2000 VOL. XIII(XXI)

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ON A PAPER BY EVERITT-KNOWLES-READ

Abstract

In this paper an example is constructed which shows the essence of the Sturm-Lioville differential expression with the main signalternating coefficient on the interval under consideration. That is although all the conditions of famous Everitt-Knowles-Read theorem on limit point hold true the constructed operator does not correspond to limit point, but to limit circle.

H.Weyl's theory (see [5] and [6]) on a limit point and limit circle on $[0,+\infty)$ for a Sturm-Liouville operator further was used by various author (see [2], [3] and [4]) on more general intervals $-\infty \le a < b \le +\infty$ and on more general differential operators.

The terminology of limit points - limit originates from the idea of considering of self-adjointness problem for the operator

$$-\frac{d^2}{dx^2}+q(x)$$

on $(a,+\infty)$ as a limit of corresponding problems on intervals (a,b) for $b\to+\infty$. Let $\varphi(x)$ and $\psi(x)$ be the solutions of the equation

$$-y''(x)+q(x)y=iy(x)$$

on $(a,+\infty)$, satisfying the conditions $\varphi(a)=\psi'(a)=0,\quad -\varphi'(a)=\psi(a)=1$. At a fixed b a set of that $z\in C$, for which $\eta(x)=\varphi(x)+z\psi(x)$ satisfies the condition $\cos\alpha\eta(b)+\sin\eta'(b)=0$ with some $\alpha\in[0,2\pi)$, from the circle C_b. For $b\to+\infty$ this circle either converges to some limit circle, or contracts to some limit circle or contracts to some limit-point.

Recently, by using Weyl's (see [3], p.175) criterion on essential self-adjointness of Sturm-Liouville's operator, these notions are defined by the quantity of solutions of differential equations belonging to L_2 near the singular ends of segments (a,b). In particular, in paper [1], for differential expression

$$l(u)=\frac{1}{W(x)}\left\{-\left(p(x)u'(x)\right)'+q(x)u(x)\right\},\ x\in[a,b), \tag{1}$$

where $-\infty<a<b\le+\infty$, $W(x)>0$ and

$$P^{-1}(x),\ q(x) \text{ and } W(x)\in L_{1,loc}[a,b) \tag{2}$$

the notion of limit-point and limit-circle are introduced as follows:

Definition. *We say that a differential expression $l(u)$ responds to the limit-point case at the point B, if for some u and consequently for all complex λ even if one solution of the equation*

$$\frac{1}{W(x)}\left\{-\left(p(x)u'(x)\right)'+q(x)u(x)\right\}=\lambda u$$

doesn't belong to $L^2_W[a,b)$. Otherwise we say that $l(u)$ responds to the limit-circle case.

Note that a chief coefficient $p(x)$ is a signalternating function, more exactly for some partition $\{\gamma_n\}_{n=0}^{\infty}$ of the segment $[a,b)$

$$a=\gamma_0<\gamma_1<...<\gamma_n<...<b \qquad (3)$$

the function $P_n(x)$ is non-negative on the interval $(\gamma_{2n},\gamma_{2n+1})$ and non-positive on the interval $(\gamma_{2n+1},\gamma_{2n+2})$ $(n=0,1,2,...,)$.

In [1] it is proved the following

Theorem (by Everitt-Knowles-Read). *By $\{I_m\}_{m=1}^{\infty}$ we denote a succession of such mutually exclusive intervals from $[a,b)$ that for each I_m there exists such n that $I_m\subset(\gamma_n,\gamma_{n+1})$ (where $\{\gamma_n\}_{n=0}^{\infty}$ - subsequence is determined as in (3)). Assume that a function $q(x)$ is represented in the form $q(x)=q_1(x)-q_2(x)$ $(q_i\in L_{1,loc}[a,b),\ i=1,2)$ and for each interval I_m there exist functions $Q_m(x)\in L_1(I_m)$, non-negative local-absolutely conditions function $\sigma_m(x)$ with support from I_m, a non-negative function $k_m(x)$ and such constants $\delta>0$, $k>0$, $\alpha(0\le\alpha\le 1)$ and $G_m>0$ at the following conditions are fulfilled:*

1) $\{1+k_1(\alpha,\delta)\}|p|(\sigma_m')^2-\alpha_m\sigma_m^2q_1+(1+\delta)\sigma_m^{2\alpha}H_m^2/|p|\le k_m$

almost everywhere in I_m where $H_m=\int Q_m(x)\sigma_m^{1-\alpha}(x)dx$,

$$\alpha_m=\begin{cases}1, & \text{если } p\ge 0 \text{ на } I_m,\\ -1, & \text{если } p\le 0 \text{ на } I_m\end{cases}$$

and

$$k_1(\alpha,\delta)>\frac{(\alpha+3)^2}{4\delta};.$$

2) For any interval J form I_m

$$\int_J(-\alpha_mq_1+Q_m)dx\ge -kG_mp_m^{1/2},$$

where

$$P_m=\inf\{|p(t)|:t\in I_m\};$$

3) $$\sum_{m=1}^{\infty}\Phi_m^{-1}\int_{I_m}\sigma_m\left(\alpha_mq_1\sigma_m^2+k_m\right)^{1/2}|p|^{-1/2}dx=+\infty,$$

$$\Phi_m=\sup_{I_m}\frac{1}{W}\left[G_m^2\sigma_m^2+k_m\right].$$

Then the differential expression $l(u)$ responds to the limit point case.

Our aim is to show the essence of the singularity of the differential expression $l(u)$ on the end of $x=b$.

Remind that the end b is singular, if $b=+\infty$ or if b is finite, but the condition of summability of coefficients is not fulfilled in the interval of the form $[c,b]$ $(a\le c<b)$.

Semi-segment $[0,1)$ divide into parts by the point $\gamma_n=1-\dfrac{1}{n+1}$, $n=0,1,2,...$,

$$0=\gamma_0<\gamma_1<...<\gamma_n<...<1.$$

On the semi-segment $[0,1)$ consider a step function (with infinite number of steps)

$$P(x)=\begin{cases}1, & \text{если } x\in(\gamma_{2n},\gamma_{2n+1}),\\ -1, & \text{если } x\in(\gamma_{2n+1},\gamma_{2n+2}),\end{cases}\quad n=0,1,2,.... \qquad (4)$$

Choose a system of segments

$$I_n = [a_n, b_n] \subset \left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right),$$

where $a_n = 1 - \dfrac{4n+3}{4n(n+1)}$ и $b_n = 1 - \dfrac{2n+1}{2n(n+1)}$ $(n = 1,2,\ldots)$.

It is obvious that the measure of the set I_m equals $\dfrac{1}{2n(n+1)}$.

Assuming in the formula (1) $W(x) = 1$, $q(x) = 0$ and choosing $p(x)$ as in the formula (4), in the space $L_2[0,1)$ consider a minimal operator L_0, generated by the differential expression

$$ly = -(p(x)y'(x))'. \tag{5}$$

By virtue of regularity of the differential expression (5) on the semi-segment $[0,1)$ the deficiency index of the operator L_0 is $(2,2)$.

Assuming

$$Q_n(x) = 0,\ \sigma_n(x) = 1,\ k_n(x) = \frac{1}{4n^2(n+1)^2},\ n = 1,3,\ldots, x \in I_n$$

and

$$G_n = \frac{1}{2n(n+1)},\ n = 1,2,\ldots,$$

we can be convinced that all the conditions of Everitt-Knowles-Read theorem are fulfilled. To check the conditions 1) and 2) is not difficult. Verify the condition 3).

The validity and condition 3) follows from the equality

$$\sum_{n=1}^{\infty} \frac{1}{\sup\limits_{x \in I_n}(G_n^2 + K_n(x))} \int_{I_n} \sqrt{K_n(x)}dx =$$

$$= \sum_{n=1}^{\infty} \frac{1}{\dfrac{1}{4n^2(n+1)^2} + \dfrac{1}{4n^2(n+1)^2}} \cdot \frac{1}{2n(n+1)} \cdot \frac{1}{2n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{2} = +\infty.$$

This example shows that though the fulfillment of all the conditions of Everitt-Knowles-Read, because of regularity the differential expression (5) responds not to limit-point but to limit-circle.

References

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Received April 4, 2000; Revised October 11, 2000.
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