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AN INEQUALITY OF A.D. ALEKSANDROV TYPE FOR DEGENERATE
ELLIPTIC-PARABOLIC OPERATORS OF THE SECOND ORDER

Abstract

The second order elliptic-parabolic operator of non-divergence structure whose coefficients are degenerated on the boundary of domain is considered. The conditions on the velocity of degeneration are found under which an inequality of A.D. Aleksandrov is valid for this operator.

Let \mathbf{R}_{n+1} be $(n+1)$ dimensional Euclidean space of points $(x, t) = (x_1, \dots, x_n, t)$, $Q_T = \Omega \times (0, T)$ be a cylindrical domain in \mathbf{R}_{n+1} where Ω is the bounded n -dimensional domain with boundary $\partial\Omega$, $T \in (0, \infty)$. Further, let $Q_0 = \{(x, t): x \in \Omega, t = 0\}$, $\Gamma(Q_T) = Q_0 \cup (\partial\Omega \times [0, T])$ be parabolic boundary of Q_T . Let us consider in Q_T the degenerate elliptic-parabolic operator of the second order

$$L = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j} + \omega(x, t) \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t},$$

where $\|a_{ij}(x, t)\|$ is the real symmetric matrix with measured elements in Q_T such that for all $(x, t) \in Q_T$ and any n -dimensional vector ξ

$$\alpha |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq \alpha^{-1} |\xi|^2, \quad \alpha \in (0, 1] - \text{const} \quad (1)$$

and $\omega(x, t) = \psi_1(\rho) \psi_2(t) \varphi(T-t)$. Here $\rho = \text{dist}(x, \partial\Omega)$, ψ_1, ψ_2 and φ are continuous, non-negative and non-decreasing functions of their arguments, moreover

$$\int_0^T \left(\frac{\varphi(v)}{v^2} \right)^{n+1} dv < \infty. \quad (2)$$

Let's denote by $W_{\omega}^{2,2}(Q_T)$ the Banach space of functions $u(x, t)$ given on Q_T with the finite norm

$$\begin{aligned} \|u\|_{W_{\omega}^{2,2}(Q_T)} = & \|u\|_{C(Q_T)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L_{n+1}(Q_T)} + \sum_{i,j=1}^n \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L_{n+1}(Q_T)} + \\ & + \left\| \frac{\partial u}{\partial t} \right\|_{L_{n+1}(Q_T)} + \left\| \omega \frac{\partial^2 u}{\partial t^2} \right\|_{L_{n+1}(Q_T)}, \end{aligned}$$

and let $\dot{W}_{\omega}^{2,2}(Q_T)$ be subspace $W_{\omega}^{2,2}(Q_T)$ where the dense set is totality of all functions from $C^{\infty}(\bar{Q}_T)$ vanishing on $\Gamma(Q_T)$.

Purpose of the present paper is to obtain the inequality

$$\|u\|_{C(Q_T)} \leq C_1 \|Lu\|_{L_{n+1}(Q_T)} \quad (3)$$

for arbitrary function $u \in \dot{W}_{\omega}^{2,2}(Q_T)$ with constant C_1 independent on u .

Let us remark that for elliptic operators of the second order the analogous estimations were established in the classic works by A.D. Aleksandrov [1-3] and

I.Ya.Bakel'man [4]. Concerning estimations of type (3) for parabolic operators of the second order, let's mention the works by N.V.Krylov [5-6] and Kaising Tso [7]. In the case of power degeneration of function $\omega(x, t)$ on the boundary of the domain Q_T estimation of form (3) was established by I.G.Gasanov [8].

Theorem. Let with respect to coefficients of operator L in domain $Q_T \subset \mathbf{R}_{n+1}$ the conditions (1)-(2) be fulfilled. Then for any function $u(x, t) \in \dot{W}_\omega^{2,2}(Q_T)$ estimation (3) is valid and constant C_1 depends only on λ, n and $d = \text{diam } Q_T$.

Proof. It is obvious that it is sufficient to prove estimation (3) for functions $u(x, t) \in C^\infty(\overline{Q_T})$ vanishing on $\Gamma(Q_T)$. Moreover, without loss of generality, we will consider, that $a_{ij}(x, t) \in C^\infty(\overline{Q_T})$, $(i, j = 1, \dots, n)$ and $\sup_{Q_T} |u| = \sup_{Q_T} u = M > 0$. Let's denote by

(x°, t°) point $\overline{Q_T}$ where $(x^\circ, t^\circ) = M$. Two cases are possible:

- i) $(x^\circ, t^\circ) \in Q_T$,
- ii) $(x^\circ, t^\circ) = (x^\circ, T)$, $x^\circ \in \Omega$.

Suppose first, that the alternative i) takes place. Let's denote for $i, j = 1, \dots, n$ $\frac{\partial u}{\partial x_i}$ by u_i and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ by u_{ij} . Let A_u be set $\{(x, t) : (x, t) \in Q_T, u(x, t) \geq 0, u_i(x, t) \geq 0, u_{ii}(x, t) \leq 0, \text{ matrix } \|u_{ij}(x, t)\| \text{ be determined non-positive}\}$.

Since (x°, t°) is the interior point of Q_T , so it belongs to one of the connected components of set A_u . Let's denote for simplicity this connected component again by A_u . Associate to every point $(x, t) \in A_u$ the vector

$$F(x, t) = \left(u_1(x, t), \dots, u_n(x, t), u(x, t) - \sum_{i=1}^n (x_i - x_i^\circ) u_i(x, t) \right).$$

Let's consider in \mathbf{R}_{n+1} the set

$$G = \left\{ (\xi, h) : |\xi| < \frac{M}{d}, d|\xi| < h < M \right\}.$$

Let's prove that $G \subset F(A_u)$. With this purpose let's fix arbitrary point $(\xi, h) \in G$ and $t' \in (0, T)$ and consider the graph of function $\tilde{u}_{t'}(x) = u(x, t')$. Draw in space (x, Y) the hyperplane P :

$$Y = \langle \xi, x - x^\circ \rangle + h.$$

It is clear, that $Y|_{\partial\Omega} \geq h$. So, moving the graph of function $\tilde{u}_{t'}(x)$ along axis t let's find such t , that hyperplane P will be tangential to graph $\tilde{u}_{t'}(x)$ in point x . But, it is obvious, in this point $F(x, t) = (\xi, h)$. Thus $G \subset F(A_u)$. Let's calculate now Jacobian J of transformation F . We have

$$J = \begin{vmatrix} u_{11} & u_{21} & \dots & u_{n1} & -\sum_{i=1}^n (x_i - x_i^0) u_{i1} \\ u_{12} & u_{22} & \dots & u_{n2} & -\sum_{i=1}^n (x_i - x_i^0) u_{i2} \\ - & - & \dots & - & - \\ u_{1n} & u_{2n} & \dots & u_{nn} & -\sum_{i=1}^n (x_i - x_i^0) u_{in} \\ u_{1t} & u_{2t} & \dots & u_{nt} & u_t - \sum_{i=1}^n (x_i - x_i^0) u_{it} \end{vmatrix}.$$

Let's multiply the first column by $x_1 - x_1^0$, the second column by $x_2 - x_2^0, \dots$, the n -th column by $x_n - x_n^0$ and add their sum to the last column. We obtain

$$J = \begin{vmatrix} u_{11} & u_{21} & \dots & u_{n1} & 0 \\ u_{12} & u_{22} & \dots & u_{n2} & 0 \\ - & - & \dots & - & 0 \\ u_{1n} & u_{2n} & \dots & u_{nn} & 0 \\ u_{1t} & u_{2t} & \dots & u_{nt} & u_t \end{vmatrix} = u_t \det \|u_{ij}\|.$$

Therefore

$$\text{mes } G \leq \int_{Au} u_t \det \|u_{ij}\| dx dt. \quad (4)$$

On the other hand

$$\begin{aligned} \text{mes } G &= \int_{|\xi| < \frac{M}{d}} d\xi \int_{d|\xi|}^M dh = \int_{|\xi| < \frac{M}{d}} (M - d|\xi|) d\xi = \omega_n \int_0^{M/d} (M - \rho d) \rho^{n-1} d\rho = \\ &= \omega_n M \int_0^{M/d} \rho^{n-1} d\rho - \omega_n d \int_0^{M/d} \rho^n d\rho = \frac{\omega_n M^{n+1}}{d^n} \cdot \frac{1}{n(n+1)}, \end{aligned} \quad (5)$$

where $\omega_n = \int_{|\xi|=1} d\xi$. From (4)-(5) we obtain

$$\begin{aligned} M^{n+1} &\leq C_2 \int_{Au} u_t \det \|u_{ij}\| dx dt = C_2 \int_{Au} \frac{u_t \det \|a_{ij} u_{ij}\|}{\det \|a_{ij}\|} dx dt \leq \\ &\leq \frac{C_2}{\alpha^n} \int_{Au} u_t \det \|a_{ij} u_{ij}\| dx dt, \end{aligned} \quad (6)$$

where constant C_2 depends only on n and d .

Note, that matrix $\|a_{ij}(x, t) u_{ij}(x, t)\|$ is determined non-positive for $(x, t) \in A_u$ and its trace is equal to $\sum_{i,j=1}^n a_{ij}(x, t) u_{ij}(x, t)$.

Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of matrix $\|a_{ij} u_{ij}\|$. By virtue of the above said $\lambda_i \leq 0, i = 1, \dots, n$. From (6) we have

$$M^{n+1} \leq \frac{C_2}{\alpha^n} \int_{Au} u_t \lambda_1 \dots \lambda_n dx dt = \frac{C_2}{\alpha^n} \int_{Au} u_t (-\lambda_1) \dots (-\lambda_n) dx dt \leq$$

$$\begin{aligned} &\leq \frac{C_2}{\alpha^n(n+1)^{n+1}} \int_{A_u} (u_t - \lambda_1 - \dots - \lambda_n)^{n+1} dx dt = \frac{C_2}{\alpha^n(n+1)^{n+1}} \int_{A_u} \left(u_t - \sum_{i,j=1}^n a_{ij} u_{ij} \right)^{n+1} dx dt \leq \\ &\leq \frac{C_2}{\alpha^n(n+1)^{n+1}} \int_{A_u} \left(u_t - \sum_{i,j=1}^n a_{ij} u_{ij} - \omega(x,t) u_{tt} \right)^{n+1} dx dt \leq C_3 \int_{Q_T} |Lu|^{n+1} dx dt, \end{aligned}$$

where $C_3 = \frac{C_2}{\alpha^n(n+1)^{n+1}}$. Thus, if alternative i) takes place, then estimation (3) is proved

with $C_1 = C_3^{\frac{1}{n+1}}$.

Now suppose, that alternative ii) is fulfilled. Let's fix arbitrary $\varepsilon > 0$ and consider function $v^\varepsilon(x,t) = (T-t)^\varepsilon u(x,t)$. It is not difficult to see that the least upper bound of function $v^\varepsilon(x,t)$ in Q_T can not be reached for $t = T$. Let's denote $\sup_{Q_T} v^\varepsilon(x,t)$

by M_ε . According to the earlier proved

$$M_\varepsilon^{n+1} \leq C_3 \int_{A_{v^\varepsilon}} \left(v_t^\varepsilon - \sum_{i,j=1}^n a_{ij} v_{ij}^\varepsilon - \omega(x,t) v_{tt}^\varepsilon \right)^{n+1} dx dt, \quad (7)$$

where set A_{v^ε} for function $v^\varepsilon(x,t)$ is determined as set A_u for function $u(x,t)$. On the other hand for $(x,t) \in A_{v^\varepsilon}$

$$v_t^\varepsilon = -\varepsilon(T-t)^{\varepsilon-1} u + (T-t)^\varepsilon u_t \leq (T-t)^\varepsilon u_t, \quad (8)$$

$$v_{tt}^\varepsilon = -\varepsilon(1-\varepsilon)(T-t)^{\varepsilon-2} u - 2\varepsilon(T-t)^{\varepsilon-1} u_t + (T-t)^\varepsilon u_{tt}. \quad (9)$$

Using (8)-(9) in (7) and applying Hölder's inequality we obtain

$$\begin{aligned} M_\varepsilon^{n+1} &\leq 3^n C_3 \int_{A_{v^\varepsilon}} (T-t)^{\varepsilon(n+1)} \left(u_t - \sum_{i,j=1}^n a_{ij} u_{ij} - \omega(x,t) u_{tt} \right)^{n+1} dx dt + \\ &+ 3^n C_3 \varepsilon^{n+1} (1-\varepsilon)^{n+1} \int_{A_{v^\varepsilon}} [\omega(x,t)]^{n+1} (T-t)^{(\varepsilon-2)(n+1)} u^{n+1} dx dt + \\ &+ C_3 2^{n+1} \cdot 3^n \varepsilon^{n+1} \int_{A_{v^\varepsilon}} [\omega(x,t)]^{n+1} (T-t)^{(\varepsilon-1)(n+1)} u_t^{n+1} dx dt. \end{aligned} \quad (10)$$

Further, we have

$$\begin{aligned} &\int_{A_{v^\varepsilon}} [\omega(x,t)]^{n+1} (T-t)^{(\varepsilon-2)(n+1)} u^{n+1} dx dt \leq M_\varepsilon^{n+1} [\psi_1(d) \psi_2(T)]^{n+1} \times \\ &\times T^{\varepsilon(n+1)} \int_{Q_T} (T-t)^{-2(n+1)} (\varphi(T-t))^{n+1} dx dt \leq \\ &\leq M_\varepsilon^{n+1} [\psi_1(d) \psi_2(T)]^{n+1} T^{\varepsilon(n+1)} d^n \omega_n \int_0^T \left(\frac{\varphi(z)}{z^2} \right)^{n+1} dz. \end{aligned} \quad (11)$$

Since $u(x,t) \in C^\infty(\overline{Q_T})$, then $|u_t(x,t)| \leq C_4$ for $(x,t) \in Q_T$, where constant C_4 depends on function u . So

$$\int_{A_{v^\varepsilon}} [\omega(x,t)]^{n+1} (T-t)^{(\varepsilon-1)(n+1)} u_t^{n+1} dx dt \leq C_4^{n+1} [\psi_1(d) \psi_2(T)]^{n+1} \times$$

$$\times T^{\varepsilon(n+1)} d^n \omega_n \int_0^T \left(\frac{\varphi(z)}{z} \right)^{n+1} dz. \quad (12)$$

Let's denote for $\sigma \in (0, T)$ $\sup_{Q_T \cap \{(x,t) | t \leq T - \sigma\}} u$ by $M(\sigma)$. Then taking into account condition (2) and inequalities (11)-(12) in (10) and directing ε to zero we obtain

$$(M(\sigma))^{n+1} \leq 3^n C_3 \int_{Q_T} |Lu|^{n+1} dx dt.$$

Now it is sufficient to direct σ to zero and we come to the request estimation (3) with $C_1 = 3^{\frac{n}{n+1}} C_3^{\frac{1}{n+1}}$. Theorem has been proved.

Remark. As it seen from the proof of the theorem constant C_1 in estimation (3) has the following form

$$C_1 = C_5 \left(\frac{d}{\alpha} \right)^{\frac{n}{n+1}},$$

where constant C_5 depends only on n .

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