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ZOHRI A., SHAHBAZOV A.I.

WEIGHTED COMPOSITION OPERATORS ON SOME BANACH MODULES OF ANALYTIC FUNCTIONS

Abstract

In this paper we consider a weighted composition operator on "small" and "big" Hardy spaces and also on some Banach modules of analytic functions these spaces and compactness and nuclearity of this operator will be studied.

1. Introduction.

If we consider the commutative semisimple Banach algebra A as the algebra of continuous functions on the compact maximal ideal spaces of A, then for any endomorphism (in particular, automorphism) $T: A \to A$ corresponds a continuous mapping (homeomorphism) $\phi : m_A \to m_A$ such that T can be represented as a weighted composition operator induced by ϕ. For this reason (and also for the other reason that, every bounded linear operator on Banach space may have this form), weighted composition operators and their sums are very important in solving and existence of solution of functional and differential equations. This kind of operators on classical Banach algebras are being studied by Nordgreen, Shapiro, Taylor, Kamowitz, Zorboska and others. These operators on general uniform spaces have been studied by Takagi, Gorin and Shahbazov and Mirzakarimi and Sedighi, and their sums studied by Gorin and Shahbazov and by Shahbazov and Dehghan on uniform spaces.

In this paper we will investigate weighted composition operators and their "summation" on some weighted Hardy spaces which we call them "big" and "small" Hardy spaces, and on the Banach spaces of analytic functions which are Banach modules on $H^2(\beta)$. Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\beta_0 = 1$ and $\frac{\beta_n + 1}{\beta_n} \to 1$ when $n \to \infty$. We define $H^2(\beta)$ as the set of all complex analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$ on D- open unit disk in $\mathbf{C}$. We will consider compactness and nuclearity of weighted composition operators and also "their sums" on "small" and "big" Hardy spaces and some $H^2(\beta)$- Banach modules of analytic functions on D.

2. Compact weighted composition operators on Banach modules on "small" Hardy spaces.

Let $D \subset C$ be open unit disk, as above; by $(Hol(D), k)$ we denote the space of all holomorphic functions on D with cloopen topology k and

$$A(D) = C(\overline{D}) \cap Hol(D),$$

where $C(\overline{D})$ is the algebra of all complex valued continuous functions on $\overline{D}$ equipped with sup-norm $\|f\| = \sup\{|f(z)| : z \in \overline{D}\}$. It is clear that $A(D)$ is a Banach algebra with sup-norm which is called disk algebra. Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ when $n \to \infty$. The set of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$ on D is a Hilbert space with the inner product

$$< f, g >_\beta = \sum_{n=0}^{\infty} a_n \overline{b}_n \beta_n^2$$

when $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

We will consider the Banach space $(X, \|\cdot\|_X)$ with the norm $\|\cdot\|$ such that $(H^2(\beta), \|\cdot\|_\beta) \subset (X, \|\cdot\|_X) \subset (Hol(D), k)$ (where $\|\cdot\|_\beta$ is the norm induced by $<,>_\beta$) and X is a $H^2(\beta)$- Banach module with respect to ordinary pointwise multiplication of functions such that for any $f \in X$ and $a \in H^2(\beta)$ the function $af \in X$ and

$$\|af\|_X \le c\|a\|_\beta \cdot \|f\|_X, \tag{1}$$

where the constant c on the right side of (1) is independent of a and f. Also we assume that natural embedding $(X, \|\cdot\|_X) \subset (Hol(D), k)$ is continuous, i.e. for any compact $K \subset D$, there exists a constant $c_K > 0$ such that

$$\|f\|_{K,\infty} = \sup_{x \in K} |f(x)| \le c_K \|f\|_X \tag{2}$$

holds for any $f \in X$.

Definition 1. *Let X be a uniform function space on X, a function $\varphi: D \to D$ is called a compositor on X if $f \circ \varphi \in X$ whenever $f \in X$. By $comp(X)$ we mean the set of all compositors on X.*

Definition 2. *A closed subset $E \subset D$ is called a peak set with respect to X if there exists a sequence $\{f_n\}$, $f_n \in X$ $(n = 1,2,...)$ such that $\|f_n\|_X = f_n(x) = 1$ for all n and $x \in E$ moreover, outside any neighborhood of the set E the sequence $\{f_n\}$ tends to 0 uniformly. A peak set consisting of only one point is called peak point.*

We will consider the weighted composition operator on X of the form $T: X \to X; f \mapsto u\, f(\varphi)$, where $u \in H^2(\beta)$ is a fixed function and $\varphi: \overline{D} \to \overline{D}$ is a continuous and holomorphic function on D and $\varphi \in \mathrm{comp}(X)$. If X *is uniformly closed* subspaces of $C(D)$ (for example $X = A(D)$, or close subspace of $A(D)$ or a closed spaces containing $A(D)$ and ...), then theorem 1.5 of [6] has the following form (we consider the non-trivial case, when $u \ne 0$ and $\varphi \not\equiv const$).

Theorem 2.1. *If the operator $T: X \to X$ of the form $f \mapsto u \cdot f(\varphi)$ (where $u \in H^2(\beta)$ and $\varphi: D \to D$ is a holomorphic function) is compact, then for every connected compact subset $Y \subset \{z \in D | u(z) \ne 0\}$ and for every peak set E with respect to $H^2(\beta)$ we have either $\varphi(Y) \cap E = \phi$ or $\varphi(Y) \subset E$.*

Proof. Easily obtained from 1.5 of [6], since every peak set with respect to $H^2(\beta)$ is also a peak set with respect to X.

Now we consider "small" weighted Hardy spaces.

Lemma 2.1. *If $\{\beta_n\}_{n=0}^{\infty}$ is a sequence such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}<\infty$, then $H^2(\beta)$ can be continuously embedded in $A(D)$, i.e. $H^2(\beta)\subset A(D)$ and $\|f\|_\infty \le C\|f\|_\beta$.*

Note. $H^2(\beta)$ is called small Hardy space when $H^2(\beta)\subset A(D)$.

Proof. Since $|z|\le 1$ then $\|f\|_\infty \le \sup_{z\in\overline{D}}\sum_{n=0}^{\infty}|a_n||z_n|^n \le \sum_{n=0}^{\infty}|a_n|$ and since $\beta_n \ne 0$ we can write

$$\sum_{n=0}^{\infty}|a_n| = \sum_{n=0}^{\infty}\frac{|a_n|\beta_n}{\beta_n} \le \left(\sum_{n=0}^{\infty}|a_n|^2\beta_n^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\right)^{1/2}$$

by Cauchy-Schwartz inequality. Because $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}$ is convergent, if we denote its limit by c we have $\|f\|_\infty < c\|f\|_\beta$, therefore $\sum_{n=0}^{\infty}a_n z^n$ is absolutely convergent on ∂D so $H^2(\beta)\subset A(D)$. ⊔

From the above lemma, and the last theorem 2.1 we have

Theorem 2.2. *If*

$$\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}<\infty \text{ and } A(D)\subset X$$

and $T: X\to X$ is compact then $|\varphi(z)|<1$ when $u(z)\ne 0$ (we assume that $u\ne 0$ and $\varphi \ne const$).

Proof. We assume that the operator is compact, then it is clear that the set $Y=\{z: u(z)\ne 0\}$ is compactly connected set. Since the boundary ∂D of the disk consists of peak points for the X (because $A(D)\subset X$), so if $|\varphi(z_0)|=1$ for some point $z_0\in Y$, then $\varphi|_Y \equiv const$ (by the theorem 2.1). Therefore, if $u(z)\ne 0$, then $|\varphi(z)|<1$. ⊔

Remark 1. In general if $\varphi(\overline{D})\subset D$ then it is easy to see that φ is compact. Indeed, if $\varphi(\overline{D})\subset D$ then $f\circ\varphi\in A(D)$ for every $f\in Hol(D)$ and moreover $\|f\circ\varphi\|_\infty \le \sup_{z\in D}|f(z)| = \|f\|_\infty$.

So the mapping $\varphi: Hol(D)\to A(D)$ of the form $\Phi(f)=f\circ\varphi$ is a continuous linear operator. Since the space $Hol(D)$ is Montelev space (i.e. any bounded subset is relatively compact); then this operator is compact (i.e. for any bounded set $Q\subset Hol(D)$ the set $\varphi(Q)$ is relatively compact in $A(D)$).

We fix some function $u\in X$ and we will consider $T: X\to X, Tf = u\cdot f\circ\varphi$, $f\in X$ and φ as above.

This operator has the following representation

$$T = M_u \circ \Phi \circ i$$

$$X \to_i Hol(D) \to_\Phi H^2(\beta) \to_{M_u} X$$

i is natural embedding $(X,\|\ \|_x)\to(Hol(D),k)$ and M_u is multiplication operator $M_u: H^2(\beta)\to X$; $M_u(f)=uf$. Since by assuming that i is continuous and by (1) we have that M_u is also linear continuous operator then we obtain that operator T is compact.

Corollary. *If the operator $T: X \to X$ has the form $f \mapsto u \cdot f(\varphi)$ (where $u \in H^2(\beta)$ and φ is in* $\mathrm{comp}(X)$*) is compact then we have $\|\varphi\| < 1$, when $u(z) \neq 0$ on the ∂D.*

Proof. It is clear. Indeed, by above theorem if $u \neq 0$ on ∂D then $|\varphi(z)| < 1$ for every $z \in \overline{D}$ in other words $\|\varphi\| < 1$ because φ is a continuous function on $\overline{D}$. □

From corollary we obtained that φ has a fixed point in $\operatorname{int} D$ when $u \neq 0$ on ∂D. This is a type of (for X) the theorem 2 of [3] (in particular when $u = 1$ then T is composition operator).

In this case (also in the case in the remark 2.1) if $z_0 \in D$ is a fixed point for transformation φ, then by using the theorem of [7] we have: the spectrum $\sigma(T)$ of the operator T is equal to

$$\{u(z_0)(\varphi'(z_0))^n : n = 0,1,2,\dots\} \cup \{0\}.$$

3. Weighted composition operators on Bnach modules on "big" Hardy spaces.

In this section we will consider "big" Hardy spaces, i.e. $\{\beta_n\}$ has this condition that $H^2(\beta) \supset A(D)$. In this case $H^2(\beta)$ is called big Hardy space. The following condition on the sequence $\{\beta_n\}$ is sufficient for "big"ness of $H^2(\beta)$.

Lemma 3.1. *If the sequence $\beta = \{\beta\}_{n=0}^{\infty}$ is bounded then $A(D) \subset H^2(\beta)$.*

Proof. Suppose that there exists N such that $\beta_n \le N$ $(n = 0,1,2,\dots)$ and let $f(z) = \sum_{k=0} a_k z^k$; if $f \in H^2$ then $\sum_0^{\infty} |a_k|^2 < \infty$ then $\sum_{k=0}^{\infty} |a_k|^2 \beta_k^2 \le N^2 \sum_{k=0}^{\infty} |a_k|^2 < \infty$, then $f \in H^2(\beta)$ thus $A(D) \subset H^2 \subset H^2(\beta)$, so $H^2(\beta)$ is big Hardy space. If $H^2(\beta)$ is a big Hardy space and X is a $H^2(\beta)$- Banach module the generalization of Kamowitz theorem is obtained which was stated for $A(D)$ as follows:

Theorem 3.1. *If uniformly closed subspace $X \subset C(X)$ is $H^2(\beta)$- Banach module on the big Hardy space then the operator $T: X \to X$ as above is compact then, we have $|\varphi(z)| < 1$ for any $z \in \overline{D}$ such that $u(z) \neq 0$.*

Proof. Since peak set of X contains ∂D then the proof is a result of theorem 1.5 of [6].

Remark 2. If we consider $T: X \to X$ as above, then it is clear that, if $u(z) \neq 0$ on ∂D then we have that $\|\varphi\| < 1$ φ has its fixed point in D. This is a type of theorem in [3], which was proved for $H^2(D)$ and we consider it for X-Banach modules.

4. Nuclear sum of weighted composition operators.

In this section we use the foregoing compactness criterion to investigate the nuclearity of sums of weighted composition operators in case that weighted functions are not vanish anywhere on ∂D. We remember that $T \in L(E,F)$ $(L(E,F))$ is the space of all bounded liner operators from Banach space E to Banach space F) is called nuclear operator if we can represent it in the form $T=\sum_{i=1}^{\infty} a_i \otimes f_i$ where $a_i \in E^*$ (E^* is the dual space of E) and $f_i \in F$ such that $\sum_{n=1}^{\infty}\|a_i\|\|f_i\|<\infty$, i.e. if we can show that the series $\sum_{i=1}^{\infty} T_i$ is absolutely convergent in $L(E,F)$, where $T_i=<.,a_i>f_i$ is a one dimensional operator. Consequently nuclear operators are compact ones. By Gorin and Shahbazov theorem [2] we obtain following theorem (the trivial case $u_i \equiv 0, \varphi=\text{constant}$ is not considered).

Theorem 4.1. *If $u_1 \neq 0, u_2 \neq 0$ and $u_1+u_2 \neq 0$ on ∂D, then operator T of the form $f \mapsto u_1(f \circ \varphi_1)+u_2(f \circ \varphi_2)$, where $\varphi_i \in H^2(\beta)$ and $\|\varphi_i\| \leq 1$, $i=1,2$ is nuclear iff there exists constant $c \in (0,1)$ such that $\|\varphi_i\| \leq c$.*

Proof. Necessity. If T is nuclear, by definition above T is compact, then by Gorin and Shahbazov result $\|\varphi_i\|$ $(i=1,2)$ on ∂D less than 1 and since φ_i is continuous function, then there exists $c \in (0,1)$ such that $\|\varphi_i\|<c$.

Sufficiency. Let $\|\varphi_i\|<c<1$ and $f(z)=\sum_{k=0}^{\infty} a_k(f)z^k \in H^2(\beta)$ then

$$Tf(z)=u_1(z)f(\varphi_1(z))+u_2(z)f(\varphi_2(z))=$$
$$=u_1(z)\sum_{k=0}^{\infty} a_k(f)(\varphi_1(z))^k+u_2(z)\sum_{k=0}^{\infty} a_k(f)(\varphi_2(z))^k=$$
$$=\sum_{k=0}^{\infty} a_k(f)\left(u_1(z)\varphi_1(z)^k+u_2(z)\varphi_2(z)^k\right)$$

since $u_1(z)\varphi_1(z)^k+u_2(z)\varphi_2(z)^k \in A(D)$ and $a_k \in A(D)^*$, and $\|a_k\|=1$ we must show that $\sum\|u_1\varphi_1^k+u_2\varphi_2^k\|<\infty$. Since

$$\sum_{k=0}^{\infty}\|u_1\varphi_1^k+u_2\varphi_2^k\| \leq \sum\|u_1\|\|\varphi_1\|^k+\|u_2\|\|\varphi_2\|^k \leq$$
$$=\sum\left(\|u_1\|+\|u_2\|\right)c^k \text{ and } 0<c<1,$$

which is convergent as a geometric series.

Remark 3. Since $\sum\frac{1}{\beta_n^2}<\infty$ implies that $H^2(\beta) \subset A(D)$ we obtain from above theorem that the result hold for $H^2(\beta)$ (i.e. for $T: H^2(\beta) \rightarrow H^2(\beta)$) in particular, if $\varphi_i \in \text{comp}H^2(\beta)\, i=1,2$ and $u_i\varphi_i^k$ in $H^2(\beta)$ for any $k=0,1,2,\ldots$ then from theorem we obtain same nuclearity criteria for $T: H^2(\beta) \rightarrow H^2(\beta)$.

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Zohri A.
Payame-noor University.
Ardabil, Iran.

Shahbazov A.I.
Institute of Mathematics & Mechanics of NAS of Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20(off.).

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