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ON THE RIEMANN-EARNSHAW INVARIANTS FOR SOME MODELS OF DYNAMICS OF NON-HOMOGENEOUS GASES

Abstract

In this paper, using system of equations consisting of 2-forms for the description of one-dimensional flows of gases, the author constructs Riemann-Earnshaw invariants for the following models of: a) isothermal flows of an ideal non-homogeneous gas under certain connection between the molecular weight of the gas and its temperature; b) adiabatic flows of polytropic non-homogeneous gases under certain connection between the density of non-perturbation gas and its entropy. Also have been constructed Riemann-Earnshaw invariants for flows of homogeneous gases situated in a gravitation field.

As is known the analysis of equations of gas dynamics written in the Riemann-Earnshaw invariants gives valuable information on flows of gases ([1]-[3]). Usually applied method the determination of these invariants for a system of partial differential equation is based on the coefficients for equations of this system. Describing dynamics of one dimensional continuum of a system of equation consisting of 2-forms, it succeeds to get more systematic method of determination of Riemann-Earnshaw invariants and corresponding to them characteristic velocities that was shown in paper [4]. In the present paper the models of 1) adiabatic flows of poly-tropic non-homogeneous gases with variable entropy along particle of the ideal non-homogeneous gas with variable temperature along particle of the gas being continuation of [4] are considered. The Riemann-Earnshaw invariants for flows of homogeneous gases situated in homogeneous gravitational field are also constructed.

Consider the system of the equations in differentials of coordinate of space of states describing one-dimensional flows of gases (see [5])

$$\begin{cases} dq \wedge d\xi - dp \wedge dt + \dfrac{\partial h}{\partial x} dt \wedge d\xi = 0, \\ d\left(\dfrac{\partial h}{\partial p}\right) \wedge d\xi + d\left(\dfrac{\partial h}{\partial q}\right) \wedge dt = 0, \end{cases} \tag{1}$$

where t is time, ξ is a lagrangian coordinate, x is an Euler coordinate, q is density of impulse, p is pressure $(p > 0)$. The quantity h for adiabatic flows of polytropic gases is sum of densities of kinematic energy of macroscopic motion of gas and enthalpy of gas $([h] = \mathrm{J/m^3})$. If the gas is situated in a homogeneous gravitation field, then follows to add a term being the density of potential of this field

$$h = \frac{1}{2\rho_0(\xi)} q^2 + \frac{\gamma}{\gamma - 1} B(\xi) p_0 \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} + \rho_0(\xi) g x, \tag{2}$$

where $\rho_0(\xi)$ is density of gas in non-perturbed state, γ is Poisson coefficient $(\gamma > 1)$; $B(\xi) \equiv \exp\left(\dfrac{S(\xi) - S_0}{C_p(\xi)}\right)$, where $S(\xi)$ is entropy of gas, and $C_p(\xi)$ is heat capacity of gas for constant pressure, g is acceleration of gravity. For isothermal flows of ideal gases situated in a homogeneous gravitation field, the quantity h is sum of densities of kinematics energy of macroscopic motion of gas, Gibb's thermodynamic potential and potential of a gravitation field

$$h=\frac{1}{2\rho_0(\xi)}q^2+\rho_0(\xi)\frac{kT(\xi)}{m(\xi)}\ln\left(\frac{p}{p_0}\right)+\rho_0(\xi)gx, \tag{3}$$

where $m(\xi)$ is mass of molecules of gas, $T(\xi)$ is absolute temperature of gas, k is a Boltsman constant. We'll call the equalities (2) and (3), the equations of determinant surface (in phase space) of models of gases.

As it is shown in paper [4] for the hyperbolic models $\left(h_{pq}^2-h_{pp}h_{qq}>0\right)$ we can construct a system of equations containing only simple 2-forms which are equivalent to the system of equations (1). For determination of coefficients of factors of these simple 2-forms we use an easily verifiable identity (in which it is taken into account that for the functions (2) and (3) the equality $h_{pq}=0$ is satisfied)

$$\begin{aligned}&\widetilde{a}\wedge\widetilde{b}\overset{def}{=}\left(a_1dt+d\xi+a_3dp+a_4dq\right)\wedge\left(b_1dt+b_2d\xi+dp+b_4dq\right)\equiv\\&\equiv\lambda_1\left(dq\wedge d\xi-dp\wedge dt+h_x dt\wedge d\xi\right)+\lambda_2\left(h_{pp}dp\wedge d\xi+h_{qq}dq\wedge dt+h_{q\xi}d\xi\wedge dt\right)+\\&+\left(a_1b_2-b_1-\lambda_1h_x+\lambda_2h_{q\xi}\right)dt\wedge d\xi+\left(a_1-a_3b_1-\lambda_1\right)dt\wedge dp+\\&\left(a_1b_4-a_4b_1+\lambda_2h_{qq}\right)dt\wedge dq+\left(1-a_3b_2+\lambda_2h_{pp}\right)d\xi\wedge dp+\left(b_4-a_4b_2+\lambda_1\right)\times\\&\times d\xi\wedge dq+\left(a_3b_4-a_4\right)dp\wedge dq\end{aligned}$$

in addition it is necessary to put a condition of linear independence of 1 forms $\widetilde{a}$ and $\widetilde{b}$.

Remark. The 2-forms contained in the system of equation (1) are determined to within arbitrary (different from zero) factor, therefore in 1-forms $\widetilde{a}$ and $\widetilde{b}$ one of coefficients is chosen to equal to unit.

From above written identity we obtain a system of six algebraically equations for the coefficients 1 form $\widetilde{a}$ and $\widetilde{b}$ and the coefficients λ_1 and λ_2. Excepting the quantities λ_1 and λ_2 from this system of equations, we obtain the system of the equations

$$\begin{cases}a_3b_4-a_4=0,\\a_1-a_3b_1+b_4-a_4b_2=0,\\h_{pp}\left(a_1b_4-a_4b_1\right)-h_{qq}\left(1-a_3b_2\right)=0,\\a_1b_2-b_1-h_x\left(a_1-a_3b_1\right)+\dfrac{h_{q\xi}}{h_{qq}}\left(a_4b_1-a_1b_4\right)=0.\end{cases} \tag{4}$$

We evaluate in the fourth equation of the system (4) the coefficient b_1 by the other quantities

$$b_1=\frac{a_1\left(h_x+\mu b_4-b_2\right)}{a_3h_x+\mu a_4-1}\qquad\left(\mu\equiv\frac{h_{q\xi}}{h_{qq}}\right). \tag{5}$$

Substituting the expression (5) in the second and third equations of the system (4) and granting that 1-forms $\widetilde{a}$ and $\widetilde{b}$ must be linear independent we obtain the system of equations

$$\begin{cases}a_1+b_4=a_3b_4\left(h_x+\mu b_4\right), & (6)\\h_{pp}a_1b_4=h_{qq}-a_3h_{qq}\left(h_x+\mu b_4\right), & (7)\end{cases}$$

which is a generalization of the system of equations (7) from paper [4]. Excepting the quantity $h_x + \mu b_4$ from the equations (6), (7) and assuming $a_1 \neq 0$ we obtain

$$b_4^{\pm} = \pm\sqrt{-\frac{h_{qq}}{h_{pp}}}. \qquad (8)$$

It is easy to check that for the obtained values $b_4^{\pm}$ by the equations of the system (6), (7) are equivalent. Consequently, choosing one by them, we get the relation between the coefficients a_1 and a_3. We can obtain the relation between the coefficients b_1 and b_2 using the equalities (5) and (6)

$$b_1 = b_4\left(h_x + \mu b_4 - b_2\right). \qquad (9)$$

Thus for hyperbolic models in linear subspace of 2-forms constructed on two 2-forms being in the left parts of the equations of the system (1), the two and simple 2-forms exist

$$\widetilde{a}_{\pm} \wedge \widetilde{b}_{\pm} \overset{def}{=} \left(a_1^{\pm}dt + d\xi + a_3^{\pm}dp + a_3^{\pm}b_4^{\pm}dq\right) \wedge \left(b_1^{\pm}dt + b_2^{\pm}d\xi + dp + b_4^{\pm}dq\right),$$

where $b_4^{\mp}$ are determined by the equality (8), the coefficients $a_1^{\pm}$ and $a_3^{\pm}$ are connected by the equality (6), and the coefficients $b_1^{\pm}$ and $b_2^{\pm}$ with the equality (9). Since these two simple 2-forms are linear independent ([4]), then we can choose them for basis of above mentioned subspace of 2-forms.

Now we find the proportionality condition of 1-forms $\widetilde{b}_{\pm}$ accurately to 1-forms and we find the potentials of these precise 1-forms, which will be Riemann-Earnshaw unknown invariants. Solving these questions we consider separately the following flows:
1) adiabatic (isothermal) flows of non-homogeneous gas with entropy (temperature) along particle on which external field of force doesn't effect;
2) the flows of homogeneous gas with constant entropy (temperature) situated in a gravitation field.

For the first group of flows we have $h_x = 0$. In this case allowing for the relation (9) between the coefficients b_1 and b_2, we write 1-form $\widetilde{b}$ in the next form

$$\widetilde{b} = b_1 dt + \left(\mu b_4 - \frac{b_1}{b_4}\right)d\xi + dp + b_4 dq.$$

As it is known for determination of proportionality condition of 1-form $\widetilde{b}$ precise in 1-form it is necessary to construct the 3-form $\widetilde{b} \wedge d\widetilde{b}$ and to equal all its coefficients to zero. Satisfying this operation we obtain the system of the conditions

$$\begin{cases} b_1\dfrac{\partial}{\partial p}\left(\mu b_4 - \dfrac{b_1}{b_4}\right) - \left(\mu b_4 - \dfrac{b_1}{b_4}\right)\dfrac{\partial}{\partial p}b_1 + \dfrac{\partial}{\partial \xi}b_1 - \dfrac{\partial}{\partial t}\left(\mu b_4 - \dfrac{b_1}{b_4}\right) = 0, & (10) \\ b_4^2\dfrac{\partial}{\partial \xi}\left(\dfrac{b_1}{b_4}\right) - b_2^2\dfrac{\partial}{\partial q}\left(\dfrac{b_1}{b_2}\right) - b_4\dfrac{\partial}{\partial t}b_2 = 0, & (11) \\ b_1\dfrac{\partial}{\partial p}b_4 + \dfrac{\partial}{\partial q}b_1 - b_4\dfrac{\partial}{\partial p}b_1 = 0, & (12) \\ \left(\mu b_4 - \dfrac{b_1}{b_4}\right)\dfrac{\partial}{\partial p}b_4 - b_4\dfrac{\partial}{\partial p}\left(\mu b_4 - \dfrac{b_1}{b_4}\right) + \dfrac{\partial}{\partial q}\left(\mu b_4 - \dfrac{b_1}{b_4}\right) - \dfrac{\partial}{\partial \xi}b_4 = 0. & (13) \end{cases}$$

In the present paper the analysis of the conditions (10)-(13) isn't led, and only it is shown that the simplest hypothesis $b_1 \equiv 0$ gives possibility to construct the Riemann-Earnshaw invariants for some flows of non-homogeneous gas. Really, for $b_1 \equiv 0$ the conditions

(10)-(12) are satisfied identically, since for $b_1 = 0$ the coefficient $b_2^{\pm} = \mu b_4^{\pm}$ and $\mu b_4^{\pm} \equiv \pm \frac{h_{q\xi}}{\sqrt{-h_{pp}h_{qq}}}$ doesn't depend on t. Further from the equations of determinant surface of models (2), (3) it follows that $b_{4,q}^{\pm} \equiv 0$, and $\mu_p \equiv \frac{\partial}{\partial p}\left(\frac{h_{q\xi}}{h_{qq}}\right) = 0$. Allowing for the equalities from (13) we obtain

$$b_4 \mu_q - b_{4,\xi} = 0. \qquad (13')$$

It's easy to check that the condition (13') for the models determined by determinants of surfaces (2) and (3) is reduced to the conditions

$$\frac{d}{\partial \xi}\left(\frac{B(\xi)}{\rho_0(\xi)}\right) \equiv \frac{\partial}{\partial \xi}\left(\frac{1}{\rho_0(\xi)}\exp\left(\frac{S(\xi)-S_0}{C_p(\xi)}\right)\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial \xi}\left(\frac{kT(\xi)}{m(\xi)}\right) = 0 \qquad (14)$$

consequently.

For models determined by the determinant surface (2) (where $g = 0$) we obtain

$$\widetilde{b}_{\pm} = \mu b_4^{\pm} d\xi + dp + b_4^{\pm} dq =$$

$$= b_4^{\pm}\left(\frac{B(\xi)\rho_0(\xi)}{\gamma\, p_0}\right)^{1/2}\left[-\left(\frac{\gamma p_0}{B(\xi)\rho_0(\xi)}\right)^{1/2}\frac{\rho_{0,\xi}}{\rho_0(\xi)} q\, d\xi \pm \left(\frac{p}{p_0}\right)^{-\frac{\gamma+1}{2\gamma}} dp + \left(\frac{\gamma p_0}{B(\xi)\rho_0(\xi)}\right)^{1/2} dq\right],$$

where the 1-form being in square brackets by virtue of the first condition from (14), is a precise 1-form, whose potential is easily found

$$\widetilde{b}_{\pm} = b_4^{\pm}\left(\frac{B(\xi)\rho_0(\xi)}{\gamma\, p_0}\right)^{1/2} p_0 d\left(\left(\frac{\gamma \rho_0(\xi)}{p_0 B(\xi)}\right)^{1/2}\frac{q}{\rho_0(\xi)} \pm \frac{2\gamma}{\gamma-1}\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right) \equiv$$

$$\equiv b_4^{\pm}\left(\frac{B(\xi)\rho_0(\xi)}{\gamma p_0}\right)^{1/2} p_0 dR_1^{\pm}.$$

If in obtained simple 2-forms we choose $a_3^{\pm} = 0$ then the system of equation equivalent to the system of equations (1) will have the form

$$\begin{cases}\left(a_1^{+} dt + d\xi\right) \wedge d\Phi^{+}\left(R_1^{+}\right) = 0,\\ \left(a_1^{-} dt + d\xi\right) \wedge d\Phi^{-}\left(R_1^{-}\right) = 0,\end{cases} \qquad (15)$$

where $a_1^{\pm} = -b_4^{\pm}$, $b_4^{\pm} = \pm\left(\frac{\gamma p_0}{B(\xi)\rho_0(\xi)}\right)^{1/2}\left(\frac{p}{p_0}\right)^{\frac{\gamma+1}{2\gamma}}$, $\Phi^{+}(\cdot)$ and $\Phi^{-}(\cdot)$ are arbitrary smooth functions (in applications we usually take $\Phi^{\pm}(\eta) \equiv c\eta$, where c is constant). Note that for $a_3^{\pm} = 0$ we can interpret the quantity $a_1^{\pm}$ as characteristics velocity depending on the coordinate ξ.

For the models determined by the determinant surface (3) (where $g = 0$) we have

$$\widetilde{b}_{\pm} = \mu b_4^{\pm} d\xi + dp + b_4^{\pm} dq =$$

$$= b_4^{\pm}\rho_0(\xi)\left(\frac{kT(\xi)}{m(\xi)}\right)^{1/2}\left[-\left(\frac{m(\xi)}{kT(\xi)}\right)^{1/2}\frac{\rho_{0,\xi}}{\rho_0^2(\xi)} q\, d\xi \pm \frac{1}{p} dp + \frac{1}{\rho_0(\xi)}\left(\frac{m(\xi)}{kT(\xi)}\right)^{1/2} dq\right],$$

where the 1-form being in square brackets by virtue of the second condition from (14) is a precise 1-form, whose potential is easily found

$$\tilde{b}_{\pm} = b_4^{\pm}\rho_0(\xi)\left(\frac{kT(\xi)}{m(\xi)}\right)^{1/2} d\left(\left(\frac{m(\xi)}{kT(\xi)}\right)^{1/2}\frac{q}{\rho_0(\xi)} \pm \ln\left(\frac{p}{p_0}\right)\right) \equiv b_4^{\pm}\rho_0(\xi)\left(\frac{kT(\xi)}{m(\xi)}\right)^{1/2} dR_2^{\pm}.$$

The system of equations written in Riemann-Eadnshaw invariants for this model, has the form of the system of the equations (15) in which $a_1^{\pm} = -b_4^{\pm} = \mp\left(\frac{m(\xi)}{kT(\xi)}\right)^{1/2}\frac{1}{\rho_0(\xi)}p$ and it follows to substitute the function $R_1^{\pm}$ by $R_2^{\mp}$. From the system of equations (15) it is easy to get both partial differential equations by the coordinate (t,ξ) with the unknown functions $\Phi^+(R^+)$ and $\Phi^-(R^-)$ and partial differential equations by the coordinate $(\Phi^+(R^+), \Phi^-(R^-))$ with respect to the unknown functions $t = t(\Phi^+(R^+), \Phi^-(R^-))$ and $\xi = \xi(\Phi^+(R^+), \Phi^-(R^-))$.

Now consider the flows of homogeneous gas in a gravitation field. In this case for models determined by the determinant surfaces (2) and (3) the next equalities $h_{x\xi} = h_{xq} = h_{q\xi} = 0$ and $b_{4,\xi} = b_{4,q} = 0$ are satisfied. Allowing for these equalities and equating to zero the coefficients of 3-form $\tilde{b} \wedge d\tilde{b}$ we obtain the system of the conditions

$$\begin{cases} Db_{2,p} - b_2 D_p - b_4 b_{2,\xi} - b_{2,t} = 0 & (16)\\ Db_{2,q} - b_2 D_q - b_4^2 b_{2,\xi} - b_4 b_{2,t} = 0 & (17)\\ Db_{4,p} - b_4 D_p - b_4 b_{2,q} = 0 & (18)\\ b_2 b_{4,p} - b_4 b_{2,p} + b_{2,q} = 0 & (19)\end{cases}$$

where $D \equiv b_4(h_x - b_2)$. From the conditions (16), (17) we obtain

$$b_4(p)\frac{\partial}{\partial p}\left(\frac{b_2}{D}\right) - \frac{\partial}{\partial q}\left(\frac{b_2}{D}\right) = 0.$$

Consequently, $\frac{b_2}{D} = \Phi\left(q + \left(\widetilde{\frac{1}{b_4(p)}}\right), t, \xi\right)$, where $\Phi(\cdot)$ is an arbitrary smooth function, and $\left(\widetilde{\frac{1}{b_4(p)}}\right)$ is primitive of the function $\frac{1}{b_4(p)}$. Thus the coefficient b_2 has the form

$$b_2 = \frac{h_x b_4 \Phi(\cdot)}{1 + b_4\Phi(\cdot)}.$$

Substituting this expression of b_2 to the conditions (18), (19) we obtain

$$b_{4,p}\Phi(\cdot) = 0 \text{ and } b_{4,p}\Phi^2(\cdot) = 0$$

consequently. Since for considered models $b_{4,p} \neq 0$ then we obtain $\Phi(\cdot) = 0$, hence it follows that $b_2 = 0$. The fulfillment of the conditions (16), (17) for the obtained values b_2 is obvious. Considering for $b_2 = 0$ the 1-form $\tilde{b} = h_x b_4 dt + dp + b_4 dq$ it is easy to find an integrating factor and potential obtained after the production by integrating factor of precise 1-form

$$\tilde{b}_{\pm} = b_4^{\pm}(p)\left(\rho_0 g dt + \frac{1}{b_4^{\pm}(p)}dp + dq\right) = b_4^{\pm}(p) d\left(q + \left(\frac{\tilde{1}}{b_4^{\pm}(p)}\right) + \rho_0 g t\right) \equiv b_4^{\pm}(p) dR_g^{\pm}.$$

Thus, we can write the flows of gas situated in field of force with density of potential equaled to $\rho_0 gx$ by the system of the equations

$$\begin{cases} \left(a_1^+ dt + d\xi + a_3^+ dp + a_3^+ b_4^+ dq\right) \wedge d\Phi^+\left(R_g^+\right) = 0 \ , \\ \left(a_1^- dt + d\xi + a_3^- dp + a_3^- b_4^- dq\right) \wedge d\Phi^-\left(R_g^-\right) = 0 \ , \end{cases} \tag{20}$$

where a_1 and a_3 are connected by the equality (6) (where $\mu = 0$), and $\Phi^{\pm}(\cdot)$ are arbitrary smooth functions. For determination from the system (20), the partial differential equations by the coordinate (t, ξ) for the unknown functions $\Phi^+\left(R_g^+\right)$ and $\Phi^-\left(R_g^-\right)$ it follows to choose $a_3^{\pm} = 0$. The coefficients $a_1^{\pm}$ for such choice of the values $a_3^{\pm}$ take the values of ordinary characteristic velocities.

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