

GARAKHANOVA N.N.

INTEGRAL CHARACTERISTICS OF $B_{k,n}$ MAXIMAL FUNCTIONS

Abstract

In this work we consider the generalized Bessel-Fourier shift operator, by means of which Hardy-Littlewood-Bessel-Fourier maximal functions ($B_{k,n}$ -maximal functions) are defined and investigated. The boundedness of $B_{k,n}$ -maximal functions from $L_1^{\gamma_{k,n}}(1 + \ln^+ L_1^{\gamma_{k,n}})(R_{k,+}^n) = L_1(1 + \ln^+ L_1)(R_{k,+}^n, x_{k,n}^{\gamma_{k,n}} dx)$ space to space $L_1^{\gamma_{k,n}}(R_{k,+}^n) \equiv L_1(R_{k,+}^n, x_{k,n}^{\gamma_{k,n}} dx)$ is proved.

In the theory of functions the shift operator $f(x) \rightarrow f(x+y)$ and the connected with it techniques of Fourier analysis a important role. Natural generalization of shift operators on R are the Delsart-Levitan generalized [1] shift operators (GSO), particularly Bessel's GSO which can be constructed by arbitrary Sturm-Liouville differential operator on R . Generalized shift operators form oneparametrical family, but nevertheless many problems of harmonic analysis can be generalized if we use generalized shift instead of ordinary ones.

In the work we consider the Bessel-Fourier generalized shift operator, by means of which are determined and investigated $B_{k,n}$ -maximal functions. The boundedness of $B_{k,n}$ -maximal functions from space $L_1^{\gamma_{k,n}}(1 + \ln^+ L_1^{\gamma_{k,n}})(R_{k,+}^n) = L_1(1 + \ln^+ L_1)(R_{k,+}^n, x_{k,n}^{\gamma_{k,n}} dx)$, $0 \leq k \leq n-1$ to space $L_1^{\gamma_{k,n}}(R_{k,+}^n)$ is proved. For Hardy-Littlewood maximal functions the analogous result was expressed in [2]. In the case $k=0$ $B_{0,n} \equiv B$ maximal functions were introduced and investigated by V.S. Guliev [3].

Let R^n be n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, $1 \leq k \leq n-1$, $x' = x_{1,k} = (x_1, \dots, x_k) \in R^k$, $x'' = x_{k,n} = (x_{k+1}, \dots, x_n) \in R^{n-k}$, $x = (x', x'') = (x_{1,k}, x_{k,n}) \in R^n$, $R_{k,+}^n = \{x = (x_{1,k}, x_{k,n}) \in R^n; x_{k+1} > 0, \dots, x_n > 0\}$, $B_{k,+}(x, r) = \{y - x \in R_{k,+}^n; |x - y| < r\}$, $\gamma_{k,n} = (\gamma_{k+1}, \dots, \gamma_n)$, $\gamma_{k+1} > 0, \dots, \gamma_n > 0$, $x_{k,n}^{\gamma_{k,n}} = x_{k+1}^{\gamma_{k+1}} \cdot \dots \cdot x_n^{\gamma_n}$.

In the case $k=0$ $x = x'' = x_{0,n} \in R^n$, $R_{0,+}^n \equiv R_{0,+}^n = \{x \in R^n; x_1 > 0, \dots, x_n > 0\}$, $\gamma = \gamma_{0,n} = (\gamma_1, \dots, \gamma_n)$.

We will denote by $L_p^{\gamma_{k,n}} = L_p^{\gamma_{k,n}}(R_{k,+}^n)$ the space of measurable functions $f(x)$, $x \in R_{k,+}^n$.

$$\|f\|_{L_p^{\gamma_{k,n}}(R_{k,+}^n)} = \|f\|_{p,\gamma_{k,n}} = \left(\int_{R_{k,+}^n} |f(x)|^p x_{k,n}^{\gamma_{k,n}} dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Suppose $L_\infty^{\gamma_{k,n}}(R_{k,+}^n) = L_\infty(R_{k,+}^n)$, where $L_\infty(R_{k,+}^n)$ is the class of all essential bounded functions f with the norm

$$\|f\|_{L_\infty^{\gamma_{k,n}}(R_{k,+}^n)} = \|f\|_{L_\infty(R_{k,+}^n)} = \operatorname{ess\,sup}_{x \in R_{k,+}^n} |f(x)|.$$

Generalized shift operator or Bessel shift (in the short form GSO or $B_{k,n}$ -shift) is determined by the following way (see [4], [5]):

$$\begin{aligned} T^\gamma f(x) &= \frac{\prod_{i=k+1}^n \Gamma\left(\frac{\gamma_{i+1}}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-k} \prod_{i=k+1}^n \Gamma\left(\frac{\gamma_i}{2}\right)} \int_0^\pi \cdots \int_0^\pi f(x_1 - y_1, \dots, x_k - y_k, \\ &\quad \sqrt{x_{k+1}^2 - 2x_{k+1}y_{k+1}\cos\alpha_{k+1} + y_{k+1}^2}, \dots, \sqrt{x_n^2 - 2x_ny_n\cos\alpha_n + y_n^2}) \times \\ &\quad \times \prod_{i=k+1}^n \sin^{\gamma_{i-1}} \alpha_i d\alpha_{k+1} \dots d\alpha_n. \end{aligned}$$

We will denote by B_j Bessel's singular differential operator $B_j = \frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}$,

$\gamma_j > 0, j = k+1, \dots, n$, $B_{k,n} = (B_{k+1}, \dots, B_n)$, and by $\Delta_{B_{k,n}}$ - Laplace-Bessel type operator which is determined by the following way:

$$\Delta_{B_{k,n}} = \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2} + \sum_{i=k+1}^n B_i.$$

Let's determine $B_{k,n}$ -maximal function by the following way:

$$M_{B_{k,n}} f(x) = \sup_{\varepsilon > 0} |B_{k,+}(0, \varepsilon)|^{-1} \int_{B_{k,+}(0, \varepsilon)} T^\gamma f(x) y_{k,n}^{\gamma_{k,n}} dy.$$

Here $B_{k,+}(0, \varepsilon) = \{y \in R_{k,+}^n; |y| < \varepsilon\}$, $|E|_{\gamma_{k,n}} = \int_E x_{k,n}^{\gamma_{k,n}} dx$, $E \subset R_{k,+}^n$.

The following theorem was proved in [6].

Theorem 1.

1) If $f \in L_1^{\gamma_{k,n}}(R_{k,+}^n)$, then for any $\alpha > 0$

$$\left| \left\{ x \in R_{k,+}^n : M_{B_{k,n}} f(x) > \alpha \right\} \right|_{\gamma_{k,n}} \leq \frac{C_1}{\alpha} \int_{R_{k,+}^n} f(x) x_{k,n}^{\gamma_{k,n}} dx, \quad (1)$$

where C does not depend on f .

2) If $f \in L_p^{\gamma_{k,n}}(R_{k,+}^n), 1 < p \leq \infty$, then $M_{B_{k,n}} f(x) \in L_p^{\gamma_{k,n}}(R_{k,+}^n)$ and

$$\|M_{B_{k,n}} f\|_{p, \gamma_{k,n}} \leq C_{p, \gamma_{k,n}} \|f\|_{p, \gamma_{k,n}},$$

where $C_{p, \gamma_{k,n}}$ depends only on p , $\gamma_{k,n}$ and dimension n .

Let's note that the first part of theorem 1 may be amplified. Namely, it is valid

Lemma 1. Let $\varepsilon \in (0, 1)$, $C_2 = C_1 / (1 - \varepsilon)$, then for any $t \in (0, \infty)$ the following inequality takes place

$$\left| \left\{ x \in R_{k,+}^n : M_{B_{k,n}} f(x) > t \right\} \right|_{\gamma_{k,n}} \leq \frac{C_2}{t} \int_{\{x \in R_{k,+}^n; f(x) > \alpha\}} f(x) x_{k,n}^{\gamma_{k,n}} dx.$$

Proof. For $\varepsilon = 0$ this is inequality (1). For $0 < \varepsilon < 1$ let's determine $f_1(x)$ so

$$f_1(x) = \begin{cases} f(x), & |f(x)| > \varepsilon t, \\ 0, & |f(x)| \leq \varepsilon t. \end{cases}$$

Then $|f(x)| \leq |f_1(x)| + \varepsilon t$ and $M_{B_{k,n}} f(x) \leq M_{B_{k,n}} f_1(x) + \varepsilon t$. So, if $M_{B_{k,n}} f(x) > t$, then $M_{B_{k,n}} f_1(x) > (1 - \varepsilon)t$, i.e.

$$\left\{ x \in R_{k,+}^n : M_{B_{k,n}} f(x) > t \right\} \subset \left\{ x \in R_{k,+}^n : M_{B_{k,n}} f_1(x) > (1 - \varepsilon)t \right\}.$$

So, according to (1)

$$\begin{aligned} \left| \left\{ x \in R_{k,+}^n : M_{B_{k,n}} f(x) > t \right\} \right|_{\gamma_{k,n}} &\leq \left| \left\{ x \in R_{k,+}^n : M_{B_{k,n}} f_1(x) > (1 - \varepsilon)t \right\} \right|_{\gamma_{k,n}} \leq \\ &\leq \frac{(1 - \varepsilon)^{-1} C_1}{t} \int_{\{x \in R_{k,+}^n, |f(x)| > \varepsilon t\}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx = C_2/t \int_{\{x \in R_{k,+}^n, |f(x)| > \varepsilon t\}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx. \end{aligned}$$

The lemma 1 has been proved.

Definition 1. Let function $f(x)$ be determined and measured on $R_{k,+}^n$. Let call $B_{k,n}$ -function of distribution of function f the function $f_{*,\gamma_{k,n}}$ determined $\forall t \in [0, \infty)$ by the equality

$$f_{*,\gamma_{k,n}}(t) = \left| \left\{ x \in R_{k,+}^n : |f(x)| > t \right\} \right|_{\gamma_{k,n}}.$$

The following theorem is valid

Theorem 2 [7]. Let f be measured on $R_{k,+}^n$ function. Then for $1 \leq p < \infty$

$$\|f\|_{L_p^{\gamma_{k,n}}(R_{k,+}^n)} = \left(p \int_0^\infty t^{p-1} f_{*,\gamma_{k,n}}(t) dt \right)^{\frac{1}{p}} = \left(- \int_0^\infty t^p df_{*,\gamma_{k,n}}(t) \right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|f\|_{L_\infty^{\gamma_{k,n}}(R_{k,+}^n)} = \inf \{t : f_{*,\gamma_{k,n}}(t) = 0\}.$$

The particular case ($p = 1$):

$$\|f\|_{L_1^{\gamma_{k,n}}(R_{k,+}^n)} = \|f_{*,\gamma_{k,n}}\|_{L_1^{\gamma_{k,n}}(0,\infty)}. \quad (2)$$

The following lemma is valid:

Lemma 2. Let f be measured on $R_{k,+}^n$ function, then for any $s \in (0, \infty)$

$$\int_{\{x \in R_{k,+}^n, |f(x)| > s\}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx = s f_{*,\gamma_{k,n}}(s) + \int_s^\infty f_{*,\gamma_{k,n}}(t) dt.$$

Proof. Let

$$g(x) = \begin{cases} f(x), & |f(x)| > s, \\ 0, & |f(x)| \leq s. \end{cases}$$

For any $t \geq s$

$$g_{*,\gamma_{k,n}}(t) = \left| \left\{ x \in R_{k,+}^n : |g(x)| > t \right\} \right|_{\gamma_{k,n}} = \left| \left\{ x \in R_{k,+}^n : |f(x)| > t \right\} \right|_{\gamma_{k,n}} = f_{*,\gamma_{k,n}}(t). \quad (3)$$

For any $t \in (0, s)$

$$g_{*,\gamma_{k,n}}(t) = \left| \left\{ x \in R_{k,+}^n : |g(x)| > t \right\} \right|_{\gamma_{k,n}} = \left| \left\{ x \in R_{k,+}^n : |f(x)| > s \right\} \right|_{\gamma_{k,n}} = f_{*,\gamma_{k,n}}(s). \quad (4)$$

Then taking into account (2), (3), (4) we have

$$\begin{aligned}
& \int_{\{x \in R_{k,+}^n, f(x) > s\}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx = \int_{R_{k,+}^n} g(x) x_{k,n}^{\gamma_{k,n}} dx = \int_0^\infty g_{*,\gamma_{k,n}}(t) dt = \\
& = \int_0^s g_{*,\gamma_{k,n}}(t) dt + \int_s^\infty g_{*,\gamma_{k,n}}(t) dt = f_{*,\gamma_{k,n}}(s) \int_0^s dt + \int_s^\infty f_{*,\gamma_{k,n}}(t) dt = \\
& = sf_{*,\gamma_{k,n}}(s) + \int_s^\infty f_{*,\gamma_{k,n}}(t) dt.
\end{aligned}$$

Lemma 2 has been proved.

Let's prove the following

Theorem 3. For any $r > 0$ and any measured on $R_{k,+}^n$ function f , for which $\text{supp } f \subset B_{k,+}(0, r)$ the following inequality holds:

$$\|M_{B_{k,n}} f\|_{L_{\gamma_{k,n}}^{\gamma_{k,n}}(B_{k,+}(0, r))} \leq C \int_{B_{k,+}(0, r)} |f(x)| (1 + \ln^+ |f(x)|) x_{k,n}^{\gamma_{k,n}} dx + |B_{k,+}(0, r)|_{\gamma_{k,n}},$$

where C does not depend on f .

Proof. By virtue of lemma 1 and lemma 2 we have

$$\begin{aligned}
& \int_{B_{k,+}(0, r)} M_{B_{k,n}} f(x) x_{k,n}^{\gamma_{k,n}} dx \leq \int_{\{x \in B_{k,+}(0, r), M_{B_{k,n}} f(x) \leq 1\}} M_{B_{k,n}} f(x) x_{k,n}^{\gamma_{k,n}} dx + \int_{\{x \in B_{k,+}(0, r), M_{B_{k,n}} f(x) > 1\}} M_{B_{k,n}} f(x) x_{k,n}^{\gamma_{k,n}} dx \leq \\
& \leq \int_{\{x \in B_{k,+}(0, r), M_{B_{k,n}} f(x) \leq 1\}} x_{k,n}^{\gamma_{k,n}} dx + \int_{\{x \in B_{k,+}(0, r), M_{B_{k,n}} f(x) > 1\}} M_{B_{k,n}} f(x) x_{k,n}^{\gamma_{k,n}} dx \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + (M_{B_{k,n}} f)_{*,\gamma_{k,n}}(1) + \\
& + \int_1^\infty (M_{B_{k,n}} f)_{*,\gamma_{k,n}}(t) dt \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + \left| \left\{ x \in B_{k,+}(0, r) : |M_{B_{k,n}} f(x)| > 1 \right\} \right|_{\gamma_{k,n}} + \\
& + \int_1^\infty (M_{B_{k,n}} f)_{*,\gamma_{k,n}}(t) dt \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + c \int_1^\infty \frac{1}{t/2} \left(\int_{\{x \in B_{k,+}(0, r), |f(x)| > \frac{t}{2}\}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx \right) dt + \\
& + c \int_{B_{k,+}(0, r)} |f(x)| x_{k,n}^{\gamma_{k,n}} dx \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + C \int_1^\infty \frac{1}{t} \left(\int_{\{x \in B_{k,+}(0, r), |f(x)| > \frac{t}{2}\}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx \right) dt + \\
& + c \int_{B_{k,+}(0, r)} |f(x)| x_{k,n}^{\gamma_{k,n}} dx \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + C \int_{\{x \in B_{k,+}(0, r), |f(x)| > \frac{1}{2}\}} |f(x)| \left(\int_1^{2|f(x)|} \frac{dt}{t} \right) x_{k,n}^{\gamma_{k,n}} dx + \\
& + c \int_{B_{k,+}(0, r)} |f(x)| x_{k,n}^{\gamma_{k,n}} dx \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + C \int_{\{x \in B_{k,+}(0, r), |f(x)| > \frac{1}{2}\}} |f(x)| \ln(2|f(x)|) x_{k,n}^{\gamma_{k,n}} dx + \\
& + c \int_{B_{k,+}(0, r)} |f(x)| x_{k,n}^{\gamma_{k,n}} dx \leq |B_{k,+}(0, r)|_{\gamma_{k,n}} + C \int_{\{x \in B_{k,+}(0, r), |f(x)| > \frac{1}{2}\}} |f(x)| (1 + \ln^+ |f(x)|) x_{k,n}^{\gamma_{k,n}} dx.
\end{aligned}$$

Theorem 3 has been proved.

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Garakhanova N.N.

Baku State University named after E.M. Rasulzadeh.
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

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