MAMEDOV I.T., GUSEYNOV S.T.

ON WEAK SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR SECOND ORDER NON-UNIFORMLY DEGENERATE PARABOLIC EQUATIONS IN DIVERGENCE FORM

Abstract

In paper the class of second order divergent parabolic equations with non-uniform power degeneration is considered. Unique week solvability of the first boundary value problem for such equations in weighted Sobolev spaces is proved.

Let $\mathbf{E}_n$ and $\mathbf{R}_{n+1}$ be the Euclidean spaces of the points $x=(x_1,...,x_n)$ and $(x,t)=(x_1,...,x_n,t)$ respectively, $\Omega\subset\mathbf{E}_n$ be a bounded domain with boundary $\partial\Omega$, $0\in\Omega$, T_0 and T be positive numbers, Q_T be cylinder $\Omega\times(-T_0,T)$, $S_T=\partial\Omega\times[-T_0,T]$, $Q_0=\{(x,t):x\in\Omega,t=-T_0\}$. Consider in Q_T the first boundary value problem

$$\mathcal{L}u=\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)=f(x,t), \tag{1}$$

$$u\big|_{S_T}=0, \qquad u\big|_{Q_0}=0, \tag{2}$$

in assumption that $f(x,t)\in L_2(Q_T)$, $\|a_{ij}(x,t)\|$ is a real symmetrical matrix with measurable elements in Q_T, for $(x,t)\in Q_T$ and $\xi\in\mathbf{E}_n$ the condition

$$\mu\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\mu^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \tag{3}$$

is satisfied.

Here $\mu\in(0,1]$ is a constant, $\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}$,

$$|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\overline{\alpha}_i}, \quad \overline{\alpha}_i=\frac{2}{2+\alpha_i}, \quad \alpha=(\alpha_1,...,\alpha_n), \quad \alpha_i\ge 0, \ i=1,...,n.$$

The aim of present paper is a proof of unique week solvability of the boundary value problem (1)-(2). Note that in case of uniformly parabolic equations analogous result is established in [1]. As to parabolic qaution with weak logarithmic degeration we'll indicate the work [2]. Regarding to question of apriori estimation of solutions of degenerate parabolic equations we'll note the works [3-6]. In addition with weak solvability of boundary equations we'll refer the works [7-8]. The more complete review of investigations by theory of boundary value problems for parabolic equations it is possible to find in monographs [1] and [9].

Now we'll note some denotations and definitions.

Let $\alpha^-=\min\{\alpha_1,...,\alpha_n\}$, $\alpha^+=\max\{\alpha_1,...,\alpha_n\}$. Denote by $A(Q_T)$ the set of all functions $u(x,t)\in C^\infty(\overline{Q}_T)$, such that for any from them there can be found the domain $\Omega(u)$, $\overline{\Omega}(u)\subset\Omega$ and supp $u\subset\Omega(u)\times[-T_0,T]$. Let further $\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ and $\overset{\circ}{W}{}^{1,1}_{2,\alpha}(Q_T)$ be completions of $A(Q_T)$ by norms

$$\|u\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)} = \left[\operatorname{vrai}\max_{t\in[-T_0,T]} \int_\Omega u^2 dx + \sum_{i=1}^n \int_{Q_T} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i}\right)^2 dxdt \right]^{1/2}$$

and

$$\|u\|_{\overset{\circ}{W}{}^{1,1}_{2,\alpha}(Q_T)} = \left(\int_{Q_T} \left[u^2 + \sum_{i=1}^n \int_{Q_T} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 \right] dxdt \right)^{1/2}$$

respectively. Everywhere further we'll keep the denotations $u_t = \dfrac{\partial u}{\partial t}, u_i = \dfrac{\partial u}{\partial x_i}$; $i = 1,\dots,n$.

The function $u(x,t) \in \overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ is called a weak solution of the boundary value problem (1)-(2), if for any function $\eta(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\alpha}(Q_T)$ and for any $t_1 \in (-T_0, T]$ the integral identity

$$\int_\Omega u(x,t_1)\eta(x,t_1)dx - \int_{Q_{t_1}} u\eta_t dxdt + \int_{Q_{t_1}} \sum_{i,j=1}^n a_{ij}(x,t) u_i \eta_j dxdt = \int_{Q_{t_1}} f\eta dxdt, \tag{4}$$

where $Q_{t_1} = \Omega \times (-T_0, t_1)$, is satisfied.

$C(\cdots)$ means that the positive constant C depends only on content of brackets.

Lemma. *If $\alpha^- < 2$, then for any function $u(x,t) \in \overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ and for any $t_1 \in (-T_0, T]$ the estimation*

$$\int_{Q_{t_1}} u^2 dxdt \le C_1(\alpha, \Omega) \int_{Q_{t_1}} \sum_{i=1}^n \lambda_i(x,t)\, u_i^2 dxdt. \tag{5}$$

is true.

Proof. It is sufficient to establish (5) for functions $u(x,t) \in A(Q_T)$. By virtue of boundedness of domain Ω there can be found $R = R(\Omega)$ such that $\overline{\Omega} \subset B_R = \{x : |x_i| < R,\ i = 1,\dots,n\}$.

Let's fix arbitrary $t' \in (-T_0, t_1)$, extend the function $u(x,t')$ by zero in $B_R \setminus \Omega$ and denote obtained extension again by $u(x,t')$.

Without loss of generality, we can take $\alpha^- = \alpha_1$. Let $(x_2,\dots,x_n) = x'$. We have for $x_1 \in (-R, R)$

$$u(x_1,x',t') = \int_{-R}^{x_1} u_1(\tau,x',t')d\tau \le \left(\int_{-R}^{x_1} \frac{d\tau}{\lambda_1(\tau,x',t')} \right)^{1/2} \left(\int_{-R}^{x_1} \lambda_1(\tau,x',t') u_1^2(\tau,x',t')d\tau \right)^{1/2}. \tag{6}$$

On the other hand

$$\int_{-R}^{x_1} \frac{d\tau}{\lambda_1(\tau,x',t')} \le \int_{-R}^{R} \frac{d\tau}{\left(|\tau|^{\overline{\alpha}_1} + \sum_{i=2}^n |x_i|^{\overline{\alpha}_i} + \sqrt{|t'|} \right)^{\alpha_1}} \le \int_{-R}^{R} \frac{d\tau}{|\tau|^{\frac{2\alpha_1}{2+\alpha_1}}}.$$

Since $\alpha_1 < 2$, then $\dfrac{2\alpha_1}{2+\alpha_1} < 1$ and so

$$\int_{-R}^{x_1} \frac{d\tau}{\lambda_1(\tau,x',t')} \le C_2(\alpha, R). \tag{7}$$

Taking into account (7) in (6) we obtain

$$u^2(x_1,x',t')\le C_2\int_{-R}^{R}\lambda_1(\tau,x',t')u_1^2(\tau,x',t')d\tau .$$

Integrate the last inequality by B_R and take into account that $u\equiv 0$ in $B_R\setminus\Omega$. We have

$$\int_\Omega u^2(x,t')dx\le 2C_2R\int_\Omega\lambda_1(x,t')u_1^2(x,t')dx\le 2C_2R\int_\Omega\sum_{i=1}^{n}\lambda_i(x,t')\,u_i^2(x,t')dx . \tag{8}$$

Now it is sufficient to integrate (8) with respect to t' from $-T_0$ to t_1 and we obtain the required estimation (5) with $C_1=2C_2R$.

The lemma is proved.

Theorem. *Let relative to coefficients of the operator $\mathcal{L}$ the condition (3) is satisfied. Then, if $\alpha^+<2$, then the first boundary value problem (1)-(2) is uniquely weak solvable in the space $\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ for any $f(x,t)\in L_2(Q_T)$. Thus for the solution $u(x,t)$ of this problem the estimate*

$$\|u\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)}\le C_3(\mu,\alpha,\Omega)\|f\|_{L_2(Q_T)} \tag{9}$$

is true.

Proof. Let for natural m Ω_m be extending sequence of domains, approximating domain Ω from within, i.e. $\Omega_m\to\Omega$ when $m\to\infty$ and for any m $\partial\Omega_m\in C^2$. Denote $\Omega_m\times(-T_0,T)$, $\partial\Omega_m\times[-T_0,T]$ and $\{(x,t):x\in\Omega_m,t=-T_0\}$ by Q_T^m, S_T^m and Q_0^m, respectively. Let further for natural p and $(x,t)\in Q_T$ $\lambda_i^p(x,t)=\max\{p^{-\alpha_i},\lambda_i(x,t)\}$; $i=1,\dots,n$. It is easy to see that for any natural p for functions $\lambda_i(x,t)$ the uniform Hölder condition in Q_T is satisfied. Let $\|a_{ij}^p(x,t)\|$ be symmetrical matrix with measurable elements in Q_T, such that $a_{ij}^p(x,t)\to a_{ij}(x,t)$ when $p\to\infty$, almost everywhere in Q_T and for $(x,t)\in Q_T$, $\xi\in\mathbf{E}_n$

$$\mu\sum_{i=1}^{n}\lambda_i^p(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}^p(x,t)\xi_i\xi_j\le\mu^{-1}\sum_{i=1}^{n}\lambda_i^p(x,t)\xi_i^2 . \tag{10}$$

Finally for fixed p and natural k $a_{ij}^{p,k}(x,t)$ and $f^k(x,t)$ be averages by Friedrichs with parameter $\frac{1}{k}$ of the functions $a_{ij}^p(x,t)$ and $f(x,t)$ respectively $(i,j=1,\dots,n)$. We assume that the function $a_{ij}^p(x,t)$ is extended in $\mathbf{R}_{n+1}$ with preservation of condition (10) (for example, $a_{ij}^p(x,t)=\delta_{ij}\lambda_i^p(x,t)$ for $(x,t)\in\mathbf{R}_{n+1}\setminus Q_T$; $i,j=1,\dots,n$, where δ_{ij} is Cronecker's symbol) and the function $f(x,t)$ is extended by zero in $\mathbf{R}_{n+1}\setminus Q_T$. It is easy to see that according to (10) for $(x,t)\in Q_T$, $\xi\in\mathbf{E}_n$

$$\mu\sum_{i=1}^{n}\lambda_i^p(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}^{p,k}(x,t)\xi_i\xi_j\le\mu^{-1}\sum_{i=1}^{n}\lambda_i^p(x,t)\xi_i^2 . \tag{11}$$

Let

$$\mathcal{L}^{p,k}=\frac{\partial}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}^{p,k}(x,t)\frac{\partial}{\partial x_j}\right).$$

Let's fix arbitrary natural p and k and for natural m consider the sequence of the auxiliary boundary value problems

$$\left.\begin{aligned} \mathcal{L}^{p,k} u^{p,k,m} = f^k \ ; \quad (x,t) \in Q_T^m \ , \\ u^{p,k,m}\Big|_{S_T^m} = 0, \ \ u^{p,k,m}\Big|_{Q_0^m} = 0 \ . \end{aligned}\right\} \tag{12}$$

It is clear that for any fixed natural p, k and m the boundary value problem (12) has the unique classical solution $u^{p,k,m}(x,t) \in C^{2,1}\left(\overline{Q}_T^m\right)$. Then for any function $\vartheta(x,t) \in C^{2,1}\left(\overline{Q}_T^m\right)$, vanishing on $S_T^m \bigcup Q_0^m$ the equality

$$\int_{Q_T^m} u_t^{p,k,m} \vartheta \, dxdt + \int_{Q_T^m} \sum_{i,j=1}^n a_{ij}^{p,k}(x,t) u_i^{p,k,m} \vartheta_j dxdt = \int_{Q_T^m} f^k \vartheta \ dxdt \tag{13}$$

is true.

Put in (13) $\vartheta(x,t) = u^{p,k,m}(x,t)$. We obtain

$$\int_{Q_T^m} u_t^{p,k,m} u^{p,k,m} dxdt + \int_{Q_T^m} \sum_{i,j=1}^n a_{ij}^{p,k}(x,t) u_i^{p,k,m} u_j^{p,k,m} dxdt = \int_{Q_T^m} f^k u^{p,k,m} dxdt \, . \tag{14}$$

From (14) and condition (11) follows that

$$\mu \int_{Q_T^m} \sum_{i=1}^n \lambda_i^p(x,t) \left(u_i^{p,k,m}\right)^2 dxdt \le \int_{Q_T^m} f^k u^{p,k,m} dxdt \, .$$

Since for any natural p $\lambda_i(x,t) \le \lambda_i^p(x,t)$, $i = 1,...,n$, then from the last inequality we obtain that

$$\int_{Q_T^m} \sum_{i=1}^n \lambda_i(x,t) \left(u_i^{p,k,m}\right)^2 dxdt \le \frac{1}{\mu} \int_{Q_T^m} f^k u^{p,k,m} dxdt \, . \tag{15}$$

For any $\beta > 0$ from (15) with regard to lemma we have

$$\begin{aligned} \int_{Q_T^m} \sum_{i=1}^n \lambda_i(x,t) \left(u_i^{p,k,m}\right)^2 dxdt &\le \frac{\beta}{2\mu} \int_{Q_T^m} \left(u^{p,k,m}\right)^2 dxdt + \frac{1}{2\beta\mu} \int_{Q_T^m} \left(f^k\right)^2 dxdt \le \\ &\le \frac{C_1\beta}{2\mu} \int_{Q_T^m} \sum_{i=1}^n \lambda_i(x,t) \left(u_i^{p,k,m}\right)^2 dxdt + \frac{1}{2\beta\mu} \int_{Q_T^m} \left(f^k\right)^2 dxdt \, . \end{aligned} \tag{16}$$

Put now in (16) $\beta = \dfrac{\mu}{C_1}$. We obtain

$$\begin{aligned} \int_{Q_T^m} \sum_{i=1}^n \lambda_i(x,t) \left(u_i^{p,k,m}\right)^2 dxdt &\le \frac{C_1}{\mu^2} \int_{Q_T^m} \left(f^k\right)^2 dxdt \le \frac{C_1}{\mu^2} \int_{Q_T} \left(f^k\right)^2 dxdt \le \\ &\le \frac{2C_1}{\mu^2} \int_{Q_T} f^2 dxdt + \frac{2C_1}{\mu^2} \int_{Q_T} \left(f^k - f\right)^2 dxdt \, . \end{aligned} \tag{17}$$

In the beginning suppose that the function $f(x,t)$ doesn't coincide with identical zero a.e. in Q_T. Then from (17) follows existence of natural $k_0 = k_0(f)$ such that for $k \ge k_0$ and any natural p and m the estimate

$$\int_{Q_T^m} \sum_{i=1}^n \lambda_i(x,t) \left(u_i^{p,k,m}\right)^2 dxdt \le \frac{4C_1}{\mu^2} \int_{Q_T} f^2 dxdt \tag{18}$$

holds.

Let's fix an arbitrary $t_1 \in (-T_0, T]$ and let $Q_{t_1}^m = \Omega_m \times (-T_0, t_1)$. It is easy to see that the equality (4) is true in cylinder $Q_{t_1}^m$, too. Then from (14) and (18) with regard to lemma we conclude

$$\frac{1}{2}\int_{\Omega_m} \left(u^{p,k,m}(x,t_1)\right)^2 dx \le \int_{Q_{t_1}^m} f^k u^{p,k,m} dxdt \le \frac{1}{2}\int_{Q_{t_1}^m} \left[\left(f^k\right)^2 + \left(u^{p,k,m}\right)^2\right] dxdt \le$$

$$\le \frac{1}{2}\int_{Q_{t_1}^m} \left(f^k\right)^2 dxdt + \frac{C_1}{2}\int_{Q_{t_1}^m} \sum_{i=1}^n \lambda_i(x,t)\left(u_i^{p,k,m}\right)^2 dxdt \le \frac{1}{2}\int_{Q_T} \left(f^k\right)^2 dxdt +$$

$$+ \frac{2C_1^2}{\mu^2}\int_{Q_T} f^2 dxdt \le \int_{Q_T} \left(f^k - f\right)^2 dxdt + \left(\frac{2C_1^2}{\mu^2} + 1\right)\int_{Q_T} f^2 dxdt \le \left(\frac{2C_1^2}{\mu^2} + 1\right)\int_{Q_T} f^2 dxdt \quad (19)$$

if only $k \ge k_0$. From (18)-(19) follows that

$$\left\| u^{p,k,m} \right\|_{\mathring{W}_{2,\alpha}^{1,0}(Q_T^m)} \le C_3 \|f\|_{L_2(Q_T)}, \quad (20)$$

where $C_3 = \left(\frac{4C_1}{\mu^2} + \frac{4C_1^2}{\mu^2} + 4\right)^{1/2}$. Note that if $f(x,t) = 0$ a.e. in Q_T, then $u^{p,k,m}(x,t) \equiv 0$ in Q_T^m and the inequality (20) is fulfilled obviously. We extend the function $u^{p,k,m}(x,t)$ by zero in $Q_T \setminus Q_T^m$ and denote the extended function again by $u^{p,k,m}(x,t)$. It is clear that $u^{p,k,m}(x,t) \in \mathring{W}_{2,\alpha}^{1,0}(Q_T)$, and from (20) it follows that

$$\left\| u^{p,k,m} \right\|_{\mathring{W}_{2,\alpha}^{1,0}(Q_T)} \le C_3 \|f\|_{L_2(Q_T)}. \quad (21)$$

Let's fix arbitrary natural p and $k \ge k_0$. Then from (21) follows weak compactness of the sequence $\left\{u^{p,k,m}(x,t)\right\}$ by m. In other words there exist a subsequence of natural numbers $m_s \to \infty$ when $s \to \infty$ and the function $z^{p,k}(x,t) \in \mathring{W}_{2,\alpha}^{1,0}(Q_T)$ such that $u^{p,k,m_s}(x,t) \to \ \to z^{p,k}(x,t)$ when $s \to \infty$ weakly in $\mathring{W}_{2,\alpha}^{1,0}(Q_T)$.

It is easy to see that the function $z^{p,k}(x,t)$ is a weak solution of the first boundary value problem

$$\left.\begin{aligned} \mathcal{L}^{p,k} z^{p,k} = f^k \ ; \quad (x,t) \in Q_T \ , \\ z^{p,k}\Big|_{S_T} = 0, \quad z^{p,k}\Big|_{Q_0} = 0 \ . \end{aligned}\right\}$$

In addition according to (21)

$$\left\| z^{p,k} \right\|_{\mathring{W}_{2,\alpha}^{1,0}(Q_T)} \le C_3 \|f\|_{L_2(Q_T)}. \quad (22)$$

Let now p be arbitrary fixed natural number, and $k \ge k_0$ be any natural number. From the estimation (2) we conclude existence of subsequence of natural numbers $k_s \to \infty$ when $s \to \infty$ and the function $w^p(x,t) \in \mathring{W}_{2,\alpha}^{1,0}(Q_T)$ such that $z^{p,k_s}(x,t) \to w^p(x,t)$ when $s \to \infty$ weakly in $\mathring{W}_{2,\alpha}^{1,0}(Q_T)$. In addition the function $w^p(x,t)$ is a weak solution of the first boundary value problem

$$\left.\begin{aligned} \mathcal{L}^p w^p = f\,; \quad (x,t)\in Q_T\,, \\ w^p\Big|_{S_T} = 0,\ \ w^p\Big|_{Q_0} = 0\,, \end{aligned}\right\}$$

where $\mathcal{L}^p = \dfrac{\partial}{\partial t} - \sum\limits_{i,j=1}^{n}\dfrac{\partial}{\partial x_i}\left(a_{ij}^p(x,t)\dfrac{\partial}{\partial x_j}\right)$. Besides

$$\left\|w^p\right\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)} \le C_3\|f\|_{L_2(Q_T)}. \tag{23}$$

Now from (23) follows existence of sequence of natural numbers $p_s \to \infty$ for $s\to\infty$ and the function $u(x,t)\in \overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ such that $w^{p_s}(x,t)\to u(x,t)$ when $s\to\infty$ weakly in $\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$. In addition according to (23)

$$\|u\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)} \le C_3\|f\|_{L_2(Q_T)}. \tag{24}$$

Prove that the function $u(x,t)$ is a weak solution of the first boundary value problem (1)-(2). For any function $\eta(x,t)\in \overset{\circ}{W}{}^{1,1}_{2,\alpha}(Q_T)$ for any $t_1\in(-T_0,T]$ we have

$$\int_\Omega w^{p_s}(x,t_1)\eta(x,t_1)dx - \int_{Q_{t_1}} w^{p_s}\eta_t\,dxdt + \int_{Q_{t_1}}\sum_{i,j=1}^{n}a_{ij}^{p_s}(x,t)w_i^{p_s}\eta_j\,dxdt = \int_{Q_{t_1}} f\,\eta\,dxdt\,.$$

It is clear that

$$\lim_{s\to\infty}\int_\Omega w^{p_s}(x,t_1)\eta(x,t_1)dx = \int_\Omega u(x,t_1)\eta(x,t_1)dx\,;\ \lim_{s\to\infty}\int_{Q_{t_1}} w^{p_s}\eta_t\,dxdt = \int_{Q_{t_1}} u\eta_t\,dxdt\,.$$

Therefore for proof of required fact it is sufficient to show that

$$\lim_{s\to\infty}\int_{Q_{t_1}}\sum_{i,j=1}^{n}a_{ij}^{p_s}(x,t)w_i^{p_s}\eta_j\,dxdt = \int_{Q_{t_1}}\sum_{i,j=1}^{n}a_{ij}(x,t)u_i\eta_j\,dxdt\,. \tag{25}$$

But

$$\int_{Q_{t_1}}\sum_{i,j=1}^{n}a_{ij}^{p_s}(x,t)w_i^{p_s}\eta_j\,dxdt = \int_{Q_{t_1}}\sum_{i,j=1}^{n}a_{ij}(x,t)w_i^{p_s}\eta_j\,dxddt + \int_{Q_{t_1}}\sum_{i,j=1}^{n}\left[a_{ij}^{p_s}(x,t)-a_{ij}(x,t)\right]\times$$
$$\times\, w_i^{p_s}\eta_j\,dxdt = I_1^s + I_2^s\,. \tag{26}$$

It is easy to see that

$$\lim_{s\to\infty} I_1^s = \int_{Q_{t_1}}\sum_{i,j=1}^{n}a_{ij}(x,t)u_i\eta_j\,dxdt\,. \tag{27}$$

Thus according to (25)-(27) we needed to prove that

$$\lim_{s\to\infty} I_2^s = 0\,. \tag{28}$$

We show the correctness (28) for functions $\eta(x,t)\in A(Q_T)$. We have with regard to (29)

$$\left|I_2^s\right| \le C_4(n)\left(\int_{Q_{t_1}}\sum_{i=1}^{n}\lambda_i(x,t)\left(w_i^{p_s}\right)^2dxdt\right)^{1/2}\left(\int_{Q_{t_1}}\sum_{i,j=1}^{n}\lambda_i^{-1}(x,t)\left|a_{ij}^{p_s}(x,t)-a_{ij}(x,t)\right|^2\left(\eta_j\right)^2dxdt\right)^{1/2} \le$$

$$\le C_5(n)C_3\|f\|_{L_2(Q_T)}\left(\int_{Q_{t_1}}\sum_{i=1}^{n}\lambda_i^{-q}(x,t)dxdt\right)^{1/2q}\left(\int_{Q_{t_1}}\sum_{i,j=1}^{n}\left|a_{ij}^{p_s}(x,t)-a_{ij}(x,t)\right|^{2q'}\left|\eta_j\right|^{2q'}dxdt\right)^{1/2q'}, \tag{29}$$

where the number $q>1$ will be chosen later, and $q'=\frac{q}{q-1}$. On the other side for $i=1,\dots,n$

$$\int_{Q_{t_1}}\lambda_i^{-q}(x,t)dxdt\le(t_1+T_0)(2R)^{n-1}\int_{-R}^{R}\frac{d\tau}{|\tau|^{\frac{2\alpha_i q}{2+\alpha_i}}},\tag{30}$$

where R has the same meaning as in proof of lemma. Since $\alpha^+<2$, then for $i=1,\dots,n$ $\frac{2\alpha_i}{2+\alpha_i}<1$. Therefore for any $i=1,\dots,n$ there exists $q_i>1$ such that $\frac{2\alpha_i q_i}{2+\alpha_i}<1$. Let's fix $q=\min\{q_1,\dots,q_n\}$. Now it is sufficient to consider that for $i,j=1,\dots,n$ $a_{ij}(x,t)\to a_{ij}(x,t)$, when $s\to\infty$ a.e. in Q_T, and from (29)-(30) follows the required limit equality (28). The existence of weak solution of the first boundary value problem (1)-(2) is proved.

Let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ be two solutions of the boundary value problem (1)-(2), $u(x,t)=u^{(1)}(x,t)-u^{(2)}(x,t)$. Then for any function $\eta(x,t)\in\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ vanishing at $t=T$ (look [1]) the equality

$$-\int_{Q_T}u\eta_t dxdt+\int_{Q_T}\sum_{i,j=1}^{n}a_{ij}(x,t)u_i\eta_j dxdt=0\tag{31}$$

is hold.

Let's fix an arbitrary $\delta\in(0,T_0'+T)$ and assume that the function $\eta(x,t)\in\overset{\circ}{W}{}^{1,1}_{2,\alpha}(Q_T)$ vanishes at $t\le -T_0$ and $t\ge T-\delta$. Denote for $h\in(0,\delta]$ $\frac{1}{h}\int_{t-h}^{t}\eta(x,\tau)d\tau$ by $\eta_{\bar h}(x,t)$ and substitute in (31) the function $\eta_{\bar h}(x,t)$ instead of $\eta(x,t)$. We obtain

$$-\int_{Q_T}u(\eta_{\bar h})_t dxdt+\int_{Q_T}\sum_{i,j=1}^{n}a_{ij}(x,t)u_i(\eta_{\bar h})_j dxdt=0.\tag{32}$$

Taking in to account that

$$(\eta_{\bar h})_t=(\eta_t)_{\bar h}\ ,\quad (\eta_{\bar h})_j=(\eta_j)_{\bar h}\ ;\quad j=1,\dots,n$$

and the equalities

$$-\int_{Q_T}u(\eta_t)_{\bar h}\,dxdt=-\int_{Q_{T-\delta}}u_h\eta_t dxdt=\int_{Q_{T-\delta}}(u_h)_t\,\eta dxdt,$$

$$\int_{Q_T}\sum_{i,j=1}^{n}a_{ij}(x,t)u_i(\eta_j)_{\bar h}\,dxdt=\int_{Q_{T-\delta}}\sum_{i,j=1}^{n}(a_{ij}(x,t)u_i)_h\eta_j dxdt,$$

from (32) we conclude

$$\int_{Q_{T-\delta}}(u_h)_t\eta\, dxdt+\int_{Q_{T-\delta}}\sum_{i,j=1}^{n}(a_{ij}(x,t)u_i)_h\eta_j dxdt=0.\tag{33}$$

Here $u_h(x,t)=\frac{1}{h}\int_{t}^{t+h}u(x,\tau)d\tau$.

It is easy to see that the equality (33) indeed is true for any function $\eta(x,t)\in\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_{T-\delta})$ ([1]). Therefore, assuming in (33) $\eta(x,t)=u_h(x,t)$ and tending h to zero, we obtain

$$\frac{1}{2}\int_{\Omega} u^2(x,T-\delta)dx + \int_{Q_{T-\delta}} \sum_{i,j=1}^{n} a_{ij}(x,t)u_i u_j dxdt = 0 .$$

According to the condition (3) from the last equality follows that

$$\int_{\Omega} u^2(x,T-\delta)dx = 0 .$$

Now consider arbitrariness of δ in the interval $(0,T_0+T)$. We obtain

$$\int_{Q_T} u^2(x,t)dxdt = 0 ,$$

i.e. $u(x,t)=0$ a.e. in Q_T. The uniqueness of the solution of the problem (1)-(2) is proved. Now the estimation (9) follows from the inequality (24).

References

[1]. Ладыженская О.А., Солонников В.А., Уральцева Н.Н. *Линейные и квазилинейные уравнения параболического типа.* М., «Наука», 1967, 736с.

[2]. Джалилов К.А. *Первая краевая задача для неравномерно вырождающихся дивергентных эллиптических и параболических уравнений 2-го порядка.* Канд. дисс., Баку, 1987, 126с.

[3]. Скрыпник И.И. *Регулярность решений вырождающихся квазилинейных параболических уравнений (весовой случай).* Укр. мат. журнал, 1996, т. 48, №7, с.972-988.

[4]. Nobutoshi I., Yasumaro K. *The Dirichlet problem for a certain degenerate parabolic equation.* J. Fac. Sci. Shinshu Univ., 1997, v.32, №1, p.1-13.

[5]. Brown R.M., Hu Wei, Lieberman G.M. *Weak solutions of parabolic equations on non-cylindrical domains.* Proc. Amer. Math. Soc., 1997, v.125, №6, p.1785-1792.

[6]. Fabricant A., Marinov M., Rangelov Ts. *Some properties of nonlinear degenerate parabolic equations.* Math. Balkan., 1994, v.8, №1, p.59-73.

[7]. Fernandes J.C., Franchi B. *Existence and properties of the Green function for a class of degenerate parabolic equations.* Rev. Mat. Iberoamer., 1996, v. 12, №2, p.491-525.

[8]. Терсенов А.С. *Об одном классе вырождающихся неравномерно параболических уравнений.* Вестн. МГУ, сер.1, 1996, №6, с.94-97.

[9]. Крылов Н.В. *Нелинейные эллиптические и параболические уравнения второго порядка.* М., «Наука», 1985, 376с.

Mamedov I.T., Guseynov S.T.

Institute of Mathematics & Mechanics AS of Azerbaijan Republic,

9, F. Agayev str., 370141, Baku, Azerbaijan.

Tel.: 39-39-27 (off.).

Received June 15, 2000; Revised September 25, 2000.

Translated by Mirzoyeva K.S.