

HASANOV S.G.

ON THE LOCAL SMOOTHNESS OF A CONJUGATE FUNCTION

Abstract

In this paper we introduce local smoothness modulus of k -th order for a continuous 2π -periodic function and obtain an estimate of local smoothness modulus of k -th order of conjugate function in terms of k -th order local smoothness modulus of its density.

In this paper we introduce a local smoothness modulus of k -th order for a continuous 2π -periodic function and obtain, an estimate on local smoothness modulus of k -th order of conjugate function in terms of k -th order local smoothness modulus of its density.

Consider

$$\tilde{f}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \operatorname{ctg} \frac{\tau - x}{2} d\tau. \quad (1)$$

Local properties of the conjugate function (1) in terms of continuity modulus, smoothness modulus and their local modifications were studied by many authors [1], [2], [3] and others. In paper [1] it was introduced a two-parametric characteristics $\omega_f^{z_0}(\delta, \eta)$, where z_0 is a fixed point, local continuity modulus in whose terms a problem on local \forall properties of a conjugate function is studied by means of obtaining Zigmund type estimates of local continuity modulus density.

Denote by $C_{2\pi}$ a space of continuous and 2π -periodic functions. Put for $f \in C_{2\pi}$

$$\omega_f^{k, x_0}(\delta, \eta) = \sup_{\substack{x \in (x_0 - \eta, x_0 + \eta) \\ 0 < t \leq \delta}} |\Delta_t^k(f; x)|,$$

where x_0 is a fixed point, $\delta, \eta > 0$ and $\Delta_t^k(f, x) = \sum_{m=0}^k (-1)^m C_k^m f(x + mt)$ is the difference of k -th order at the point x with the step t .

For brevity put $\omega_f^{1, x_0}(\delta, \eta) = \omega_f^{x_0}(\delta, \eta)$.

Theorem 1. Let $f \in C_{2\pi}$ and the condition

$$\int_0^\pi \frac{\omega_f^{x_0}(y, \eta)}{y} dy < +\infty, \eta \in (0, \pi]$$

be fulfilled.

Then $\tilde{f}(x) \in C_{2\pi}$ and it is valid the following estimate

$$\omega_{\tilde{f}}^{k, x_0}(\delta, \eta) \leq C \left(\int_0^\delta \frac{\omega_f^{x_0}(y, \eta + (3k+1)\delta)}{y} dy + \delta^k \int_\delta^\eta \frac{\omega_f^{x_0}(y, 2\eta)}{y^{k+1}} dy + \delta^k \int_\eta^\pi \frac{\omega_f^{x_0}(y, 2\eta)}{y^{k-1}} dy \right),$$

$$0 < \delta \leq \eta \leq \pi.$$

Proof. It is known (see [1]), $\tilde{f}(x) \in C_{2\pi}$. Prove the correctness of the estimation. Not diminishing generality, by virtue of periodicity of f we consider $x_0 = 0$. Let $x \in (-\eta, \eta)$, $t > 0$ and $\eta \in (0, \pi]$, $m = \overline{0, k}$.

Write (taking into account that $\int_{-\pi}^{\pi} \operatorname{ctg} \frac{\tau - x}{2} d\tau = 0$)

$$\tilde{f}(x + mt) = \frac{1}{\pi} \int_{-\pi+x+mt}^{\pi+x+mt} [f(\tau) - f(x + mt)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau. \quad (2)$$

It is obvious that we can write the integral at the right hand side of the equality (2) in the form of the sum of integrals by the following way

$$\begin{aligned} \pi \tilde{f}(x + mt) &= \int_{-(k+1)x+x}^{(k+1)x+x} [f(\tau) - f(x + mt)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau + \\ &+ \left(\int_{-\pi+x+mt}^{(k+1)x+x} - \int_{(k+1)x+x}^{\pi+x+mt} \right) [f(\tau) - f(x)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau + \\ &+ [f(x) - f(x + mt)] \left(\int_{-\pi+x+mt}^{(k+1)x+x} \int_{(k+1)x+x}^{\pi+x+mt} \right) \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau, \end{aligned} \quad (3)$$

where $m = \overline{0, k}$.

Calculate the latter in the sum (3). We have

$$\begin{aligned} &[f(x) - f(x + mt)] \left(\int_{-\pi+x+mt}^{(k+1)x+x} \int_{(k+1)x+x}^{\pi+x+mt} \right) \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau = \\ &= 2[f(x) - f(x + mt)] \ln \left| \sin \frac{\tau - x - mt}{2} \right| \left| \frac{-(k+1)x+x}{\pi+x+mt} + \frac{\pi+x+mt}{(k+1)x+x} \right| = \\ &= 2[f(x) - f(x + mt)] \ln \left| \frac{\sin \frac{(k+1+m)t}{2}}{\sin \frac{(k+1-m)t}{2}} \right|. \end{aligned} \quad (4)$$

Denoting $[-\pi + x, \pi + x]$ by T and taking into account the equalities (3), (4) for $\Delta_t^k(\tilde{f}; x)$ we get

$$\begin{aligned} \pi \Delta_t^k(\tilde{f}; x) &= \sum_{m=0}^k (-1)^m C_k^m \int_{-(k+1)x+x}^{(k+1)x+x} [f(\tau) - f(x + mt)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau + \\ &+ \int_{T \setminus (-(k+1)x+x, (k+1)x+x)} [f(\tau) - f(x)] \sum_{m=0}^k (-1)^m C_k^m \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau + \\ &+ \sum_{m=0}^k (-1)^m C_k^m \left(\int_{\pi+x}^{\pi+x+mt} + \int_{-\pi+x+mt}^{-\pi+x} \right) [f(\tau) - f(x)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau + \\ &+ \sum_{m=0}^k (-1)^m C_k^m 2[f(x) - f(x + mt)] \ln \left| \frac{\sin \frac{k+1+m}{2} t}{\sin \frac{k+1-m}{2} t} \right| = \sum_{i=1}^4 I_i. \end{aligned} \quad (5)$$

Estimate each term of I_i in turn. We get

$$\begin{aligned}
& \int_{-(k+1)t+x}^{(k+1)t+x} [f(\tau) - f(x+mt)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau = \\
& = \int_{-(k+1)t+x}^{(k+1-m)t+x} [f(\tau + mt) - f(x+mt)] \operatorname{ctg} \frac{\tau - x}{2} d\tau \leq \\
& \leq \int_{-(k+1+m)t+x}^{(k+1+m)t+x} \frac{\omega_f^0(|\tau - x|, \max\{|\tau + mt|, |x + mt|\})}{|\tau - x|} d\tau \leq \\
& \leq 2 \int_0^{(k+1+m)t} \frac{\omega_f^0(|\tau - x|, \max\{|\tau + mt|, |x + mt|\})}{|\tau - x|} d\tau = \\
& = 2 \int_0^{(k+1+m)t} \frac{\omega_f^0(y, y + x + mt)}{y} dy \leq 2 \int_0^{(2k+1)t} \frac{\omega_f^0(y, \eta + (3k+1)t)}{y} dy.
\end{aligned}$$

So

$$\begin{aligned}
|I_1| &= \left| \sum_{m=0}^k (-1)^m C_k^m \int_{-(k+1)t+x}^{(k+1)t+x} [f(\tau) - f(x+mt)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau \right| \leq \\
&\leq 2 \sum_{m=0}^k C_k^m \int_0^{(2k+1)t} \frac{\omega_f^0(y, \eta + (3k+1)t)}{y} dy. \tag{6}
\end{aligned}$$

Now consider the next expression

$$\sum_{m=0}^k (-1)^m C_k^m \operatorname{ctg} \frac{\tau - x - mt}{2} \stackrel{\text{def}}{=} \Delta_t^k(\operatorname{ctg}; x).$$

By virtue of the theorem (see [4]) after its k -fold application we see that there exists $\theta \in (0,1)$ at which $\Delta_t^k(\operatorname{ctg}; x) = t^k \operatorname{ctg}^{(k)} \frac{\tau - x - k\theta t}{2}$, $0 < \theta < 1$.

By differentiating k times we get

$$\Delta_t^k(\operatorname{ctg}; x) = t^k \frac{P\left(\sin \frac{\tau - x - \theta kt}{2}, \cos \frac{\tau - x - \theta kt}{2}\right)}{\sin^{k+1} \frac{\tau - x - \theta kt}{2}},$$

where $P\left(\sin \frac{\tau - x - \theta kt}{2}, \cos \frac{\tau - x - \theta kt}{2}\right)$ is a trigonometric polynomial.

Thus

$$|\Delta_t^k(\operatorname{ctg}; x)| \leq \frac{t^k}{|\tau - x|^{k+1}},$$

where C depends only on k .

Now estimate I_2 . We have

$$\begin{aligned}
|I_2| &\leq Ct^k \left(\int_{-\pi+x}^{-(k+1)t+x} + \int_{(k+1)t+x}^{\pi+x} \right) \frac{\omega_f^0(|\tau - x|, \max\{|\tau|, |x|\})}{|\tau - x|^{k+1}} d\tau = \\
&= Ct^k \left(\int_{-\pi}^{-(k+1)t} + \int_{(k+1)t}^{\pi} \right) \frac{\omega_f^0(|y|, \max\{|y + x|, |x|\})}{|y|^{k+1}} dy \leq
\end{aligned}$$

$$\leq 2Ct^k \int_{(k+1)t}^{\pi} \frac{\omega_f^0(y, y+\eta)}{y^{k+1}} dy \leq Ct^k \int_{(k+1)t}^{\pi} \frac{\omega_f^0(y, \eta+y)}{y^{k+1}} dy + 2Ct^k \int_{\eta}^{\pi} \frac{\omega_f^0(y, y+\eta)}{y^{k+1}} dy.$$

We get

$$\begin{aligned} |I_2| &= \left| \int_{T \setminus (-k+1)t+x(k+1)t+x)} [f(\tau) - f(x)] \sum_{m=0}^k (-1)^m C_k^m \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau \right| \leq \\ &\leq Ct^k \left(\int_{\eta}^{\pi} \frac{\omega_f^0(y, \eta+y)}{y^{k+1}} dy + \int_{\eta}^{\pi} \frac{\omega_f^0(y, y+\eta)}{y^{k+1}} dy \right). \end{aligned} \quad (7)$$

Estimate I_3 . We have

$$\begin{aligned} &\int_{-\pi+x+mt}^{-\pi+x} [f(\tau) - f(x)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau \stackrel{\tau = y+x-\pi}{=} \\ &= \int_0^0 [f(y+x-\pi) - f(x)] \operatorname{ctg} \frac{y - \pi - mt}{2} dy = \\ &= \int_0^{mt} [f(\pi+y+x) - f(x)] \operatorname{ctg} \frac{\pi + mt - y}{2} dy, \\ &\int_{\pi+x}^{\pi+x+mt} [f(\tau) - f(x)] \operatorname{ctg} \frac{\tau - x - mt}{2} d\tau \stackrel{\tau = y+x-\pi}{=} \\ &= \int_0^{mt} [f(\pi+x+y) - f(x)] \operatorname{ctg} \frac{\pi - (mt-y)}{2} dy. \end{aligned}$$

Thus

$$|I_3| \leq 2 \sum_{m=0}^k C_k^m \int_0^{mt} |f(\pi+x+y) - f(x)| \left| \operatorname{tg} \frac{mt-y}{2} \right| dy \leq C m^2 t^2 \omega_f^0(\pi+mt, \eta+\pi+mt).$$

Estimate I_4 . We have

$$|I_4| = \sum_{m=0}^k (-1)^m C_k^m 2 [f(x) - f(x+mt)] \ln \left| \frac{\sin \frac{k+1+m}{2} t}{\sin \frac{k+1-m}{2} t} \right| \leq C \omega_f^0(mt, \eta+mt). \quad (8)$$

By summing the estimates (6)-(8) we get the statement of the theorem.

The theorem is proved.

In the next theorem it is proved an estimation being a local analogy of the known Maršo inequality connecting the smoothness modulus of different orders, and analogies of inequalities (see Guseynov E.G. [5]), connecting a local mixed smoothness modulus with a local smoothness modulus are given as corollaries.

Introduce a local mixed smoothness modulus of orders $i, k \geq 1$ of the form

$$\omega_f^{i,k,x_0}(s, \delta, \eta) = \sup_{\substack{|h_1| \leq s, |h_2| \leq \delta \\ x \in O_\eta(x_0)}} |\Delta_{h_1}^i \Delta_{h_2}^k f(x)|, \quad x_0 \in [-\pi, \pi].$$

Theorem 2. 1) Let $f \in C_{2\pi}$, $k, j \in N$, $j = \overline{1, k}$, $h \leq s \leq \pi$, $x_0 \in [-\pi, \pi]$. Then $\forall x \in O_\eta(x_0)$ it is valid the estimation

$$\begin{aligned} |\Delta_h^j(f; x)| &\leq Ch^j \left(\int_{\frac{x}{h}}^{\frac{x+x_0}{h}} \frac{\omega_f^{k+1, x_0} \left(\tau, \eta + \frac{1}{2}(k+j-2)(k-j+1)\tau \right)}{\tau^{j+1}} d\tau + \right. \\ &\quad \left. + \frac{\omega_f^{j, x_0} \left(s, \eta + (k+j-2)(k-j+1)s \right)}{s^j} \right) \leq \\ &\leq Ch^j \left(\int_{\frac{x}{h}}^{\frac{\pi}{h}} \frac{\omega_f^{k+1, x_0} \left(\tau, \eta + \frac{1}{2}(k+j-2)(k-j+1)\tau \right)}{\tau^{j+1}} d\tau + \|f\|_c \right). \end{aligned}$$

2) Let $f \in C_{2\pi}$, i, k, l be natural numbers, where $l > k$. Then for $\delta \leq s$ it is valid the estimation

$$\begin{aligned} \omega_f^{i, k, x_0} (s, \delta, \eta) &\leq C\delta^k \left(\int_{\delta}^s \frac{\omega_f^{l, x_0} \left(\tau, \eta + \frac{l+k-3}{2}(l-k)\tau \right)}{\tau^{k+1}} d\tau + \right. \\ &\quad \left. + \frac{\omega_f^{i, k, x_0} (s, s, \eta + (l+k-3)(l-k)s)}{s^k} \right). \end{aligned}$$

3) Let $f \in C_{2\pi}$, k, l be natural numbers, where $l > k$. Then it is valid the following estimation ($0 < \delta \leq \eta \leq \pi$)

$$\begin{aligned} \omega_f^{k, l, x_0} (\delta, \delta, \eta) &\leq C\delta^{k+1} \left(\int_{\frac{\delta}{\pi}}^{\frac{\pi}{\delta}} \frac{d\tau}{\tau^2} \int_{\delta}^{\pi} \frac{\omega_f^{l, x_0} \left(y, \eta + \left[\frac{1}{2}(l-k)(l-1) + \frac{l+k-3}{2}(l-k) \right] \tau \right)}{y^{k+1}} dy + \right. \\ &\quad \left. + \int_{\delta}^{\pi} \frac{\omega_f^{l, x_0} (\tau, \pi)}{\tau^{k+2}} d\tau + \|f\|_c \right). \end{aligned}$$

Proof of the theorem:

Prove 1. Consider the case $j = k$. The case $j < k$ is established by induction. By the scheme of Marcho theorem (see [4], p.165) we write

$$\begin{aligned} |\Delta_{2h}^k(f; x) - 2^k \Delta_h^k(f; x)| &= \left| \sum_{v=0}^k (-1)^v C_k^v \Delta_h^k [f(x + vh) - f(x)] \right| = \\ &= \left| \sum_{v=1}^k (-1)^v C_k^v \sum_{j=0}^{v-1} \Delta_h^{k+1} f(x + jh) \right|. \end{aligned}$$

Whence provided $x \in O_\eta(x_0)$, we get

$$\begin{aligned} |\Delta_{2h}^k(f; x) - 2^k \Delta_h^k(f; x)| &\leq k 2^{k-1} \omega_f^{k+1, x_0} (h, |x + kh - x_0|) \leq \\ &\leq k 2^{k-1} \omega_f^{k+1, x_0} (h, \eta + (k-1)h). \end{aligned}$$

So at any natural r

$$\begin{aligned} |\Delta_h^k(f; x)| &\leq \frac{k}{2} \omega_f^{k+1, x_0}(h, \eta + (k-1)h) + \frac{1}{2^k} |\Delta_{2h}^k(f; x)| \leq \\ &\leq \frac{k}{2} \omega_f^{k+1, x_0}(h, \eta + (k-1)h) + \frac{\omega_f^{k, x_0}(2h, \eta)}{2^k} \leq \dots \leq \\ &\leq \frac{\omega_f^{k, x_0}(2^r h, \eta)}{2^{kr}} + \frac{k}{2} \sum_{j=0}^{r-1} \frac{\omega_f^{k+1, x_0}(2^j h, \eta + (k-1)2^j h)}{2^{jk}}. \end{aligned}$$

At any entire $j \geq 0$ it holds the inequality

$$\frac{\omega_f^{k+1, x_0}(2^j h, \eta + (k-1)2^j h)}{2k \cdot 2^{jk}} \leq h^k \int_{2^j h}^{2^{j+1} h} \frac{\omega_f^{k+1, x_0}(u, \eta + (k-1)u)}{u^{k+1}} du.$$

By choosing $r = \left\lceil \log_2 \frac{s}{h} \right\rceil + 1$ we get

$$|\Delta_h^k(f; x)| \leq C 2^k h^k \frac{\omega_f^{k+1, x_0}(s, \eta + 2(k-1)s)}{s^k} + k^2 h^k \int_h^s \frac{\omega_f^{k+1, x_0}(u, \eta + (k-1)u)}{u^{k+1}} du.$$

Taking into account $\omega_f^{k, x_0}(s, \eta + 2(k-1)s) \leq 2^k \|f\|_c$, we get

$$|\Delta_h^k(f; x)| \leq k^2 h^k \int_h^\pi \frac{\omega_f^{k+1, x_0}(u, \eta + (k-1)u)}{u^{k+1}} du + Ch^k \|f\|_c.$$

Prove 2. Denote $g(x) = \Delta_h^1 f(x)$. Applying the statement 1) of Theorem 2 to the function $g(x)$ we have

$$\omega_g^{k, x_0}(\delta, \eta) \leq C \delta^k \left(\int_\delta^s \frac{\omega_g^{l, x_0}\left(\tau, \eta + \frac{l+k-3}{2}(l-k)\tau\right)}{\tau^{k+1}} d\tau + \frac{\omega_g^{k, x_0}(s, \eta + (l+k-3)(l-k)s)}{s^k} \right). (*)$$

Taking into account the following relation

$$\begin{aligned} \omega_g^{l, x_0}\left(\tau, \eta + \frac{l+k-3}{2}(l-k)\tau\right) &\leq 2^l \omega_f^{l, x_0}\left(\tau, \eta + \frac{l+k-3}{2}(l-k)\tau\right), \\ \omega_g^{k, x_0}(s, \eta + (l+k-3)(l-k)s) &\leq \omega_f^{k, x_0}(s, s, \eta + (l+k-3)(l-k)s) \end{aligned}$$

we get

$$\begin{aligned} \sup_{|h_j| \leq s} \omega_g^{k, x_0}(\delta, \eta) &= \omega_f^{k, x_0}(\delta, s, \eta) \leq \\ &\leq C \delta^k \left(2^k \int_\delta^s \frac{\omega_f^{l, x_0}\left(\tau, \eta + \frac{l+k-3}{2}(l-k)\tau\right)}{\tau^{k+1}} d\tau + \frac{\omega_f^{k, x_0}(s, s, \eta + (l+k-3)(l-k)s)}{s^k} \right). \end{aligned}$$

Prove 3. By statement 1) of the theorem for $s = \pi$, we have

$$\omega_g^{x_0}(\delta, \eta) \leq C \delta \left(\int_\delta^\pi \frac{\omega_f^{l, x_0}\left(\tau, \eta + \frac{1}{2}(l-2)(l-1)\tau\right)}{\tau^2} d\tau + \frac{1}{\pi} \omega_f^{x_0}(\pi, \eta + (l-2)(l-1)\pi) \right).$$

Taking into account that $\omega_g^{l,x_0}(\tau, \eta + \frac{1}{2}(l-2)(l-1)\tau) \leq \omega_f^{l,k,x_0}(\tau, \delta, \eta + \frac{1}{2}(l-2)(l-1)\tau)$

and applying statement 2) of the theorem we get

$$\begin{aligned} & \omega_f^{l,k,x_0}(\tau, \delta, \eta + \frac{1}{2}(l-2)(l-1)\tau) \leq \\ & \leq C\delta^k \left(\int_{\delta}^{\tau} \frac{\omega_f^{l,x_0}\left(y, \eta + \frac{1}{2}(l-2)(l-1)\tau + \frac{l+k-3}{2}(l-k)y\right)}{y^{k+1}} dy + \right. \\ & \quad \left. + \frac{\omega_f^{l,k,x_0}\left(\tau, \tau, \eta + \left[\frac{1}{2}(l-2)(l-1) + (l+k-3)(l-k)\right]\tau\right)}{\tau^k} \right). \end{aligned}$$

Remark that

$$\begin{aligned} \omega_g^{x_0}(\pi, \eta + (l-2)(l-1)\pi) & \leq 2\omega_f^{k,x_0}(\delta, \eta + (l-2)(l-1)\pi) \text{ and} \\ \omega_f^{l,k,x_0}\left(\tau, \tau, \eta + \left[\frac{1}{2}(l-2)(l-1) + (l+k-3)(l-k)\right]\tau\right) & \leq \\ & \leq 2^k \omega_f^{l,x_0}\left(\tau, \eta + \left[\frac{1}{2}(l-2)(l-1) + (l+k-3)(l-k)\right]\tau\right). \end{aligned}$$

Taking these relations into account we have

$$\begin{aligned} \omega_g^{x_0}(\delta, \eta) & \leq C\delta \left(\int_{\delta}^{\tau} \frac{\omega_f^{l,x_0}\left(y, \eta + \left[\frac{1}{2}(l-2)(l-1) + \frac{l+k-3}{2}(l-k)\right]\tau\right)}{y^{k+1}} dy \right) + \\ & + C\delta \left(\int_{\delta}^{\tau} \frac{\omega_f^{l,x_0}\left(\tau, \eta + \left[\frac{l-2}{2}(l-1) + \frac{l+k-3}{2}(l-k)\right]\tau\right)}{\tau^k} d\tau + \frac{1}{\pi} \omega_f^{k,x_0}(\delta, \eta + (l-2)(l-1)\pi) \right), \\ \omega_f^{k,x_0}(\delta, \eta + (l-2)(l-1)\pi) & \leq C\delta^k \left(\int_{\delta}^{\tau} \frac{\omega_f^{l,x_0}\left(\tau, \eta + (l-2)(l-1) + \frac{l+k-3}{2}(l-k)\tau\right)}{\tau^{k+1}} d\tau + \|f\|_c \right). \end{aligned}$$

Statement 3) is proved.

The theorem is proved completely.

Lemma 1. Let $f \in C_{2\pi}$, $x_0 \in [-\pi, \pi]$, $x \in O_\eta(x_0)$, $\eta \in (0, \pi]$ and be fulfilled the condition

$$\int_0^\pi \frac{\omega_f^{k,l,x_0}(\delta, \tau, \eta)}{\tau} d\tau < +\infty,$$

where k is natural number. Then there exists \tilde{f} and it is valid the following estimation

$$\omega_f^{2k+1, x_0}(\delta, \eta) \leq \left(\frac{\delta \omega_f^{k, 1, x_0}(\delta, \tau, \eta + (3k+4)\delta)}{\tau} d\tau + \delta^{k+1} \int_{\delta}^{\pi} \frac{\omega_f^{k, 1, x_0}(\delta, \tau, \eta + \tau)}{\tau^{k+2}} d\tau \right).$$

Proof. Write (taking into account that $\int_{-\pi+x+mh}^{\pi+x+mh} \operatorname{ctg} \frac{\tau - x - mh}{2} d\tau = 0$)

$$\begin{aligned} \pi \Delta_h^{2k+1} \tilde{f}(x) &= \\ &= \sum_{m=0}^{k+1} (-1)^m C_{k+1}^m \sum_{i=0}^k (-1)^i C_k^i \int_{-\pi+x+ih+mh}^{\pi+x+ih+mh} [f(\tau) - f(x + ih + mh)] \operatorname{ctg} \frac{\tau - x - ih - mh}{2} d\tau = \\ &= \sum_{m=0}^{k+1} (-1)^m C_{k+1}^m \int_{-\pi+x+mh}^{\pi+x+mh} \sum_{i=0}^k (-1)^i C_k^i [f(\tau + ih) - f(x + ih + mh)] \operatorname{ctg} \frac{\tau - x - mh}{2} d\tau. \end{aligned}$$

Taking into account that $\Delta_h^k f(x) = \sum_{i=0}^k (-1)^i C_k^i f(x + ih)$, we have

$$\pi \Delta_h^{2k+1} \tilde{f}(x) = \sum_{m=0}^{k+1} (-1)^m C_{k+1}^m \int_{-\pi+x+mh}^{\pi+x+mh} [\Delta_h^k f(\tau) - \Delta_h^k f(x + mh)] \operatorname{ctg} \frac{\tau - x - mh}{2} d\tau.$$

By Theorem 1 we have

$$\begin{aligned} |\Delta_h^{2k+1} \tilde{f}(x)| &\leq \frac{1}{\pi} \left| \sum_{m=0}^{k+1} (-1)^m C_{k+1}^m \int_{-\pi+x+mh}^{\pi+x+mh} \Delta_h^k [f(\tau) - f(x + mh)] \operatorname{ctg} \frac{\tau - x - mh}{2} d\tau \right| \leq \\ &\leq C \left(\frac{\delta \omega_f^{k, 1, x_0}(\delta, \tau, \eta + (3k+4)\delta)}{\tau} d\tau + \delta^{k+1} \int_{\delta}^{\pi} \frac{\omega_f^{k, 1, x_0}(\delta, \tau, \eta + \tau)}{\tau^{k+2}} d\tau \right). \end{aligned}$$

The lemma is proved.

Corollary 1. Let $f \in C_{2\pi}$, $x_0 \in [-\pi, \pi]$, $x \in O_\eta(x_0)$, $\eta \in (0, \pi]$ and be fulfilled the condition

$$\int_0^{\pi} \frac{\omega_f^{k, x_0}(\tau, \eta)}{\tau} d\tau < +\infty.$$

Then it is valid the estimation

$$\omega_f^{2k+1, x_0}(\delta, \eta) \leq C \left(\frac{\delta \omega_f^{k, x_0}(\tau, \eta + [(3k+4)+(k-2)(k+1)]\delta)}{\tau} d\tau + \delta^{k+1} \int_{\delta}^{\pi} \frac{\omega_f^{k, x_0}(\tau, \eta + \tau)}{\tau^{k+2}} d\tau \right).$$

Proof. Taking into account that

$$\omega_f^{k, 1, x_0}(\delta, \delta, \eta) \leq 2 \omega_f^{k, x_0}(\delta, \eta)$$

by Lemma 1 we have

$$\omega_f^{2k+1, x_0}(\delta, \eta) \leq C \left(\frac{\delta \omega_f^{k, 1, x_0}(\delta, \tau, \eta + (3k+4)\delta)}{\tau} d\tau + \delta^{k+1} \int_{\delta}^{\pi} \frac{\omega_f^{k, 1, x_0}(\delta, \tau, \eta + \tau)}{\tau^{k+2}} d\tau \right).$$

Using estimation 2) from Theorem 2, we have

$$\begin{aligned}
\omega_f^{2k+1,x_0}(\delta, \eta) &\leq C \left(\int_0^\delta \frac{\omega_f^{k,1,x_0}(\delta, \tau, \eta + (3k+4)\delta)}{\tau} d\tau + \delta^{k+1} \int_\delta^\pi \frac{\omega_f^{k,x_0}(\tau, \eta + \tau)}{\tau^{k+2}} d\tau \right) \leq \\
&\leq C \left(\int_0^\delta \frac{d\tau}{\tau} \int_\tau^\delta \frac{\omega_f^{k+1,x_0}(y, \eta + (3k+4)\delta + \frac{k-2}{2}(k-1)\delta)}{y^2} dy + \right. \\
&+ \left. \frac{\delta d\tau}{\tau} \frac{\omega_f^{1,k,x_0}(\delta, \delta, \eta + (3k+4)\delta + (k-2)(k-1)\delta)}{\delta} + \delta^{k+1} \int_\delta^\pi \frac{\omega_f^{k,x_0}(\tau, \eta + \tau)}{\tau^{k+2}} d\tau \right) \leq \\
&\leq C \left(\int_0^\delta \frac{\omega_f^{k+1,x_0}(y, \eta + (3k+4)\delta + (k-2)(k-1)\delta)}{y^2} dy \int_0^y d\tau + \right. \\
&+ \left. \delta^{k+1} \int_\delta^\pi \frac{\omega_f^{k,x_0}(\tau, \eta + \tau)}{\tau^{k+2}} d\tau + \omega_f^{k,x_0}(\delta, \eta + (3k+4)\delta + (k-2)(k-1)\delta) \leq \right. \\
&\leq C \left(\int_0^\delta \frac{\omega_f^{k,x_0}(\tau, \eta + (3k+4)\delta + (k-2)(k-1)\delta)}{\tau} d\tau + \delta^{k+1} \int_\delta^\pi \frac{\omega_f^{k,x_0}(\tau, \eta + \tau)}{\tau^{k+2}} d\tau \right).
\end{aligned}$$

The Corollary is proved.

Corollary 2. Under the conditions of lemma, if $\eta = \pi$ then it is valid the estimation

$$\omega_f^{2k+1}(\delta) \leq C \int_0^\delta \frac{\omega_f^k(\tau)}{\tau} d\tau.$$

Indeed, if $\eta = \pi$ then by lemma 1, we have

$$\begin{aligned}
\omega_f^{2k+1}(\delta) &\leq C \left(\int_0^\delta \frac{\omega_f^{k,1}(\delta, \tau)}{\tau} d\tau + \int_\delta^\pi \frac{\omega_f^{k,1}(\delta, \tau)}{\tau^{k+2}} d\tau \right) \leq \\
&\leq C \left(\int_0^\delta \frac{\omega_f^{k,1}(\delta, \tau)}{\tau} d\tau + \delta^{k+1} \frac{\omega_f^{k,1}(\delta, \delta)}{\delta} \int_\delta^\pi \frac{d\tau}{\tau^{k+1}} \right) \leq C \int_0^\delta \frac{\omega_f^{k,1}(\delta, \tau)}{\tau} d\tau.
\end{aligned}$$

By corollary to Marcho theorem (see [4]), we get

$$\begin{aligned}
\omega_f^{2k+1}(\delta) &\leq C \int_0^\delta \frac{\omega_f^{k,1}(\delta, \tau)}{\tau} d\tau \leq C \int_0^\delta \frac{d\tau}{\tau} \left(\int_\tau^\delta \frac{\omega_f^k(y)}{y^2} dy + \frac{\omega_f^k(\delta, \delta)}{\delta} \right) \leq \\
&\leq C \left(\int_0^\delta \frac{\omega_f^k(y)}{y^2} dy \int_0^y d\tau + \frac{\omega_f^k(\delta)}{\delta} \int_0^\delta d\tau \right) \leq C \int_0^\delta \frac{\omega_f^k(\tau)}{\tau} d\tau.
\end{aligned}$$

Corollary 3. Under the conditions of Corollary 2 it is valid the estimation

$$\omega_f^k(\delta) \leq C \left(\int_0^\delta \frac{\omega_f^k(\tau)}{\tau} d\tau + \delta^k \int_\delta^\pi \frac{\omega_f^k(\tau)}{\tau^{k+1}} d\tau + \delta^k \|f\|_c \right).$$

Indeed,

$$\begin{aligned}\omega_{\tilde{f}}^k(\delta) &\leq C\delta^k \left(\int_{-\delta}^{\pi} \frac{\omega_{\tilde{f}}^{2k+1}(\tau)}{\tau^{k+1}} d\tau + \|\tilde{f}\|_c \right) \leq C\delta^k \left(\int_{-\delta}^{\pi} \frac{d\tau}{\tau^{k+1}} \int_0^\tau \frac{\omega_f^k(y)}{y} dy + \|\tilde{f}\|_c \right) \leq \\ &\leq C \left(\int_0^{\delta} \frac{\omega_f^k(\tau)}{\tau} d\tau + \delta^k \int_{-\delta}^{\pi} \frac{\omega_f^k(\tau)}{\tau^{k+1}} d\tau + \delta^k \|\tilde{f}\|_c \right).\end{aligned}$$

Theorem 3. Let $f \in C_{2\pi}$, $x_0 \in [-\pi, \pi]$, $\eta \in (0, \pi]$, and be fulfilled the condition

$$\int_0^{\pi} \frac{\omega_f^{k, x_0}(\tau, \eta)}{\tau} d\tau < +\infty.$$

Then it is valid the following estimation

$$\begin{aligned}\omega_{\tilde{f}}^{k, x_0}(\delta, \eta) &\leq C\delta^k \left(\int_{-\delta}^{\pi} \frac{d\tau}{\tau^{k+1}} \int_0^\tau \frac{\omega_f^{k, x_0}(y, \eta + (3k+2)(k+1)\tau)}{y} dy + \right. \\ &\quad \left. + \delta^k \int_{-\delta}^{\pi} \frac{\omega_f^{k, x_0}(y, \eta + \frac{1}{2}[(3k-2)(k+1)+2]y)}{y^{k+1}} dy + \delta^k \int_0^{\pi} \frac{\omega_f^{k, x_0}(y, \pi)}{y} dy + \delta^k \|f\|_c \right).\end{aligned}$$

where $C > 0$ does not depend on δ and η .

Proof. By Corollary 1 to Lemma 1 and by Theorem 1 we write

$$\begin{aligned}\omega_{\tilde{f}}^{k, x_0}(\delta, \eta) &\leq C\delta^k \left(\int_{-\delta}^{\pi} \frac{\omega_{\tilde{f}}^{2k+1, x_0}(\tau, \eta + \frac{1}{2}[(3k-2)(k+1)+2]\tau)}{\tau^{k+1}} d\tau + \|f\|_c \right) \leq \\ &\leq C \left(\delta^k \int_{-\delta}^{\pi} \frac{d\tau}{\tau^{k+1}} \int_0^\tau \frac{\omega_f^{k, x_0}(y, \eta + (3k+2)(k+1)\tau)}{y} dy + \right. \\ &\quad \left. + \delta^k \int_{-\delta}^{\pi} \frac{d\tau}{\tau^{k+1}} \int_\tau^\pi \frac{\omega_f^{k, x_0}(y, \eta + \frac{1}{2}(3k-2)(k+1)\tau + y)}{y^{k+2}} dy + \delta^k \|\tilde{f}\|_c \right).\end{aligned}$$

Taking into account that

$$\begin{aligned}|\tilde{f}(x)| &= \left| \int_{-\pi}^{\pi} (f(\tau) - f(x)) \operatorname{ctg} \frac{\tau - x}{2} d\tau \right| \leq C \int_0^{\pi} \frac{\omega_f^{x_0}(\tau, \pi)}{\tau} d\tau \leq \\ &\leq C \left(\int_0^{\pi} \frac{d\tau}{\tau} \tau \int_{-\pi}^{\pi} \frac{\omega_f^{x_0}(y, \pi)}{y^2} dy + \|f\|_c \right) = C \left(\int_0^{\pi} \frac{\omega_f^{x_0}(y, \pi)}{y^2} dy \int_0^y d\tau + \|f\|_c \right)\end{aligned}$$

we have

$$\|\tilde{f}\|_c \leq \left(\int_0^{\pi} \frac{\omega_f^{x_0}(\tau, \pi)}{\tau} d\tau + \|f\|_c \right).$$

The theorem is proved.

Example. Let $\omega_f^{k,x_0}(\delta, \eta) \sim \delta^\alpha \eta^\beta$, $\alpha + \beta \leq k$. Show that $\omega_{\tilde{f}}^{k,x_0}(\delta, \eta) \leq C \delta^\alpha \eta^\beta$.

Really,

$$\delta^k \int_{\delta}^{\eta} \frac{d\tau}{\tau^{k+1}} \int_0^{\tau} \frac{y^\alpha \eta^\beta}{y} dy \leq C \delta^k \eta^\beta \int_{\delta}^{\eta} \frac{\tau^\alpha}{\tau^{k+1}} d\tau \leq C \delta^k \eta^\beta \frac{1}{\delta^{k-\alpha}} = C \delta^\alpha \eta^\beta,$$

$$\delta^k \int_{\delta}^{\eta} \frac{d\tau}{\tau^{k+1}} \int_0^{\tau} \frac{y^\alpha \tau^\beta}{y} dy \leq C \delta^k \int_{\eta}^{\pi} \frac{\tau^{\alpha+\beta}}{\tau^{k+1}} d\tau \leq C \delta^k \frac{1}{\eta^{k-\alpha-\beta}} \leq C \delta^\alpha \eta^\beta,$$

$$\delta^k \int_{\delta}^{\eta} \frac{y^\alpha \eta^\beta}{y^{k+1}} dy \leq C \delta^k \eta^\beta \frac{1}{\delta^{k-\alpha}} \leq C \delta^\alpha \eta^\beta,$$

$$\delta^k \int_{\eta}^{\pi} \frac{y^{\alpha+\beta}}{y^{k+1}} dy \leq C \delta^k \frac{1}{\eta^{k-\alpha-\beta}} \leq C \delta^\alpha \eta^\beta.$$

Let $h(\delta, \eta)$ be a k -th order local smoothness modulus type function. Denote

$$H_h^{k,x_0} = \left\{ f \in C_{2\pi}; \omega_f^{k,x_0}(\delta, \eta) = O(h(\delta, \eta)), 0 < \delta \leq \eta \leq \pi \right\}.$$

Introduce the norm

$$\|f\|_{H_h^{k,x_0}} = \|f\|_c + \sup_{0 < \delta \leq \eta \leq \pi} \frac{\omega_f^{k,x_0}(\delta, \eta)}{h(\delta, \eta)}.$$

Theorem 4. Let $h(\delta, \eta)$ be a k -th order local smoothness modulus type function and be fulfilled the condition

$$\begin{aligned} & \int_0^{\delta} \frac{h(\tau, \eta + (3k+2)(k+1)\delta)}{\tau} d\tau + \\ & + \delta^k \int_{\delta}^{\pi} \frac{h\left(\tau, \eta + \frac{1}{2}[(3k-2)(k+1)+2]\tau\right)}{\tau^{k+1}} d\tau = O(h(\delta, \eta)). \end{aligned}$$

Then, if $f \in H_h^{k,x_0}$ then $\tilde{f} \in H_h^{k,x_0}$ and $\|\tilde{f}\|_{H_h^{k,x_0}} \leq C \|f\|_{H_h^{k,x_0}}$.

Proof. Let $f \in H_h^{k,x_0}$. Then by definition of the norm $\omega_f^{k,x_0}(\delta, \eta) \leq \|f\|_{H_h^{k,x_0}} \cdot h(\delta, \eta)$, $0 < \delta \leq \eta \leq \pi$. It follows from the condition of the theorem that

$$\int_0^{\eta} \frac{h(\tau, \eta + (3k+2)(k+1)\tau)}{y} dy = O(h(\delta, \eta)).$$

Whence we get the following relation

$$\delta^k \int_{\delta}^{\pi} \frac{d\tau}{\tau^{k+1}} \int_0^{\tau} \frac{h(y, \eta + (3k+2)(k+1)\tau)}{y} dy = O(h(\delta, \eta)).$$

Thus, by theorem 3, we have

$$\begin{aligned} \omega_{\tilde{f}}^{k,x_0}(\delta, \eta) & \leq C \|f\|_{H_h^{k,x_0}} (\delta^k \int_{\delta}^{\pi} \frac{d\tau}{\tau^{k+1}} \int_0^{\tau} \frac{h(y, \eta + (3k+2)(k+1)\tau)}{y} dy + \\ & + \delta^k \int_0^{\pi} \frac{h\left(y, \eta + \frac{1}{2}[(3k-2)(k+1)+2]y\right)}{y^{k+1}} dy + \delta^k \int_0^{\pi} \frac{h(\tau, \pi)}{\tau} d\tau + \delta^k) \leq C \|f\|_{H_h^{k,x_0}} \cdot h(\delta, \eta). \end{aligned}$$

Now prove that \tilde{f} is bounded. It is obvious (Theorem 3) that

$$\|\tilde{f}\|_C \leq C \left(\int_0^\pi \frac{\omega_f^{k,x_0}(\tau, \pi)}{\tau} d\tau + \|f\|_C \right).$$

$$\text{So } \|\tilde{f}\|_{H_h^{k,x_0}} = \|\tilde{f}\|_C + \sup_{0 < \delta \leq \eta \leq \pi} \frac{\omega_f^{k,x_0}(\delta, \eta)}{h(\delta, \eta)} \leq C \|f\|_{H_h^{k,x_0}}.$$

The theorem is proved.

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Hasanov S.G.

Gandja Pedagogical Institute named after H.Zardabi.
187, Shah Ismayil Khataii str., Gandja, Azerbaijan.

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