2000 VOL. XIII(XXI)

JAFAROV N.J.

UNIQUE WEAK SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR A HILBARG-SERRIN PARABOLIC EQUATION IN NON-CYLINDRIC DOMAINS

Abstract

The first boundary value problem for a Hilbarg-Serrin parabolic equation is considered in the paper. Its unique weak solvability in corresponding weight spaces of Sobolev is established.

Introduction. Let G be a bounded n-dimensional domain containing the origin coordinates. Consider in G the first boundary-value problem for the Hilbarg-Serrin equation

$$\begin{cases} \Delta u + \lambda \sum\limits_{i,j=1}^{n} \dfrac{x_i x_j}{|x|^2} \dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x), \quad x \in G \\ u\big|_{\partial G} = 0, \end{cases} \tag{0.1}$$

where $\lambda > -1$ is a constant. The problems of a weak and strong solvability of the problem (0.1) in corresponding weight spaces of Sobolev have been studied in [1-3]. The goal of this work is the obtaining of analogies of these results concerning the weak solvability for the case of Hilbarg-Serrin parabolic equation in so-called P-domains. Note that in the general case, Hilbarg-Serrin parabolic equation doesn't satisfy the parabolic condition by Cardes [4]. As to the problem of solvability of boundary value problems for general parabolic equations of second order, we shall indicate them in monograph [5-6].

1^0. Notations, definitions and subsidiary statements.

Mention the notations and definitions used in this paper.

Let E_n be an n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$, D be a bounded domain in E_n with boundary ∂D, moreover $0 \in D$. By R_{n+1} we denote $(n+1)$-dimensional Euclidean space of points $(x,t) = (x_1, \ldots, x_n, t)$,

$$R_{n+1}^{-} = R_{n+1} \cap \{(x,t) : t < 0\}.$$

We call $Q \subset R_{n+1}^{-}$ the domain of paraboloid type (or P-domain), ([7]) if its section with any hyper-plane of $t = \tau$ $(\tau < 0)$ has the form:

$$\left\{ x : \frac{x}{2\sqrt{-\tau}} \in D \right\}.$$

Let

$$Q_T = Q \cap \{(x,t) : -T < t < 0\}, \quad S_T = \partial Q \cap \{(x,t) : -T < t < 0\},$$
$$D_T = Q \cap \{(x,t) : t = -T\},$$

$\Gamma(Q_T)$ is a parabolic boundary of the domain Q_T ([8]).

Consider a parabolic operator with coefficient determined on Q_T

$$L \equiv \Delta + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}; \tag{1.1}$$

where Δ is a Laplace operator, λ is a numerical parameter that satisfies the condition

$$-\frac{1}{d^2}<\lambda<\infty, \tag{1.2}$$
$$d=\sup_{y\in D}|y|.$$

Note that the condition (1.2) is non other than condition of uniform parabolicity of the operator L on the domain Q_T.

By the analogy with elliptic case the operator L we shall call Hilberg-Serrin operator.

The symbols u_i, u_{ij} everywhere denote the derivatives

$$\frac{\partial u}{\partial x_i} \text{ and } \frac{\partial^2 u}{\partial x_i \partial x_j},\ u_x=(u_1,...,u_n);$$

$$u_{xx}=(u_{ij});\ i,j=\overline{1,n};\ u_x^2=\sum_{i=1}^n u_i^2\ ;\ u_{xx}^2=\sum_{i,j=1}^n u_{ij}^2\ .$$

Let a numerical number γ satisfy the condition:

$$\gamma\in\left(\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8};+\infty\right), \tag{1.3}$$

$C_0^\infty(Q_T)$ is the space of all infinitely differentiable functions with a compact support in Q_T. $A_0^\infty(Q_T)$ is a space of infinitely differentiable finite functions in Q_T, for which the integral

$$\int_{Q_T}(-t)^{\gamma^{-1}}u^2dxdt$$

is finite. $L_{2,\gamma}(Q_T)$ is a class of measurable functions, $u(x,t)$ given in Q_T with a finite norm

$$\|u\|_{L_{2,\gamma}(Q_T)}=\left(\int_{Q_T}(-t)^\gamma u^2dxdt\right)^{1/2},$$

$W_{2,\gamma}^{1,0}(Q_T)$ is a class of measurable functions $u(x,t)$, given in Q_T with a finite norm

$$\|u\|_{W_{2,\gamma}^{1,0}(Q_T)}=\left(\int_{Q_T}(-t)^\gamma\left(u^2+u_x^2\right)dxdt\right)^{1/2},$$

$\overset{\circ}{W}{}_{2,\gamma}^{1,0}(Q_T)$ is a closed subspace of $W_{2,\gamma}^{1,0}(Q_T)$, where $A_0^\infty(Q_T)$ is a dense set.

$W_{2,\gamma}^{1,1}(Q_T)$ is a class of measurable functions of $u(x,t)$, given in Q_T with a finite norm

$$\|u\|_{W_{2,\gamma}^{1,1}(Q_T)}=\left(\int_{Q_T}(-t)^\gamma\left(u^2+u_x^2+u_t^2\right)dxdt\right)^{1/2},$$

$\overset{\circ}{W}{}_{2,\gamma}^{1,1}(Q_T)$ is a closed subspace of $W_{2,\gamma}^{1,1}(Q_T)$, where $A_0^\infty(Q_T)$ is a dense set.

In the domain Q_T consider the first boundary value problem

$$\begin{cases} Lu = \Delta u + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_k}, \\ u\big|_{\Gamma(Q_T)} = 0 \end{cases} \tag{1.4}$$

where $f \in L_{2,\gamma}(Q_T)$, $f^k \in L_{2,\gamma}(Q_T)$, $k = \overline{1,n}$.

Definition. *Under the weak solution of Hilbarg-Serrin equation with a right hand side* $f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_k}$ *we shall understand such a function* $u(x,t) \in W_{2,\gamma}^{1,0}(Q_T)$ *that satisfies the integral identity*

$$\int_{Q_T} (-t)^\gamma u \vartheta_t dxdt - \int_{Q_T} (-t)^\gamma \sum_{i,j=1}^{n} \left(\delta_{ij} + \lambda \frac{x_i x_j}{4(-t)} \right) \vartheta_i u_j dxdt + \lambda(n+1) \int_{Q_T} (-t)^\gamma \sum_{i=1}^{n} \frac{x_i}{4(-t)} \times$$
$$\times u \vartheta_i dxdt + \frac{\lambda n(n+1)}{4} \int_{Q_T} (-t)^{\gamma-1} u \vartheta dxdt - \gamma \int_{Q_T} (-t)^{\gamma-1} \vartheta u dxdt = \int_{Q_T} (-t)^\gamma f \vartheta dxdt -$$
$$- \int_{Q_T} (-t)^\gamma \sum_{k=1}^{n} f^k \vartheta_k dxdt \tag{1.5}$$

for any $\vartheta(x,t) \in \overset{\circ}{W}{}_{2,\gamma}^{1,1}(Q_T)$.

Clarify briefly the derivation principle of (1.5). For this, it is necessary to cut off the domain Q_T from above by a hyperplane $t = -\varepsilon$ for sufficiently small positive ε and write the Hilbarg-Serrin operator in the form of sum of a divergent part and small terms. Then, the equation is considered in the layer $Q_{T,\varepsilon} = Q_T \setminus \overline{Q}_\varepsilon$ and both parts of the equation are multiplied by the function $(-t)^\gamma \cdot \vartheta(x,t)$, where $\vartheta \in A_0^\infty(Q_T)$ and they vanish near the parabolic boundary $\Gamma(Q_{T,\varepsilon})$. Integrating both sides of the equation with respect to $Q_{T,\varepsilon}$ and using Ostrogodsky's formula we are led to expressions that uniformly depend on ε.

Tending ε to zero we derive the integral identity (1.5).

Fridrichs type inequality. *Let* Q_T *be a domain described above and* $u(x,t) \in \overset{\circ}{W}{}_{2,\gamma}^{1,0}(Q_T)$. *Then it is valid inequality*

$$\int_{Q_T} (-t)^\gamma u^2(x,t) dxdt \le C \cdot \int_{Q_T} (-t)^\gamma u_x^2(x,t) dxdt, \tag{1.6}$$

where a constant $C > 0$ *-depends only on the domain* Q_T.

Proof. Since the domain Q_T is bounded, then there exists a parallelepiped $K = \{(x,t): -R \le x_i \le R, -T \le t \le 0\}$ inside of which we can arrange Q_T. Let $u(x,t) \in A_0^\infty(Q_T)$. Continue the function $u(x,t)$ by zero in K.

We have

$$u(t,x_1,x') = u(t,-R,x') + \int_{-R}^{x_1} u_1(t,y,x') dy = \int_{-R}^{x_1} u_1(t,y,x') dy,$$

where $x' = (x_2,\dots,x_n)$.

Thus,

$$u^2(t,x_1,x')=\left(\int_{-R}^{x_1}u_1(t,y,x')dy\right)^2\le 2R\int_{-R}^{x_1}u_1^2dy. \tag{1.7}$$

Multiply both sides of the latter inequality by $(-t)^\gamma$ and integrate with respect to the domain K. Since in $K\setminus Q_T$ $u\equiv 0$, we get

$$\int_{Q_T}(-t)^\gamma u^2dxdt\le 2R\int_{Q_T}(-t)^\gamma\int_{-R}^{x_1}u_1^2(t,y,x')dydxdt\le 2R\int_{-R}^{R}\cdots\int_{-R}^{R}\int_{-T}^{0}(-t)^\gamma\int_{-R}^{R}u_1^2(t,y,x')dydxdt=$$
$$=2R\int_{-R}^{R}dy\int_{Q_T}(-t)^\gamma u_1^2dxdt=4R^2\int_{Q_T}(-t)^\gamma u_1^2dxdt \tag{1.8}$$

(1.6) is obtained from (1.8) with the help of the passage to the limit.

2^0. Main a priori estimate.

By deriving the main a priori estimate we use the same scheme as in deriving the main integral identity.

Let $Q_{T,\varepsilon}$ have the same meaning that above, but $A_p^\infty(Q_{T,\varepsilon})$ is a totality of all functions from $A_0^\infty(Q_T)$, vanishing near the parabolic boundary $\Gamma(Q_{T,\varepsilon})$. It is easy to see

$$Lu=\Delta u+\lambda\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{4(-t)}u_j\right)_i-\lambda(n+1)\sum_{j=1}^{n}\frac{x_j}{4(-t)}u_j-u_t. \tag{2.1}$$

For any function $u(x,t)\in A_p^\infty(Q_{T,\varepsilon})$ we have

$$-\int_{Q_{T,\varepsilon}}(-t)^\gamma uLudxdt=-\int_{Q_{T,\varepsilon}}(-t)^\gamma u\Delta udxdt-\lambda\sum_{i,j=1}^{n}\int_{Q_{T,\varepsilon}}(-t)^\gamma u\left(\frac{x_ix_j}{4(-t)}u_j\right)_i dxdt+$$
$$+\lambda\cdot(n+1)\sum_{j=1}^{n}\int_{Q_{T,\varepsilon}}(-t)^\gamma\frac{x_j}{4(-t)}u_j\cdot udxdt+\int_{Q_{T,\varepsilon}}(-t)^\gamma u\cdot u_tdxdt=J_{1,\varepsilon}+J_{2,\varepsilon}+J_{3,\varepsilon}+J_{4,\varepsilon}. \tag{2.2}$$

On the other hand,

$$J_{1,\varepsilon}=-\int_{Q_{T,\varepsilon}}(-t)^\gamma u\Delta udxdt=-\int_{Q_{T,\varepsilon}}(-t)^\gamma\sum_{i=1}^{n}u\cdot u_{ii}dxdt=\int_{Q_{T,\varepsilon}}(-t)^\gamma\sum_{i=1}^{n}u_i^2dxdt=$$
$$=\int_{Q_{T,\varepsilon}}(-t)^\gamma u_x^2dxdt; \tag{2.3}$$

$$J_{2,\varepsilon}=-\lambda\sum_{i,j=1}^{n}\int_{Q_{T,\varepsilon}}(-t)^\gamma u\left(\frac{x_ix_j}{4(-t)}u_j\right)_i dxdt=\lambda\sum_{i,j=1}^{n}\int_{Q_{T,\varepsilon}}(-t)^\gamma\frac{x_ix_j}{4(-t)}u_iu_jdxdt; \tag{2.4}$$

$$J_{3,\varepsilon}=\lambda(n+1)\sum_{j=1}^{n}\int_{Q_{T,\varepsilon}}(-t)^\gamma\frac{x_j}{4(-t)}u_ju\,dxdt=\frac{\lambda(n+1)}{2}\sum_{j=1}^{n}\int_{Q_{T,\varepsilon}}(-t)^\gamma\frac{x_j}{4(-t)}\left(u^2\right)_jdxdt=$$
$$=\frac{-\lambda n(n+1)}{2}\int_{Q_{T,\varepsilon}}(-t)^\gamma\frac{u^2}{4(-t)}dxdt; \tag{2.5}$$

$$J_{4,\varepsilon}=\int_{Q_{T,\varepsilon}}(-t)^\gamma u\,u_tdxdt=\frac{1}{2}\int_{Q_{T,\varepsilon}}(-t)^\gamma\left(u^2\right)_tdxdt. \tag{2.6}$$

Clarify briefly how we can transform (2.6). We have

$$\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma}\left(u^2\right)_t dxdt=\int\limits_{Q_{T,\varepsilon}}\frac{d}{dt}\left((-t)^{\gamma}u^2\right)dxdt-\int\limits_{Q_{T,\varepsilon}}\left[(-t)^{\gamma}\right]_t\cdot u^2dxdt. \qquad (2.7)$$

Estimate the integral $\int\limits_{Q_{T,\varepsilon}}\frac{d}{dt}\left((-t)^{\gamma}u^2\right)dxdt$.

To this end, arrange the layer $Q_{T,\varepsilon}$ in a parallelepiped $D_T\times(-T,-\varepsilon)$ and continue the function $(-t)^{\gamma}u^2$ by zero to this parallelepiped. Denote a new function by $F(t,x)$. Then

$$\left|\int\limits_{Q_{T,\varepsilon}}\frac{d}{dt}\left((-t)^{\gamma}u^2\right)dxdt\right|=\left|\int\limits_{D_T\times(-T,\varepsilon)}\frac{d}{dt}F(t,x)dxdt\right|=\left|\int\limits_{-T}^{-\varepsilon}\int\limits_{D_T}\frac{d}{dt}F(t,x)dxdt\right|=$$

$$=\left|\int\limits_{P_\varepsilon}F(-\varepsilon,x)dx-\int\limits_{D_T}F(-T,x)dx\right|,$$

where P_ε is an upper edge of the parallelepiped and $P_\varepsilon\supset D_\varepsilon$.

We see from the construction of $F(t,x)$ that

$$\int\limits_{D_T}F(-T,x)dx=0,$$

$$\left|\int\limits_{P_\varepsilon}F(-\varepsilon,x)dx\right|=\left|\int\limits_{D_\varepsilon}(-\varepsilon)^{\gamma}\left(u^2\right)\Big|_{t=-\varepsilon}dx\right|\le\left|(-\varepsilon)^{\gamma}\right|\cdot\sup_{x\in D_\varepsilon}\left|u^2\right|\cdot mes_nD_\varepsilon=o(\varepsilon),\ \varepsilon\to0+.$$

After transforming (2.7) we finally arrive at the expression

$$J_{4,\varepsilon}=\frac{\gamma}{2}\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma-1}u^2dxdt+o(\varepsilon),\ \varepsilon\to0+. \qquad (2.8)$$

In (2.2) consider (2.3), (2.4), (2.5), (2.8). We get

$$-\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma}uLudxdt=\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma}u_x^2dxdt+\lambda\sum_{i,j=1}^{n}\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma}\frac{x_ix_j}{4(-t)}u_iu_jdxdt-$$

$$-\frac{\lambda n(n+1)}{2}\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt+\frac{\gamma}{2}\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma-1}u^2dxdt+o(\varepsilon),\ \varepsilon\to0+. \qquad (2.9)$$

Pass to the limit for $\varepsilon\to0+$ and write the obtained expression in the form

$$\int\limits_{Q_{T,\varepsilon}}(-t)^{\gamma}\left(|\nabla u|^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\right)dxdt=\frac{\lambda n(n+1)-4\gamma}{2}\int\limits_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt-$$

$$-\int\limits_{Q_T}(-t)^{\gamma}uLudxdt. \qquad (2.10)$$

We see from (2.10) that for

$$\gamma\ge\frac{\lambda n(n+1)}{4}, \qquad (2.11)$$

$$\int\limits_{Q_T}(-t)^{\gamma}\left(u_x^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\right)dxdt\le-\int\limits_{Q_T}(-t)^{\gamma}uLudxdt. \qquad (2.12)$$

For $\frac{\lambda n(n+1)-4\gamma}{2}>0$ for any $\varepsilon_1>0$ we have

$$-\sum_{i=1}^{n}\int_{Q_T}(-t)^{\gamma}\frac{1}{2\sqrt{-t}}\cdot\frac{x_i}{2\sqrt{-t}}u\cdot u_i\,dxdt\le\frac{\varepsilon_1}{2}\int_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt+$$
$$+\frac{1}{2\varepsilon_1}\int_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\,dxdt. \qquad (2.13)$$

On the other hand

$$-\sum_{i=1}^{n}\int_{Q_T}(-t)^{\gamma}\frac{1}{2\sqrt{-t}}\cdot\frac{x_i}{2\sqrt{-t}}u\cdot u_i\,dxdt=-\frac{1}{2}\sum_{i=1}^{n}\int_{Q_T}(-t)^{\gamma}\frac{x_i}{4(-t)}\left(u^2\right)_i dxdt=$$
$$=\frac{n}{2}\int_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt. \qquad (2.14)$$

We get from (2.13) and (2.14)

$$\frac{n-\varepsilon_1}{2}\int_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt\le\frac{1}{2\varepsilon_1}\int_{Q_t}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\,dxdt. \qquad (2.15)$$

For $\varepsilon_1=\frac{n}{2}$, from (2.15) it follows

$$\int_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt\le\frac{4}{n^2}\int_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\,dxdt. \qquad (2.16)$$

Taking (2.16) in (2.10) we conclude

$$\int_{Q_T}(-t)^{\gamma}\left(u_x^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\right)dxdt\le\frac{2\lambda n(n+1)-8\gamma}{n^2}\times$$
$$\times\int_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\,dxdt-\int_{Q_T}(-t)^{\gamma}uLu\,dxdt. \qquad (2.17)$$

Solve the inequality

$$\frac{2\lambda n(n+1)-8\gamma}{n^2}<\frac{1}{d^2}+\lambda,$$

we obtain

$$\gamma>\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8}=\gamma_1. \qquad (2.18)$$

On the other hand $\gamma<\frac{\lambda n(n+1)}{4}=\gamma_2$. Hence we see that for $\lambda\ge-\frac{1}{d^2}$ $\gamma_1<\gamma_2$. Return to (2.12). As we had shown above (2.12) is satisfied, when

$$\gamma\ge\frac{\lambda n(n+1)}{4}.$$

But

$$\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j=\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2. \qquad (2.19)$$

If $\lambda\ge0$, then

$$\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2\ge0. \qquad (2.20)$$

But if

$$-\frac{1}{d^2}<\lambda<0,$$

then

$$\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2 \geq \lambda d^2 u_x^2. \qquad (2.21)$$

We get from (2.20) and (2.21) that there exists $\varepsilon_2>0$ such that

$$\varepsilon_2 \int_{Q_T}(-t)^\gamma u_x^2 dxdt \leq -\int_{Q_T}(-t)^\gamma u\,Ludxdt. \qquad (2.22)$$

Return to the case when $\gamma \in \left(\dfrac{n^2\left(\lambda-\dfrac{1}{d^2}\right)+2\lambda n}{8};\dfrac{\lambda n(n+1)}{4}\right)$. As we had shown above in this case (2.17) is fulfilled.

Then there exists such $\varepsilon_3>0$ that $\dfrac{2\lambda n(n+1)-8\gamma}{n^2}<\dfrac{1}{d^2}+\lambda-\dfrac{\varepsilon_3}{d^2}$, $0<\varepsilon_3<1$.
Then we obtain from (2.17)

$$\int_{Q_T}(-t)^\gamma\left(u_x^2+\left(\frac{\varepsilon_3}{d^2}-\frac{1}{d^2}\right)\sum_{i,j=1}^{n}\frac{x_i x_j}{4(-t)}u_i u_j\right)dxdt \leq -\int_{Q_T}(-t)^\gamma uLudxdt. \qquad (2.23)$$

We get from (2.17) and Fridrichs type inequality that there exists such a positive constant ε_4 that

$$\varepsilon_4 \int_{Q_T}(-t)^\gamma u^2 dxdt \leq -\int_{Q_T}(-t)^\gamma uLudxdt. \qquad (2.24)$$

Thus, we arrive at the main a priori estimate which we formulate in the form of theorem.

Theorem 2.1. *Let Q_T - P be a domain, D be its basis, moreover $O\in D$ and $\gamma \in \left(\dfrac{n^2\left(\lambda-\dfrac{1}{d^2}\right)+2\lambda n}{8};+\infty\right)$. Then for the Hilbarg-Serrin operator on Q_T and for any function $u(x,t)\in \overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$ there exists such a constant $C>0$ not depending on the choice $u\in \overset{\circ}{W}{}^{1}_{2,\gamma}(Q_T)$ that it is valid the inequality*

$$\|u\|_{W^1_{2,\gamma}(Q_T)} \leq C\|Lu\|_{L_{2,\gamma}(Q_T)}. \qquad (2.25)$$

3^0. Unique solvability of the first boundary value problem.

Theorem 3.1. *Let $\gamma \in \left(\dfrac{n^2\left(\lambda-\dfrac{1}{d^2}\right)+2\lambda n}{8};\infty\right)$. Then the problem (1.4) is unique solvable in the space $\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$.*

Proof. First we prove the existence of the solution. Assume again $Q_{T,\varepsilon} = Q_T \setminus \overline{Q}_\varepsilon$, $\varepsilon \in (0,T)$. Let $D_h \subset D$, $D_h \to D$ for $h \to 0+$ and domains D_h have sufficiently smooth boundaries. Moreover, we smooth functions f^k, $k=\overline{1,n}$ and f which are in right-hand side of (14). We denote the smoothed functions by $f^{k,\mu}$ and f^μ, where $\mu > 0$.

Let Q_T - P be a domain with basis D_n, $Q^h_{T,\varepsilon} = Q^h_T \setminus \overline{Q}^h_\varepsilon$, settle $\varepsilon > 0$ and $h > 0$, $\mu > 0$. Consider the problem (1.4) in the domain $Q^h_{T,\varepsilon}$. It is known that this problem has a unique solution $u^{h,\mu}_\varepsilon \in C^\infty\left(\overline{Q}^h_{T,\varepsilon}\right)$. We have

$$\Delta u + \lambda \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i x_j}{4(-t)} u_j\right)_t - \lambda(n+1)\sum_{i=1}^{n} \frac{x_i}{4(-t)} u_i - u_t = f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_k}, \tag{3.1}$$

where we omit the indices $u^{h,\mu}_\varepsilon$ for the reduction of notation.

Both sides of (3.1) we multiply by $(-t)^\gamma u(t,x)$ and integrate with respect to the domain $Q^h_{T,\varepsilon}$. We get

$$\int_{Q^h_{T,\varepsilon}} (-t)^\gamma \Delta u \cdot u dx dt + \lambda \int_{Q^h_{T,\varepsilon}} (-t)^\gamma u \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i x_j}{4(-t)} u_j\right) dx dt - \lambda(n+1) \times$$

$$\times \int_{Q^h_{T,\varepsilon}} (-t)^\gamma \sum_{i=1}^{n} \frac{x_i}{4(-t)} u_i u \, dx dt - \int_{Q^h_{T,\varepsilon}} (-t)^\gamma u \cdot u_t dx dt = \int_{Q^h_{T,\varepsilon}} (-t)^\gamma f \cdot u dx dt +$$

$$+ \int_{Q^h_{T,\varepsilon}} (-t)^\gamma \sum_{k=1}^{n} \frac{\partial f^k}{\partial x^k} u \, dx dt. \tag{3.2}$$

Further we have

$$\int_{Q^h_{T,\varepsilon}} (-t)^\gamma \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j\right) dx dt - \frac{\lambda(n+1) - 4\gamma}{2} \times$$

$$\times \int_{Q^h_{T,\varepsilon}} (-t)^\gamma \cdot \frac{u^2}{4(-t)} dx dt + o(\varepsilon) = \int_{Q^h_{T,\varepsilon}} (-t)^\gamma \sum_{k=1}^{n} f^k \cdot u_k dx dt - \int_{Q^h_{T,\varepsilon}} (-t)^\gamma f \cdot u dx dt, \ \varepsilon \to 0+. \tag{3.3}$$

Settle $h > 0$, $\mu > 0$ and pass to the limit for $\varepsilon \to 0+$. Denote a limit function by $u^{h,\mu}(t,x)$.

1) Let $\dfrac{\lambda n(n+1) - 4\gamma}{2} \le 0$, i.e. $\gamma \ge \dfrac{\lambda n(n+1)}{2}$. Then, the second integral at the hand side of (3.3) is positive. Besides, if $\lambda \ge 0$ then

$$\lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \ge 0.$$

But if $-\dfrac{1}{d^2} < \lambda < 0$, then

$$\lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \ge \lambda \, d^2 u_x^2.$$

2) Let $\dfrac{\lambda n(n+1) - 4\gamma}{2} > 0$, i.e. $\gamma < \dfrac{\lambda n(n+1)}{2}$. Then

$$\frac{\lambda n(n+1)-4\gamma}{2}\int_{Q_T^h}(-t)^\gamma\cdot\frac{u^2}{4(-t)}dxdt\le\frac{2\lambda\, n(n+1)-8\gamma}{n^2}\int_{Q_T^h}(-t)^\gamma\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\,dxdt. \quad (3.4)$$

Note that for $\gamma\in\left(\dfrac{n^2\left(\lambda-\dfrac{1}{d^2}\right)+2\lambda n}{8};\dfrac{\lambda n(n+1)}{2}\right)$,

$$\frac{2\lambda n(n+1)-8\gamma}{n^2}<\frac{1}{d^2}+\lambda-\frac{\mu}{d^2}, \quad (3.5)$$

where $0<\mu<1$.

Consider (3.4) and (3.5) in (3.3). We get

$$\int_{Q_T^h}(-t)^\gamma\left(u_x^2+\frac{\mu-1}{d^2}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\right)dxdt\le\int_{Q_T^h}(-t)^\gamma\left(\sum_{i=1}^{n}f^i\cdot u_i-fu\right)dxdt. \quad (3.6)$$

Since $\dfrac{\mu-1}{d^2}<0$, then

$$\frac{\mu-1}{d^2}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\ge(\mu-1)|\nabla u|^2. \quad (3.7)$$

Taking (3.7) in to account in (3.6) we deduce

$$\mu\cdot\int_{Q_T^h}(-t)^\gamma u_x^2dxdt\le\int_{Q_T^h}(-t)^\gamma\left(\sum_{k=1}^{n}f^k\cdot u_k-f\cdot u\right)dxdt. \quad (3.8)$$

If we estimate the right hand side of (3.8) in a standard way and use the Frederichs type inequality we deduce

$$\left\|u^{h,\mu}\right\|_{W_{2,\gamma}^{1,0}(Q_T^h)}\le M, \quad (3.9)$$

where a constant M doesn't depend on $u^{h,\mu}$. Continue the function $u^{h,\mu}$ by zero in $Q_T\setminus Q_T^h$. It is obvious that the continued function will be an element of the space $\overset{\circ}{W}{}_{2,\gamma}^{1,0}(Q_T)$ and the estimate (3.9) is valid in the norm $\overset{\circ}{W}{}_{2,\gamma}^{1,0}(Q_T)$. Hence it follows that there exists such a sequences $\mu_k\to0, h_m\to0$ for $k\to\infty, m\to\infty$ that u^{h_m,μ_k} tends to some function $u\in\overset{\circ}{W}{}_{2,\gamma}^{1,0}(Q_T)$ weakly in $\overset{\circ}{W}{}_{2,\gamma}^{1,0}(Q_T)$. It is easy to see that the function $u(x,t)$ is the solution of the problem (1.4).

To prove the uniqueness of the solution it is sufficient to use the procedure in proving the existence of the solution with small alternations and to obtain the estimate

$$\|u\|_{W_{2,\gamma}^{1,0}(Q_T)}\le C\left(\|f\|_{L_{2,\gamma}(Q_T)}+\sum_{k=1}^{n}\left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right)$$

with a constant C not depending on a function u.

The author wants to express his deep gratitude to his supervisor Prof. I.T.Mamedov for the problem statement and for his constant attention to the paper.

References

[1]. Bass R.F. *The Dirichlet problem for radially homogeneous elliptic operators.* Trans.of AMS, 1990, v.320, №2, p.593-61.

[2]. Мамтиев Т.Р. *О разрешимости первой краевой задачи для эллиптических уравнений 2-го порядка с однородными коэффициентами.* Деп. в Аз.НИИНТИ, 1995, №2287, Аз., 26с.

[3]. Мамедов И.Т., Мамтиев Т.Р. *Коэрцитивная оценка для эллиптических операторов 2-го порядка с однородными коэффициентами.* В сб. трудов I. Республиканской конференции по мат.и мех., Баку, «Элм», 1995, ч.II, с.140-148.

[4]. Алхутов Ю.А., Мамедов И.Т. *Первая краевая задача для недивергентных параболических уравнений второго порядка с разрывными коэффициентами.* Мат.сб., 1986, т.131(173), №4(12), с.477-500.

[5]. Ладыженская О.А., Солонников В.А., Уральцева Н.Н. *Линейные и квазилинейные уравнения параболического типа.* М., Наука, 1967, 736с.

[6]. Крылов Н.В. *Нелинейные эллиптические и параболические уравнения второго порядка.* М., «Наука», 1985, 376с.

[7]. Джафаров Н.Дж. *О гладкости вплоть до границы решений первой краевой задачи для параболических уравнений 2-го порядка в областях типа параболоида.* Доклады АН Азербайджана, 1999, т.55, №1-2, с.3-8.

[8]. Ландис Е.М. *Уравнения второго порядка эллиптического и параболического типов.* М., «Наука», 1971, 288с.

Jafarov N.J.
Institute of Mathematics & Mechanics AS of Azerbaijan Republic,
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20 (off.), 30-17-33 (apt).

Received May 19, 2000; Revised October 12, 2000.
Translated by Aliyeva E.T.