GASANOVA L.K.

ON THE SOLUTION OF GOURSAT-DARBOUX PROBLEM WITH THE BOUNDARY GENERALIZED INFLUENCES

Abstract

Theorems on the existence and uniqueness of the weak local and the weak global solutions of Goursat-Darboux problems with boundary generalized influence are proved.

In the given paper the theorems on the existence and uniqueness of the weak local and the weak global solutions of Goursat-Darboux problem with the boundary-generalized influences are proved.

Let the process be described by the system of equations

$$z_{tx}=f(z,t,x),\quad (t,x)\in D=D_{t_1}\times D_{x_1} \tag{1}$$

with the condition

$$z_t(t,x_0)=f_1(z(t,x_0),t)+G_1(t)\dot u_1(t);\ t\in\overline{D}_{t_1}=[t_0,t_1], \tag{2}$$

$$z_x(t_0,x)=f_2(z(t_0,x),x)+G_2(x)\dot u_2(x);\ x\in\overline{D}_{x_1}=[x_0,x_1], \tag{3}$$

$$z(t_0,x_0)=z^0. \tag{4}$$

Here $f(z,t,x),\ f_1(z,t),\ f_2(z,x),\ z^0$ are the given n-dimensional vector-functions; $G_1(t)$ and $G_2(x)$ are matrix functions with dimensions $n\times m_1$ and $n\times m_2$ correspondingly, $(u_1(t),u_2(x))$ is m_1+m_2-dimensional vector-function with the bounded variations, $(\dot u_1(t),\dot u_2(x))$ is a zero order distribution, being a generalized derivative of the function $(u_1(t),u_2(x))$ [1, 5]. Let us denote by $VB_{m_1}(t_0,t_1)$ the set of m_1-dimensional vector-functions of the bounded variation on segment $[t_0,t_1]$ and continuous from left on the open interval in the neighbourhood of segment $[t_0,t_1]$ ([3]).

Vector-function $z(t,x)\in\ \in VB_n(D)$ (see [2]) satisfying the integral system

$$z(t,x)=z^0+\int_{t_0}^{t}f_1(z(\tau,x_0),\tau)d\tau+\int_{x_0}^{x}f_2(z(t_0,s),s)ds+\int_{t_0}^{t}\int_{x_0}^{x}f(z(\tau,s),\tau,s)dsd\tau+$$

$$+\int_{t_0}^{t}G_1(\tau)du_1(\tau)+\int_{x_0}^{x}G_2(s)du_2(s) \tag{5}$$

is called the weak solution of problem (1)-(4) corresponding to vector-function

$$(u_1(t),u_2(x))\in VB_{m_1}(t_0,t_1)\times VB_{m_2}(x_0,x_1).$$

The last two integrals in the right-hand side of equality (5) are understood in Stieltyes sense.

Theorem 1. *Let functions $f(z,t,x),\ f_1(z,t),\ f_2(z,x),\ G_1(t),\ G_2(x)$ be continuous by totality of arguments for $t\in\overline{D}_{t_1}$, $x\in\overline{D}_{x_1}$, $z\in R_n$. Moreover, let functions $f(z,t,x)$, $f_1(z,t)$, $f_2(z,x)$ satisfy locally Lipschitz condition with respect to z, i.e. for $(t,x)\in D$, $z_1,z_2\in S_R(z^0)=\{z\in R_n,\|z-z^0\|\le R\}$ inequalities $\|f_i(z_2,s)-f_i(z_1,s)\|\le L(R)\|z_2-z_1\|$, $i=1,2$; $\|f(z_2,t,x)-f(z_1,t,x)\|\le L(R)\|z_2-z_1\|$ are valid, where $L(R)$ is the constant* *dependent on R. Then for arbitrary distributions $(\dot{u}_1(t), \dot{u}_2(x))$ of zero order the unique weak local solution of problem (1)-(4) exists.*

Proof. Let

$$Var_{t_0}^{t_1} u_1(t) = N_1,\ Var_{x_0}^{x_1} u_2(t) = N_2,\ \max_{t_0 \le t \le t_1} \|G_1(t)\| = N_{G_1},\ \max_{x_0 \le x \le x_1} \|G_2(t)\| = N_{G_2},$$

$$R = 2(N_1 N_{G_1} + N_2 N_{G_2}),\ S_R(z^0) = \{z \in R_n : \|z - z^0\| \le R\},\ \max_{S_R(z^0) \times [t_0, t_1]} \|f_1(z,t)\| = N_{f_1},$$

$$\max_{S_R(z^0) \times [x_0, x_1]} \|f_2(z,t)\| = N_{f_2},\ \max_{S_R(z^0) \times D} \|f(z,t,x)\| = N_f.$$

We choose $t_0 < t_1' \le t_1$, $x_0 < x_1' \le x_1$ from condition

$$N_{f_1}(t_1' - t_0) + N_{f_2}(x_1' - x_0) + N_f(t_1' - t_0)(x_1' - x_0) + N_1 N_{G_1} + N_2 N_{G_2} \le R.$$

Let us prove the existence and uniqueness of the weak solution of problem (1)-(4) in domain $D' = (t_0, t_1') \times (x_0, x_1')$.

Let's consider successive approximations ([4])

$$z^k(t,x) = z^0 + \int_{t_0}^{t} f_1(z^{k-1}(\tau, x_0), \tau) d\tau + \int_{x_0}^{x} f_2(z^{k-1}(t_0, s), s) ds + \int_{t_0}^{t}\int_{x_0}^{x} f(z^{k-1}(\tau, s), \tau, s) ds d\tau +$$

$$+ \int_{t_0}^{t} G_1(\tau) du_1(\tau) + \int_{x_0}^{x} G_2(s) du_2(s),\ k = 1,2,\ldots,$$

$$z^0(t,x) = z^0. \tag{6}$$

From equality (6) for $k = 1$ we obtain:

$$\|z^1(t,x) - z^0\| \le \int_{t_0}^{t} \|f_1(z^0, \tau)\| d\tau + \int_{x_0}^{x} \|f_2(z^0, s)\| ds + \int_{t_0}^{t}\int_{x_0}^{x} \|f(z^0, \tau, s)\| d\tau ds +$$

$$+ \int_{t_0}^{t} \|G_1(t)\| \|du_1(\tau)\| + \int_{x_0}^{x} \|G_2(s)\| \|du_2(s)\|.$$

Hence we have

$$\|z^1(t,x) - z^0\| \le N_{f_1}(t - t_0) + N_{f_2}(x - x_0) + N_f(t - t_0)(x_0 - x) +$$

$$+ N_{G_1} \int_{t_0}^{t} \|du_1(\tau)\| + N_{G_2} \int_{x_0}^{x} \|du_2(s)\|.$$

Since

$$\int_{t_0}^{t} \|du_1(\tau)\| \le Var_{t_0}^{t_1} u_1(t) = N_1,\ t \in [t_0, t_1],$$

$$\int_{x_0}^{x} \|du_2(s)\| \le Var_{x_0}^{x_1} u_2(x) = N_2,\ x \in [x_0, x_1],$$

then it follows from choice of t_1' and x_1' that

$$\|z^1(t,x) - z^0\| \le N_{f_1}(t_1' - t_0) + N_{f_2}(x_1' - x_0) + N_f(t_1' - t_0)(x_1' - x_0) +$$

$$+ N_{G_1} N_1 + N_{G_2} N_2 \le R.$$

By the mathematical induction method one can prove, that

$$\|z^k(t,x) - z^0\| \le R,\ (t,x) \in D',\ k = 1,2,\ldots. \tag{7}$$

Now, let us prove that sequence $\{z^k(t,x)\}$ uniformly converges to D'. Let us assume $\delta_k(t,x)=\left\|z^k(t,x)-z^{k-1}(t,x)\right\|$, $k=1,2,\ldots$, $H=\max\{1+t_1'-t_0, 1+x_1'-x_0\}$.

From equality (6) we obtain:

$$\delta_k(t,x)\le \int_{t_0}^{t}\left\|f_1\left(z^{k-1}(\tau,x_0),\tau\right)-f_1\left(z^{k-2}(\tau,x_0),\tau\right)\right\|d\tau+\int_{x_0}^{x}\left\|f_2\left(z^{k-1}(t_0,s),s\right)-f_2\left(z^{k-2}(t_0,s),s\right)\right\|ds+$$

$$+\int_{t_0}^{t}\int_{x_0}^{x}\left\|f\left(z^{k-1}(\tau,s),\tau,s\right)-f\left(z^{k-2}(\tau,s),\tau,s\right)\right\|d\tau ds.$$

Hence, using Lipschitz condition we have:

$$\delta_k(t,x)\le L(R)\left\{\int_{t_0}^{t}\delta_{k-1}(\tau,x_0)d\tau+\int_{x_0}^{x}\delta_{k-1}(t_0,s)ds+\int_{t_0}^{t}\int_{x_0}^{x}\delta_{k-1}(\tau,s)dsd\tau\right\},\ k=2,3,\ldots. \quad (8)$$

By virtue of (7): $\delta_1(t,x)=\left\|z^1(t,x)-z^0\right\|\le R$, $(t,x)\in D'$.

Then from (8) for $k=2$ we obtain:

$$\delta_2(t,x)\le RL(R)\{(t-t_0)+(x-x_0)+(t-t_0)(x-x_0)\}\le$$

$$\le RL(R)\{(t-t_0)(1+x_1'-x_0)+(x-x_0)(1+t_1'-t_0)\}\le RL(R)H\{(t-t_0)+(x-x_0)\}.$$

By the mathematical induction method one can prove, that

$$\delta_k(t,x)\le R\frac{L^{k-1}(R)H^{k-1}}{(k-1)!}\left[(t-t_0)^{k-1}+(x-x_0)^{k-1}\right],\ k=2,3,\ldots. \quad (9)$$

Indeed inequality (9) is valid for $k=2$. Assuming that it is valid for $k=n-1$, one can show its validity for $k=n$. From inequality (8) we have:

$$\delta_n(t,x)\le L(R)\left\{\int_{t_0}^{t}R\frac{L^{n-2}(R)H^{n-2}}{(n-2)!}(\tau-t_0)^{n-2}d\tau+\int_{x_0}^{x}R\frac{L^{n-2}(R)H^{n-2}}{(n-2)!}\times\right.$$

$$\left.\times(s-x_0)^{n-2}ds+\int_{t_0}^{t}\int_{x_0}^{x}R\frac{L^{n-2}(R)H^{n-2}}{(n-2)!}\left[(\tau-t_0)^{n-2}+(s-x_0)^{n-2}\right]dsd\tau\right\}=$$

$$=R\frac{L^{n-1}(R)H^{n-2}}{(n-2)!}\left\{\frac{(t-t_0)^{n-1}}{n-1}+\frac{(x-x_0)^{n-1}}{n-1}+\frac{(t-t_0)^{n-1}}{n-1}(x-x_0)+\right.$$

$$\left.+(t-t_0)\frac{(x-x_0)^{n-1}}{n-1}\right\}\le R\frac{L^{n-1}(R)H^{n-2}}{(n-1)!}\{(t-t_0)^{n-1}(1+x_1'-x_0)+$$

$$+(x-x_0)^{n-1}(1+t_1'-t_0)\}\le R\frac{L^{n-1}(R)H^{n-1}}{(n-1)!}\left[(t-t_0)^{n-1}+(x-x_0)^{n-1}\right].$$

Consequently, inequality (9) is valid.

By Dalamber's test the series, whose general term $u_n=R\dfrac{L^{n-1}(R)H^{n-1}}{(n-1)!}\times$ $\times\left[(t_1'-t_0)^{n-1}+(x_1'-x_0')^{n-1}\right]$ converges. Then by virtue of inequality (9) the series

$$z^1(t,x)+\left(z^2(t,x)-z^1(t,x)\right)+\cdots+\left(z^n(t,x)-z^{n-1}(t,x)\right)+\cdots \quad (10)$$

admits

$$\delta_1(t_1',x_1')+\delta_2(t_1',x_1')+\cdots+\delta_n(t_1',x_1')+\cdots \quad (11)$$

as a majorant.

Therefore series (10) absolutely and uniform convergences in D'.

Consequently, sequence $\{z^k(t,x)\}$ for $k \to \infty$ uniformly converges to some limit $z(t,x)$, $(t,x) \in D'$. In formulas (6) one can pass to the limit for $k \to \infty$ under the sign of integral at the result of which these formulas will pass to integral equation (5).

Consequently, vector-function $z(t,x)$ is the weak solution of problem (1)-(4), correspondingly to control $(u_1(t), u_2(x))$. Let us prove, that problem (1)-(4) has not any other solution.

Let $z^*(t,x)$ be some weak solution of problem (1)-(4), satisfying the condition $\|z^*(t,x) - z^0\| \le R$, $(t,x) \in D'$. Then

$$z^*(t,x) = z^0 + \int_{t_0}^{t} f_1(z^*(\tau,x_0),\tau)d\tau + \int_{x_0}^{x} f_2(z^*(t_0,s),s)ds +$$

$$\int_{t_0}^{t}\int_{x_0}^{x} f(z^*(\tau,s),\tau,s)dsd\tau + \int_{t_0}^{t} G_1(\tau)du_1(\tau) + \int_{x_0}^{x} G_2(s)du_2(s).$$

Subtracting from (6) this equality, taking in both sides absolute quantity and using Lipschitz condition, we obtain:

$$\|z^k(t,x) - z^*(t,x)\| \le L(R)\left\{\int_{t_0}^{t}\|z^{k-1}(\tau,x_0) - z^*(\tau,x_0)\|d\tau + \right.$$

$$\left. + \int_{x_0}^{x}\|z^{k-1}(t_0,s) - z^*(t_0,s)\|ds + \int_{t_0}^{t}\int_{x_0}^{x}\|z^k(\tau,s) - z^*(\tau,s)\|dsd\tau\right\}, \quad k = 1,2,\ldots.$$

Hence, assuming

$$\delta_k^*(t,x) = \|z^k(t,x) - z^*(t,x)\|$$

we have

$$\delta_k^*(t,x) \le L(R)\left\{\int_{t_0}^{t}\delta_{k-1}^*(\tau,x_0)d\tau + \int_{x_0}^{x}\delta_{k-1}^*(t_0,s)ds + \int_{t_0}^{t}\int_{x_0}^{x}\delta_{k-1}^*(\tau,s)dsd\tau\right\}, \quad k = 1,2,\ldots.$$

This inequality has the same form of (8), analogically to that as inequality (9) has been obtained, we'll obtain:

$$\delta_k^*(t,x) \le R\frac{(L(R)H)^k}{k!}\left((t-t_0)^k + (x-x_0)^k\right), \quad k = 1,2,\ldots.$$

Thus, $\delta_k^*(t,x)$ not exceeding the general term of the almost everywhere convergent series, necessarily tends to zero for $k \to +\infty$. Consequently, $\lim_{k\to\infty}\|z^k(t,x) - z^*(t,x)\| = 0$, $(t,x) \in D'$.

From uniqueness of the limit of the convergent sequence it follows that $z^*(t,x) = z(t,x)$, $(t,x) \in D'$, i.e. the solution is unique.

Theorem 2. *Let function $f(z,t,x)$, $f_1(z,t)$, $f_2(z,x)$, $G_1(t)$, $G_2(x)$ be continuous for $(t,x) \in D$, $z \in R^n$, $t \in [t_0,t_1]$, $x \in [x_0,x_1]$. Moreover, $f(z,t,x)$, $f_1(z,t)$, $f_2(z,x)$ satisfy Lipschitz condition with respect to $z \in R_n$, for $(t,x) \in D$. Then for arbitrary distribution $(\dot{u}_1(t), \dot{u}_2(x))$ of zero order the weak solution of problem (1)-(4)* exists in *domain D.*

Analogically to the proof of theorem 1, proving theorem 2 successive approximations (6) are considered and it is proved, that sequence of functions $\{z^k(t,x)\}$

uniformly converges on D. The limit function is the unique weak solution of (1)-(4) on D.

References

[1]. Гельфанд И.М., Шилов Г.Е. *Обобщенные функции и действия над ними.* Вып.I, М., 1958, с.440.
[2]. Математическая энциклопедия, т.I, М., 1977, с.1151.
[3]. Орлов Ю.В. *Теория оптимальных систем с обобщенными управлениями.* М., «Наука», 1988, с.190.
[4]. Трикоми Ф. *Лекции по уравнениям в частных производных.* М., 1957, с. 444.
[5]. Филлипов А.Ф. *Дифференциальные уравнения с разрывной правой частью.* М., «Наука», 1985, с.224.

Gasanova L.K.
Baku State University named after E.M. Rasulzadeh.
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

Received December 28, 1999; Revised June 14, 2000.
Translated by Nazirova S.H.