition 2. *Let* $T \in B(X,Y)$. *The following statements are equivalent:*

(i) $T \in \mathsf{F}_-(X,Y)\,(T \in \mathsf{F}_+(X,Y))$;

(ii) $Y + R(T_{\mathcal{U}}) = Y_{\mathcal{U}}$ $(T(X_{\mathcal{U}} \setminus X) \subseteq Y_{\mathcal{U}} \setminus Y)$.

3. Fredholm pairs of operators. In this section we introduce Fredholm pairs of operators and characterize such pairs in the terms of ultrapower.

Definition 1. *Let* X, Y, Z *be Banach spaces and* $S \in B(X,Y), T \in B(Y,Z)$ *be operators such that* TS *is finite-rank operator. The pair of operators* (S,T) *is called to be Fredholm pair if* $\dim(N(T))/(R(S) \cap N(T)) < \infty$ *is closed and* $R(T)$ *is closed.*

This notion generalies semi-Fredholm operators: for $X = \{0\}$ we obtain that T is in $\mathsf{F}_+(Y,Z)$, if $Z = \{0\}$ then S is in $\mathsf{F}_-(X,Y)$. For $TS = 0$ we have got Fredholm sequence (see [2, section 2.2])

$$X \xrightarrow{S} Y \xrightarrow{T} Z,$$

where $\dim(N(T)/R(S) < \infty)$ and $R(T)$ is closed. Fredholm complex consists of Fredholm sequences and it can be written as the following sequence of Banach spaces and operators

$$\cdots \to X^{n-1} \xrightarrow{T^{n-1}} X^n \xrightarrow{T^n} X^{n+} \to \cdots,$$

where $T^n T^{n-1} = 0$ and $\dim\left(N(T^n)/R(T^{n-1})\right) < \infty$ for all n.

Fix an infinite set I and an ultrafilter $\mathcal{U}$ on I.

Lemma 1. *Let* $S \in B(X,Y)$, $T \in B(Y,Z)$ *be operators such that* TS *is finite-rank operator, and let* $S|_{N(TS)} \in B(N(TS), N(T))$ *be the restriction of* S. *Then*

$$\left(S|_{N(TS)}\right)_{\mathcal{U}} = S_{\mathcal{U}}|_{N(T_{\mathcal{U}}S_{\mathcal{U}})}, \quad N(T_{\mathcal{U}}) \cap R(S)_{\mathcal{U}} \subseteq (N(T) \cap R(S))_{\mathcal{U}}.$$

Moreover, (S,T) *is Fredholm pair iff* $S|_{N(TS)} \in \mathsf{F}_-(N(TS), N(T))$ *and* $R(T)$ *is closed.*

Proof. By assumption $R(TS)$ is finite dimensional and $X = N(TS) \oplus E$, where $\dim(E) < \infty$. One easily check that $R\left(S|_{N(TS)}\right) = R(S) \cap N(T)$, and by using proposition 1, for $x_i \in N(TS)$, $i \in I$, we have

$$\left(S|_{N(TS)}\right)_{\mathcal{U}}[x_i] = \left[S|_{N(TS)} x_i\right] = [Sx_i] = S_{\mathcal{U}}[x_i] = S_{\mathcal{U}}|_{N(T_{\mathcal{U}}S_{\mathcal{U}})}[x_i].$$

Now, let $[Sx_i] \in N(T_{\mathcal{U}}) \cap R(S)_{\mathcal{U}}$. Then $T_{\mathcal{U}}[Sx_i] = 0$, i.e. $\lim_{\mathcal{U}} TSx_i = 0$. Let $x_i = a_i + e_i$, where $a_i \in N(TS)$, $e_i \in E$, $i \in I$. We have

$$\lim_{\mathcal{U}} TSe_i = \lim_{\mathcal{U}} TSx_i = 0 .$$

But $\dim(E) < \infty$, then $\lim_{\mathcal{U}} e_i = 0$ and

$$TSx_i = TSe_i ,\ i \in I .$$

Next, let $y_i = Sx_i - Se_i \in N(T) \cap R(S)$. Since $(Sx_i)_{i\in I}$ is bounded, it follows that $(y_i)_{i\in I}$ is also bounded and $[Sx_i] = [Se_i] + [y_i] = [y_i] \in (N(T) \cap R(S))_{\mathcal{U}}$. Thus

$$N(T_{\mathcal{U}}) \cap R(S)_{\mathcal{U}} \subseteq (N(T) \cap R(S))_{\mathcal{U}} .$$

Now assume that $R(T)$ is closed. Then (S,T) is Fredholm pair iff

$$\dim(N(T)/R(S) \cap N(T)) = \dim\left(N(T)/R\left(S|_{N(TS)}\right)\right) < \infty .$$

But it means that $S|_{N(TS)} \in \mathsf{F}_-(N(TS), N(T))$. □

Theorem 1. *Let $S \in B(X,Y), T \in B(Y,Z)$ be operators such that TS is finite-rank operator. The following statements are equivalent:*

(i) (S,T) *is a Fredholm pair;*

(ii) $N(T_{\mathcal{U}}) = N(T) + R(S_{\mathcal{U}}) \cap N(T_{\mathcal{U}})$.

Proof. (i)⇒(ii). By lemma 1 $S|_{N(TS)} \in \mathsf{F}_-(N(TS), N(T))$ and by proposition 1 we have $N(T)_{\mathcal{U}} = N(T_{\mathcal{U}})$ and $N(TS)_{\mathcal{U}} = N(T_{\mathcal{U}} S_{\mathcal{U}})$. Then $N(T) + R\left(\left(S|_{N(TS)}\right)_{\mathcal{U}}\right) = N(T_{\mathcal{U}})$ by proposition 2. By using again lemma 1, we obtain that

$$R\left(\left(S|_{N(TS)}\right)_{\mathcal{U}}\right) = R\left(S_{\mathcal{U}}|_{N(T_{\mathcal{U}} S_{\mathcal{U}})}\right) = R(S_{\mathcal{U}}) \cap N(T_{\mathcal{U}}).$$

Thus (ii) is satisfied.

(ii)⇒(i) Let $N(T_{\mathcal{U}}) = N(T) + R(S_{\mathcal{U}}) \cap N(T_{\mathcal{U}})$. By proposition 1, $R(S_{\mathcal{U}}) \subseteq R(S)_{\mathcal{U}}$, $N(T)_{\mathcal{U}} \subseteq N(T_{\mathcal{U}})$. By lemma 1, $R(S)_{\mathcal{U}} \cap N(T_{\mathcal{U}}) \subseteq N(T)_{\mathcal{U}}$. Thus we have the following inclusions

$$N(T) + R(S_{\mathcal{U}}) \cap N(T_{\mathcal{U}}) \subseteq N(T) + R(S)_{\mathcal{U}} \cap N(T_{\mathcal{U}}) \subseteq N(T)_{\mathcal{U}} \subseteq N(T_{\mathcal{U}}).$$

It follows that $N(T)_{\mathcal{U}} = N(T_{\mathcal{U}})$ and, by proposition 1, $R(T)$ is closed.

It remains to prove that $\dim(N(T)/R(S) \cap N(T)) < \infty$. We consider again the operator $S|_{N(TS)}$. By assumption, $N(TS)_{\mathcal{U}} = N(T_{\mathcal{U}} S_{\mathcal{U}})$ and $R\left(S_{\mathcal{U}}|_{N(T_{\mathcal{U}} S_{\mathcal{U}})}\right) = R(S_{\mathcal{U}}) \cap N(T_{\mathcal{U}})$. By lemma 1, we have

$$N(T) + R\left(\left(S|_{N(TS)}\right)_{\mathcal{U}}\right) = N(T)_{\mathcal{U}} .$$

By using proposition 2, we see that $S|_{N(TS)} \in \mathsf{F}_-(N(TS), N(T))$. It means that $N(T)/R\left(S|_{N(TS)}\right)$ is finite dimensional and $R\left(S|_{N(TS)}\right) = R(S) \cap N(T)$. □

Corollary 1. *The sequence $X \xrightarrow{S} Y \xrightarrow{T} Z, TS = 0$ is a Fredholm sequence iff $N(T_{\mathcal{U}}) = N(T) + R(S_{\mathcal{U}})$.*

If, in corollary 1, we set that $Z = \{0\}$ and $S \in \mathsf{F}_-(X,Y)$, then $Y_{\mathcal{U}} = Y + R(S_{\mathcal{U}})$. Thus our assertion is reduced to the proposition 2. If $X = \{0\}$ and $T \in \mathsf{F}_-(Y,X)$, then by corollary 1, $N(T_{\mathcal{U}}) = N(T)$. Hence, $N(T) = N(T)_{\mathcal{U}} = N(T_{\mathcal{U}})$ (see proposition 1). Note that $N(T) = N(T)_{\mathcal{U}}$ iff $\dim(N(T)) < \infty$ (see [4, proposition 7]).

Theorem 2. *Let $S \in B(X,Y), T \in B(Y,Z)$ be operators such that TS is finite-rank operator. (S,T) is a Fredholm pair if and only if $(S_{\mathcal{U}}, T_{\mathcal{U}})$ is a Fredholm pair.*

Proof. (i)⇒(ii) Since $\dim(R(TS))<\infty$, so $R(T_{\mathcal{U}}S_{\mathcal{U}})=R(TS)$. Thus $T_{\mathcal{U}}S_{\mathcal{U}}$ is a finite-rank operator. We can apply theorem 1 for the product of ultrafilters $\mathcal{U}\times\mathcal{U}$ (see [5, chapter 13]). Then

$$N(T_{\mathcal{U}\times\mathcal{U}})=N(T)+R(S_{\mathcal{U}\times\mathcal{U}})\cap N(T_{\mathcal{U}\times\mathcal{U}}).$$

But the spaces $X_{\mathcal{U}\times\mathcal{U}}, Y_{\mathcal{U}\times\mathcal{U}}, Z_{\mathcal{U}\times\mathcal{U}}$ and the operators $S_{\mathcal{U}\times\mathcal{U}}, T_{\mathcal{U}\times\mathcal{U}}$ are identificated with $(X_{\mathcal{U}})_{\mathcal{U}}, (Y_{\mathcal{U}})_{\mathcal{U}}, (Z_{\mathcal{U}})_{\mathcal{U}}$ and $(S_{\mathcal{U}})_{\mathcal{U}}, (T_{\mathcal{U}})_{\mathcal{U}}$ respectively. These identifications are given by means of the canonical isometric maps constructed in [4, lemma 8]. Then we have

$$N((T_{\mathcal{U}})_{\mathcal{U}})=N(T)+R((S_{\mathcal{U}})_{\mathcal{U}})\cap N((T_{\mathcal{U}})_{\mathcal{U}}).$$

By using again theorem 1, we obtain that $(S_{\mathcal{U}},T_{\mathcal{U}})$ is Fredholm pair. By the same way we can prove the opposite implication. □

Corollary 2. *The sequence*

$$\cdots\to X^{n-1}\xrightarrow{T^{n-1}}X^{n}\xrightarrow{T^{n}}X^{n+1}\to\cdots$$

is Fredholm complex if and only if the sequence

$$\cdots\to X^{n-1}_{\mathcal{U}}\xrightarrow{T^{n-1}_{\mathcal{U}}}X^{n}_{\mathcal{U}}\xrightarrow{T^{n}_{\mathcal{U}}}X^{n+1}_{\mathcal{U}}\to\cdots$$

is Fredholm complex.

Proof. It suffices to apply theorem 2 for any short part of the sequence. □

Theorem 3. *Let $S\in B(X,Y), T\in B(Y,Z)$ be operators such that TS is finite-rank operator. The following statements are equivalent:*

(i) $R(S)$ is closed;

(ii) $N(T)\cap R(S)$ is closed;

(iii) $N(T_{\mathcal{U}})\cap R(S_{\mathcal{U}})=(N(T)\cap R(S))_{\mathcal{U}}$.

In particular, if (S,T) is a Fredholm pair, then $R(S)$ is closed.

Proof. (i)⇔(ii) Since $R(TS)$ is finite dimensional, so $X=N(TS)\oplus E$, where $\dim(E)<\infty$. One easily check that

$$R(S)=R(S|_{E})+N(T)\cap R(S).$$

It follows that $R(S)$ is closed iff $N(T)\cap R(S)$ is closed.

(i)⇒(iii) Assume that $R(S)$ is closed. By proposition 1, $R(S_{\mathcal{U}})=R(S)_{\mathcal{U}}$. Since $N(T)_{\mathcal{U}}\subseteq N(T_{\mathcal{U}})$, then

$$(N(T)\cap R(S))_{\mathcal{U}}\subseteq N(T_{\mathcal{U}})\cap R(S_{\mathcal{U}})$$

and by lemma 1, the opposite inclusion is also valid. Thus (iii) is proved.

(iii)⇒(ii) We have to show that $N(T)\cap R(S)$ is closed. We consider the operator $S|_{N(TS)}\in B(N(TS),N(T))$. It is obvious that $N(T)\cap R(S)=R(S|_{N(TS)})$. By using [4, proposition 5] we have

$$\overline{N(T)\cap R(S)}=(N(T)\cap R(S))_{\mathcal{U}}\cap Y=R(S_{\mathcal{U}})\cap N(T_{\mathcal{U}})\cap Y=R(S_{\mathcal{U}})\cap N(T).$$

Let us show that

$$R((S|_{N(TS)})_{\mathcal{U}})\cap N(T)=R(S_{\mathcal{U}})\cap N(T).$$

It is obvious that $R((S|_{N(TS)})_{\mathcal{U}})\cap N(T)\subseteq R(S_{\mathcal{U}})\cap N(T)$. Let $y=S_{\mathcal{U}}[x_i]$ such that $Ty=0$ and $x_i=a_i+e_i$, $a_i\in N(TS)$, $e_i\in E$, $i\in I$. As above we have

$$\lim_{\mathcal{U}}TSe_i=\lim_{\mathcal{U}}TSx_i=T\lim_{\mathcal{U}}Sx_i=Ty=0,$$

it follows that $\lim_{\mathcal{U}}e_i=0$. Then

$$\lim_{U} TSe_i = \lim_{U} TSx_i = T\lim_{U} Sx_i = Ty = 0,$$

it follows that $\lim_U e_i = 0$. Then

$$\lim_{U}(Sa_i - y) = \lim_{U}(Sx_i - y) - \lim_{U}(Se_i) = Ty = 0$$

or $\left(S|_{N(TS)}\right)_U [a_i] = y$.

Thus

$$\overline{R\left(S|_{N(TS)}\right)} = \overline{(N(T) \cap R(S))} = R\left(\left(S|_{N(TS)}\right)_U\right) \cap N(T).$$

By using [4, proposition 18], we see that $R\left(S|_{N(TS)}\right)$ is closed. Therefore, $N(T) \cap R(S)$ is closed.

Now, assume that (S,T) is Fredholm pair. Then $\dim(N(T)/N(T) \cap R(S)) < \infty$. By Kato's theorem [6, item 7.4.11], $N(T) \cap R(S)$ is closed. Hence $R(S)$ is closed.

4. The compact perturbation. Many properties of Fredholm pairs of operators can be easily obtained from theorems 1 - 3. In this section we prove that Fredholm pairs are stable under compact perturbations.

Theorem 4. *Let* (S,T) be Fredholm pair of operators, and let $P \in B(X,Y)$, $K \in B(Y,Z)$ be compact operators. Assume that the composition $(T+K)(S+P)$ is finite-rank operator. Then $(S+P, T+K)$ is Fredholm pair of operators.

Proof. By theorem 1, we have to show that

$$N((T+K)_U) = N(T+K) + R((S+P)_U) \cap N((T+K)_U).$$

Let $[y_i] \in N(T_U + K_U)$. By [4, proposition 23] $K_U[y_i] = z$, where $z \in Z$. Then $T_U[y_i] = -z$. By Kato's theorem (see [6, item 7.4.11]) $R(S) \cap N(T)$ is closed. We consider the operator $T^{\sim} \in B(Y/R(S) \cap N(T), Z)$, $T^{\sim}(y^{\sim}) = T(y)$. By definition 1,

$$T^{\sim} \in \mathsf{F}_+(Y/R(S) \cap N(T), Z).$$

The spaces $(Y/R(S) \cap N(T))_U$, $Y_U/(R(S) \cap N(T))_U$ are canonically isometric (see [4, lemma 9]). Then

$$\left(T^{\sim}\right)_U [y_i]^{\sim} = [Ty_i] = T_U[y_i] = -z.$$

By proposition 2, $[y_i]^{\sim} = y^{\sim}$ and by theorem 3, $[y_i] - y \in N(T_U) \cap R(S_U)$. Then $[y_i] - y = S_U[x_i]$. By assumption,

$$N((T+K)_U (S+P)_U) \oplus E = X_U,$$

where $\dim(E) < \infty$. Then $[x_i] = [u_i] + e$ and

$$(S+P)_U [u_i] = [y_i] - y - Se + P_U[u_i].$$

We conclude from proposition 2, that $P_U[u_i] \in Y$. Hence $y_1 = -y - Se + P_U[u_i] \in Y$ and $y_1 = [y_i] - (S+P)_U[u_i] \in N((T+K)_U) \cap Y$. But $N((T+K)_U) \cap Y = N(T+K)$, i.e. $y_1 \in N(T+K)$ and $[y_i] = y_1 + (S+P)_U[u_i] \in N(T+K) + R((S+P)_U) \cap N((T+K)_U)$. □

As corollary we have got the well known result, that the classes of semi-Fredholm operators are closed under compact perturbations. More precisely, let $K \in B(X,Y)$ be a compact operator. If $T \in \mathsf{F}_-(X,Y)$, then $T + K \in \mathsf{F}_-(X,Y)$ and if $T \in \mathsf{F}_+(X,Y)$, then $T + K \in \mathsf{F}_+(X,Y)$.

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Dosiev A.A.
Institute of Mathematics & Mechanics of Azerbaijan AS.
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20 (off.).

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