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SPECTRAL THEORY OF MULTIPARAMETRER SYSTEM, POLYNOMIALLY DEPENDING ON PARAMETERS

Abstract

For a parameter system polynomially depending on all the parameters acting on a separable Hilbert space, the possibility of separation of the spectrum and coincidence of the root subspace of this system with the root subspaces of operators Γ_i that are well known in multiparameters references are proved under fulfilling the independence condition of the system.

F.Atkinson has generalized the results related with multiparameter spectral problems for systems with ordinary differential equations, finite difference multiparameter systems, harmonic expansions of many variable functions, multi-point boundary value problems for linear and non-linear differential equations and other in a spectral theory of multiparameter selfadjoint systems.

Atkinson led the investigation of general spectral problems for such systems to the investigation of some commutative finite set of linear selfadjoint operators (see [1], [7]).

The papers by P.J.Brown, B.D. Sleeman and others (see [2], [3], [4], [8]) play an important role in this direction. Remind that all these investigations refer to selfadjoint multiparameter systems.

Spectral theory of nonselfadjoint two-parameterr systems firstly were studied in papers [9], [10], [11].

In [5], [6] the notions of associated vectors were introduced for linear non-selfadjoint vectors and a theorem on the completeness of the system of eigen and associated vectors of the considered problem was proved.

In particular, the coincidence of eigen and associated vectors of the considered system and some linear operator connected with the system is show.

In the given paper we consider a multiparameter system with polynomial dependence on parameters, namely

$$\begin{cases} A(\lambda)x_i = \left(A_{0,i} + \sum_{k=1}^{k_{1,i}} \lambda_1^k A_{1,k,i} + \sum_{k=1}^{k_{2,i}} \lambda_2^k A_{2,k,i} + \cdots + \sum_{k=1}^{k_{n,i}} \lambda_n^k A_{n,k,i} \right) x_i = 0, \\ i = 1,2,\ldots,n, \end{cases} \tag{1}$$

where $A_{s,j,i}, A_{0,i}$ are operators bounded in separable Hilbert space H_i, $i = 1,2,\ldots,n$; $j = 1,2,\ldots,k_{s,i}$; $s = 1,2,\ldots,n$; $\max\limits_{1\le i\le n} k_{r,i} = k_r$, $k_{r,i}$ are entire positive numbers, $\lambda = (\lambda_1,\ldots,\lambda_n) \in C^n$; $H = H_1 \otimes \cdots \otimes H_n$.

For the system (1) determine the operators Δ_i, Γ_i, playing an important role in investigation of spectral properties of multiparameter systems.

To this end we introduce the operators T_0, T_1, T_2 acting in a finite-dimensional space R_2 and determined by means of the matrix

$$T_0 \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

and denotations

$$\lambda_i^s = \widetilde{\lambda}_{k_1+k_2+\ldots+k_{i-1}+s}; \quad i=1,2,\ldots,n; \quad k_0=0, \quad s=1,2,\ldots,k_i. \tag{3}$$

Complete the system (1) by the following equations

$$\begin{gathered}
\left(T_2+\widetilde{\lambda}_1 T_0+\widetilde{\lambda}_2 T_1\right)x_{n+1}=0, \\
\left(\widetilde{\lambda}_1 T_2+\widetilde{\lambda}_2 T_0+\widetilde{\lambda}_3 T_1\right)x_{n+2}=0, \\
\ldots\ \ldots\ \ldots\ \ldots \\
\left(\widetilde{\lambda}_{k_1-2} T_2+\widetilde{\lambda}_{k_1-1} T_0+\widetilde{\lambda}_{k_1} T_1\right)x_{n+k_1-1}=0, \\
\left(T_2+\widetilde{\lambda}_{k_1+1} T_0+\widetilde{\lambda}_{k_1+2} T_1\right)x_{n+k_1}=0, \\
\ldots\ \ldots\ \ldots\ \ldots \\
\left(\widetilde{\lambda}_{k_1+k_2-2} T_2+\widetilde{\lambda}_{k_1+k_2-1} T_0+\widetilde{\lambda}_{k_1+k_2} T_1\right)x_{n+k_1+k_2-2}=0, \\
\ldots\ \ldots\ \ldots\ \ldots \\
\left(\widetilde{\lambda}_{k_1+k_2+\cdots k_{n-1}-2} T_2+\widetilde{\lambda}_{k_1+k_2+\cdots+k_{n-1}-1} T_0+\widetilde{\lambda}_{k_1+k_2+\cdots+k_{n-1}} T_1\right)x_{k_1+k_2+\cdots+k_{n-1}+1}=0, \\
\ldots\ \ldots\ \ldots\ \ldots \\
\left(\widetilde{\lambda}_{k_1+k_2+\cdots k_n-2} T_2+\widetilde{\lambda}_{k_1+\cdots+k_n-1} T_0+\widetilde{\lambda}_{k_1+\cdots+k_n} T_1\right)x_{k_1+k_2+\cdots+k_1}=0,
\end{gathered} \tag{4}$$

$x_s \in R_2$, if $s>n$

The equation (4) means that (at nontriviality of x_i)

$$\widetilde{\lambda}_1=\lambda_1,\ \widetilde{\lambda}_2=\lambda_1^2,\ \widetilde{\lambda}_3=\lambda_1^3,\ \ldots,\ \widetilde{\lambda}_{k_1}=\lambda_1^{k_1},\ldots\widetilde{\lambda}_{k_1+k_2+s}=\lambda_2^s,\ldots,\widetilde{\lambda}_{k_1+k_2+\cdots+k_n}=\lambda_n^{k_n}. \tag{5}$$

Really, the first equation from (4) means that

$$\left[\begin{pmatrix}0&0\\0&1\end{pmatrix}+\widetilde{\lambda}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}+\widetilde{\lambda}_2\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\begin{pmatrix}x_{1,n+1}\\x_{2,n+1}\end{pmatrix}=0,$$

whence

$$\begin{gathered}
\widetilde{\lambda}_1 x_{2,n+1}+\widetilde{\lambda}_2 x_{1,n+1}=0, \\
x_{2,n+1}+\widetilde{\lambda}_1 x_{1,n+1}=0
\end{gathered}$$

or $\widetilde{\lambda}_2=\widetilde{\lambda}_1^2$, $\widetilde{\lambda}_2=\lambda_1^2$.

In a general case we have

$$\begin{gathered}
\left(\widetilde{\lambda}_s T_0+\widetilde{\lambda}_{s+1} T_1+\widetilde{\lambda}_{s-1} T_2\right)x=0, \quad x\in R_2, \quad x=(x_1,x_2), \\
\left[\widetilde{\lambda}_s\begin{pmatrix}0&1\\1&0\end{pmatrix}+\widetilde{\lambda}_{s+1}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\widetilde{\lambda}_{s-1}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]\begin{pmatrix}x_1\\x_2\end{pmatrix}=0, \\
\widetilde{\lambda}_s x_2+\widetilde{\lambda}_{s+1} x_1=0,\ \widetilde{\lambda}_s x_1+\widetilde{\lambda}_{s-1} x_2=0,
\end{gathered}$$

or

$$\widetilde{\lambda}_s^2=\widetilde{\lambda}_{s-1}\widetilde{\lambda}_{s+1}, \tag{6}$$

We also introduce the denotations

$$A_{r,s,i}=\widetilde{A}_{k_1+k_2+\cdots+k_{r-1}+s,i}. \tag{7}$$

In $A_{r,s,i}$ the first index indicates the number of the parameter λ_r, s is the degree of parameter λ_r, i is the index of the space H_i, where the operator $A_{r,s,i}$ acts. In the index $\widetilde{A}_{k_1+k_2+\cdots+k_{r-1}+s,i}$ indicates the index $\widetilde{\lambda}_m$, s is the degree of the parameter of $\widetilde{\lambda}_{k_1+k_2+\cdots+k_{r-1}+1}=\lambda_r$.

Taking into account (3) and (7) the system (1) is rewritten in the form

$$\begin{cases} A_i(\lambda)x_i = \left(\widetilde{A}_{0,i} + \sum_{j=1}^{k} \widetilde{\lambda}_j \widetilde{A}_{j,i}\right) x_i = 0, \\ i = 1,2,\dots,n\ . \end{cases} \tag{8}$$

The system (8) is a system with k parameters. (8) and (4) together represent a multiparameter system consisting of k equations and depending on k parameters.

Let $\widetilde{H}$ be a tensor product of spaces $H_1, H_2, \dots, H_n$, R_2 namely

$$\widetilde{H} = H_1 \otimes \cdots \otimes H_n \otimes \underbrace{R_2 \otimes \cdots \otimes R_2}_{k-n \text{ times}}.$$

By $\widetilde{A}^{+}_{i,k}$ we denote the operators induced by the operators $\widetilde{A}_{i,k}$ to the space $\widetilde{H}$ and by $\widetilde{\Delta}_i$ $(i = 0,1,\dots,k)$ abstract analogies of Kramer's determinants for the system (8) and (4), acting in the space $\widetilde{H}$, and in case of the operator $\widetilde{\Delta}_0$ has a bounded inverse, then $\widetilde{\Gamma}_i = \widetilde{\Delta}_0^{-1}\widetilde{\Delta}_i$ $(i = 1,2,\dots,k)$. The operators $\widetilde{\Gamma}_i$ mutually commutate, and in case when all the operators contained in (1) are bounded self-adjoint, they are selfadjoint [2], [3].

Introduce some denotations

1. $\lambda^0 = (\lambda_1^0, \dots, \lambda_n^0) \in C^n$ is the eigenvalue of the system (1) if there exist nonzero vectors $x_i \in H_i$, $x_{n+r} \in R_2$ $(i = 1,\dots,n;\ r = 1,2,\dots,k-n)$, that (1) is fulfilled, and decomposable tensor $x = x_1 \otimes \cdots \otimes x_k \in \widetilde{H} = H_1 \otimes \cdots \otimes H_n \otimes \underbrace{R_2 \otimes \cdots R_2}_{k-n}$ is the corresponding eigen element of the system (1).
2. The element $\widetilde{x}_{m_1,\dots,m_k} \in \widetilde{H}$ is $(m_1,\dots,m_k)$-th associated to the eigen vector $x_{0,\dots,0}$ of the system (1), if there exists such a family of vectors $(\widetilde{x}_{i_1,\dots,i_k}) \subset \widetilde{H}$, $0 \le i_s \le m_s$, $s = 1,2,\dots,k$

$$\sum_{s=1}^{k} \sum_{0 \le r \le i_s} \frac{\partial^r \widetilde{A}_i^{+}(\lambda_0)}{r!\,\partial\lambda_s^r} \widetilde{x}_{i_1,i_2,\dots,i_s-r,\dots,i_k} = 0, \quad i = 1,2,\dots,n, \quad k = \sum_{i=1}^{n} k_i\ . \tag{9}$$

If $\sum_{s=1}^{k} i_s < 0$ then in (9) we assume $x_{i_1,\dots,i_k} = 0$.

Linearly independent vectors from the family $(x_{i_1,\dots,i_k})$ form a chain of eigen and adjoint vectors responding to the eigen value λ_0.

Let be

$$\Delta = \begin{pmatrix} A^{+}_{1,k_1,1} & A^{+}_{2,k_2,1} & \cdots & A^{+}_{n,k_n,1} \\ A^{+}_{1,k_1,2} & A^{+}_{2,k_2,2} & \cdots & A^{+}_{n,k_n,2} \\ \cdots & \cdots & \cdots & \cdots \\ A^{+}_{1,k_1,n} & A^{+}_{2,k_2,n} & \cdots & A^{+}_{n,k_n,n} \end{pmatrix} \tag{10}$$

and $\widetilde{\Delta}_0'$ is the will known in multiparameter theory operator [1], [2], [3]. Remind that the operator coefficients of the parameter $\widetilde{\lambda}_i$ from the system ((8),(4)) are the elements of the i-th column of $\widetilde{\Delta}_0$.

Lemma 1. $\ker \widetilde{\Delta}_0$ *is zero if and only if* $\ker \Delta = \theta$.

Proof of lemma 1. In the determinant $\widetilde{\Delta}_0$ make a permutation of columns by arranging the columns for which the operators $\widetilde{A}_{i,k}$ with 1-st indices $k_1,\dots,k_n$ at the end of the determinant $\widetilde{\Delta}_0$.

Operator $\widetilde{\Delta}_0$ may be represented as the sum of different operators Δ'_k, acting in the $\widetilde{H}$. Δ'_k are the products of the multiolication of two operators, one of them is represented with help of the matrix standing at the intersection of the first n-tuples and any n-columns of the matrix $\widetilde{\Delta}_0$ and another operator is represented with the help of the matrix standing at the intersection of the other $k-n$ tuples and $k-n$-columns of $\widetilde{\Delta}_0$.

Besides of operator

$$\begin{pmatrix} \widetilde{A}^+_{k_1,1} & \widetilde{A}^+_{k_1+k_2,1} & \cdots & \widetilde{A}^+_{\sum_1^n k_i} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{A}^+_{k_1,n} & \widetilde{A}^+_{k_1+k_2,n} & \cdots & \widetilde{A}^+_{\sum_1^n k_i} \end{pmatrix} \otimes \begin{pmatrix} T_0^+ & T_1^+ & 0 & 0 & 0 & 0 & . & . & . & . \\ T_2^+ & T_0^+ & T_1^+ & . & . & . & . & . & . & . \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & T_0^+ & T_1^+ & 0 & \cdots & \cdots & . & . \\ 0 & 0 & \cdots & T_2^+ & T_0^+ & T_1^+ & \cdots & \cdots & . & . \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ . & . & \cdots & . & . & . & \cdots & \cdots & T_2^+ & T_0^+ \end{pmatrix} \tag{11}$$

all Δ'_k equal to zero.

The first matrix in (11) coincides with the matrix Δ having the inverse in H by virtue of condition of lemma, the second matrix in (11) as it is easy to check, is a triangle matrix on one of whose diagonals stand units, consequently it is inverse in $R_2 \otimes R_2 \otimes \cdots \otimes R_2 = R^{2^{k-n}}$.

Lemma 1 is proved.

Theorem 1. *Let an operator Δ has a bounded inverse in the space H, then root subspaces of (1) coincides with a root subspaces of each operators $\widetilde{\Gamma}_i$ $(i=1,2,\dots,k)$.*

Proof. We have from [5] and [6] that if the operator Δ_0 of the system ((1), (4)) has a bounded inverse, then a system of eigen and associated vectors ((1), (4)) coincides with the system of eigen and associated vectors of each of operator Γ_i $(i=1,2,\dots,k)$, acting in $\widetilde{H}$.

For example, if $\widetilde{\lambda}_0 = (\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_k)$ are eigen-values of the system ((1), (4)) and $\widetilde{x} \in \widetilde{H}$ is the corresponding eigen vector, then we have

$$\Gamma_s \widetilde{x} = \widetilde{\lambda}_s \widetilde{x}_s, \quad s = 1,2,\dots,k.$$

Taking the equality (5) into account we have

$$\Gamma_{k_1+k_2+\dots+k_i+s}\widetilde{x} = \lambda^s_{k_i}\widetilde{x}, \quad s = 1,2,\dots,k_{i+1}, \quad i = 0,1,\dots,n-1. \tag{12}$$

Then use the equality

$$\begin{gathered} \widetilde{A}^+_{0,s} + \widetilde{A}^+_{1,s}\Gamma_1 + \cdots + \widetilde{A}^+_{k,s}\Gamma_k = 0, \\ \cdots\ \cdots\ \cdots \\ \widetilde{T}_2^+ + \widetilde{T}_0^+\Gamma_{\sum_1^i k_r+1} + \widetilde{T}_1^+\Gamma_{\sum_1^i k_r+2} = 0, \\ \cdots\ \cdots\ \cdots \end{gathered} \tag{13}$$

$$\widetilde{T}_2^+\Gamma_{\sum_1^i k_m+r}+\widetilde{T}_0^+\Gamma_{\sum_1^i k_m+r+1}+\widetilde{T}_1^+\Gamma_{\sum_1^i k_m+r+2}=0,$$

$$r=1,2,..,k_{i+1}-2;\quad s=1,...,n,$$

substituting to the first n equations of the system (13) the eigen and associated vectors (without restricting generality) of the operator $\Gamma_{\sum_1^i k_r+s}$ $(s\le k_{i+1})$. Then for the eigenvector $\widetilde{x}_0=\widetilde{x}_{0,0,...,0}$ of the operator $\Gamma_{\sum_1^i k_r+s}$ being the eigen vector of all other operators Γ_i $(i=1,2,...,k)$ we have with regard to equalities

$$\left(\widetilde{A}_{0,s}^+ +\sum_{k=1}^{k_{1,i}}\lambda_1^k\widetilde{A}_{1,k,s}^+ +\sum_{k=1}^{k_{2,i}}\lambda_2^k\widetilde{A}_{2,k,s}^+ +\cdots+\sum_{k=1}^{k_{n,i}}\lambda_n^k\widetilde{A}_{n,k,s}^+\right)\widetilde{x}_0=0,\ s=1,2,...,n. \tag{14}$$

Now let $\widetilde{x}_0,\widetilde{x}_1,...,\widetilde{x}_r$ is a chain of eigen and adjoint vectors $\Gamma_{\sum_1^i k_r+s}$, then

$$\Gamma_{\sum_1^i k_r+s}\widetilde{x}_0=\lambda_{k_i}^s\widetilde{x}_0,$$

$$\Gamma_{\sum_1^i k_r+s}\widetilde{x}_1=\lambda_{k_i}^s\widetilde{x}_1+s\lambda_{k_i}^{s-1}\widetilde{x}_0,$$

$$\cdots\cdots\cdots$$

$$\Gamma_{\sum_1^i k_r+s}\widetilde{x}_r=\lambda_{k_i}^s\widetilde{x}_r+s\lambda_{k_i}^{s-1}\widetilde{x}_{r-1}+\cdots+\frac{s(s-1)\cdots(s-r)}{r!}\lambda_{k_i}^{s-r}\widetilde{x}_0. \tag{15}$$

Substituting successively the eigen and associated vectors of the operator $\Gamma_{\sum_1^i k_r+s}$ into the first n equations of the system (13) and taking into account that the operators Γ_i, commutate pairwise, i.e. they coincide their root subspaces. Then, if $\widetilde{x}_r$ is the r-th associated vector of the operator $\Gamma_{\sum_1^i k_r+s}$, then there exist, generally speaking, such complex numbers $\widetilde{\lambda}_1,...,\widetilde{\lambda}_{\sum_1^i k_r+s-1},\widetilde{\lambda}_{\sum_1^i k_r+s+1},...,\widetilde{\lambda}_k$ that $\widetilde{\lambda}_i$ are eigen-values of the operator Γ_i $(i=1,2,...,k)$. Then the vector $\widetilde{x}_r$ belongs to the root subspace of the operator Γ_i responding to the eigen value of λ_i.

The analysis of the proof of the theorem from [5] shows that an eigen vector for one of operators Γ_i may be an associated vector for some others.

Taking in account this fact, equalities (5) and (15) and calculating the values of operators standing at the left hand side in the first n equations from (13) on the eigen and associated vectors of the operator $\Gamma_{\sum_1^i k_r+s}$ we establish that they are eigen and associated vectors of the system (1).

Really, assume that $\widetilde{x}$ is an vector associated to the eigen vector of one of operators Γ_i. Then, let it be m_r-th associated vector of operators Γ_r $(r=1,2,...,k)$.

Denote $\widetilde{x}=\widetilde{x}_{m_1,...,m_k}$.

For any s (for definiteness assume that $s=k_1+k_2+\cdots+k_i+r$) we have

$$\Gamma_s \widetilde{x}_{m_1,\dots,m_k} = \lambda_{i+1}^r \widetilde{x}_{m_1,\dots,m_k} + r\lambda_{i+1}^{r-1} \widetilde{x}_{m_1,\dots,m_{s-1},\dots,m_k} + \frac{r(r-1)}{2!}\lambda_{i+1}^{r-2} \widetilde{x}_{m_1,\dots,m_{s-2},\dots,m_k} +$$

$$+ \frac{r(r-1)\cdots(r-m_s)}{m_s!}\lambda_{i+1}^{r-m_s} \widetilde{x}_{m_1,\dots,0,\dots,m_k} = 0 ,$$

.....

$$\Gamma_s \widetilde{x}_{m_1,\dots,1,m_{s+1},\dots,m_k} = \lambda_{i+1}^r \widetilde{x}_{m_1,\dots,1,m_{s+1},\dots,m_k} + r\lambda_{i+1}^{r-1} \widetilde{x}_{m_1,\dots,0,\dots,m_k} = 0 ,$$

$$\Gamma_s \widetilde{x}_{m_1,\dots,0,m_{s+1},\dots,m_k} = \lambda_{i+1}^r x_{m_1,\dots,m_{s-1},0,m_{s+1},\dots,m_k} . \tag{16}$$

Then

$$\left(\widetilde{A}_{0,s}^+ + \widetilde{A}_{1,s}^+\Gamma_1 + \cdots + \widetilde{A}_{k,s}^+\Gamma_k\right)x_{m_1,\dots,m_k} = 0, \quad s = 1,\dots,n , \tag{17}$$

or as it is easy to check, after substitution (16) into (17), that

$$\sum_{s=1}^{k}\sum_{0,r\le i_s} \frac{\partial^r \widetilde{A}_i^+(\lambda^0)}{r!\partial\lambda_s^r} \widetilde{x}_{i_1,i_2,\dots,i_s-r,\dots,i_k} = 0 , \ 0 \le i_s \le m_s , \ i = 1,2,\dots,n .$$

The latter means that $x_{m_1,\dots m_k}$ is an associated element of the system ((1), (4)).

Thus, eigen and associated vectors of the operator Γ_s coincide with eigen and associated vectors of the system (1).

The inverse follows from the uniqueness of the selection of vectors $\widetilde{g}_1,\dots,\widetilde{g}_k$ on vector $g_0 \in \widetilde{H}$, $\widetilde{A}_{0,s} g_0 + \widetilde{A}_{1,s}^t g_1 + \cdots + \widetilde{A}_{k,s}^t g_k = 0$, for which it is valid the equalities

$$\widetilde{T}_2^+ g_0 + \widetilde{T}_0^+ g_{\sum_1^s k_r+1} + \widetilde{T}_1^+ g_{\sum_1^s k_r+2} = 0 , \ \widetilde{T}_2^+ \widetilde{g}_{\sum_1^s k_m+r} + \widetilde{T}_0^+ \widetilde{g}_{\sum_1^s k_m+r+1} + \widetilde{T}_1^+ \widetilde{g}_{\sum_1^s k_m+r+2} = 0 ,$$

$$r = 1,2,\dots,k_s - 2 , \quad s = 1,2,\dots,n .$$

The theorem is proved.

Remark. In the case when spaces H_i are finite-dimensional, the condition $\ker\Delta = \varnothing$ is necessary and sufficient that a system of eigen and associated vectors of (1) form basis in $H_1 \otimes \cdots \otimes H_n \otimes \underbrace{R_2 \otimes \cdots \otimes R_2}_{k-n}$.

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