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ON THE NATURAL VIBRATION OF THE RIGID SUPPORTED COMPOSITE PLATE CONTAINING A CRACK

Abstract

In the framework of the exact equation of motion of the theory of elasticity for anisotropic body the natural vibration of the rigid supposed composite plate strip containing a crack is investigated. It is assumed that the crack edges are parallel to the plate plane and for the solution of the formulated problem FEM is employed. In this case through the special location of nodes in the finite elements containing crack tips the singularity order of stresses and strains at these tips is kept. The numerical results related to the influence of the crack location and of the crack length to the values of t he first mod natural frequencies of the considered plate are given.

There are a lot of investigations related to the free vibration of composite plates with a crack. The review of these investigations can be found in [1,2] and others. It should be noted that mentioned investigations have been carried out in the framework of the various approximate plate theories and relate to the clamped plates. In the present investigation the attempt is made for studying of the clamped composite palate-strip with a crack in the framework of the exact equations of motion of the theory of elasticity of the anisotropic body. Moreover in the present investigation we take the singularity order of the stresses and strains at the crack tips into account.

Thus consider the plate, which occupies the region $\Omega = \{0 \le x_1 \le l, -h/2 \le x_2 \le h/2, \quad -\infty < x_3 < +\infty\}$ and assume that this plate is fabricated from the composite material that is modeled as orthotropic material with normalized mechanical properties and contains crack. The geometry of the plate and the location of the crack are shown in Fig.1. We associate Cartesian coordinate system $Ox_1x_2x_3$ with the plate and suppose that the principal elastic symmetry axes of the plate material are the Ox_1, Ox_2 and Ox_3 axes (Fig.1). Note that the Ox_3 axis is perpendicular to the plane Ox_1, Ox_2 and is not shown in Fig.1.

We write the following governing equations, which are satisfied in the region Ω.

$$\frac{\partial}{\partial x_j}\sigma_{ij} = \rho\frac{\partial^2}{\partial t^2}u_i, \sigma_{11} = A_{11}\varepsilon_{11} + A_{12}\varepsilon_{22}, \quad \sigma_{22} = A_{12}\varepsilon_{11} + A_{22}\varepsilon_{22},$$
$$\sigma_{12} = 2A_{66}\varepsilon_{12}, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}u_i + \frac{\partial}{\partial x_i}u_j\right) \tag{1}$$

In (1) the conventional notation is used.
For the considered case the following boundary conditions are satisfied

$$u_i\Big|_{x_1=0,\, l;\; x_2\in[-h/2,h/2]} = 0, \quad \sigma_{i2}\Big|_{x_2=\pm h/2,\; x_1\in[0,l]} = 0,$$
$$\sigma_{i2}\Big|_{x_2=(h/2-h_u)\pm 0,\, x_1\in[(l-l_0)/2,(l+l_0)/2]} = 0. \tag{2}$$

The sought values are presented as

$$\{\sigma_{ij}, \varepsilon_{ij}, u_i\} = \{\bar{\sigma}_{ij}, \bar{\varepsilon}_{ij}, \bar{u}_i\}e^{i\omega t}. \tag{3}$$

Substituting (3) into (2) we obtain the following equations and boundary conditions for the amplitude of the sought values.

$$\frac{\partial}{\partial x_j}\bar{\sigma}_{ij} + \rho\omega^2\bar{u}_i = 0,\ \ \bar{\sigma}_{11} = A_{11}\bar{\varepsilon}_{11} + A_{12}\bar{\varepsilon}_{22},\ \ \ \bar{\sigma}_{22} = A_{12}\bar{\varepsilon}_{11} + A_{22}\bar{\varepsilon}_{22},$$

$$\bar{\sigma}_{12} = 2A_{66}\bar{\varepsilon}_{12},\ \ \ \ \bar{\varepsilon}_{ji} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\bar{u}_i + \frac{\partial}{\partial x_i}\bar{u}_j\right). \tag{4}$$

$$\bar{u}_i\Big|_{x_1=0,l;\ x_2\in[-h/2,h/2]} = 0,\ \ \bar{\sigma}_{i2}\Big|_{x_2=\pm h/2,\ x_1\in[0,l]} = 0,$$

$$\bar{\sigma}_{i2}\Big|_{x_2=(h/2-h_*)\pm 0,\ x_1\in[(l-l_0)/2,(l+l_0)/2]} = 0. \tag{5}$$

Since, the investigation of the free vibration of the composite plate containing the crack is reduced to the solution of the eigenvalue problem (4)-(5).

Using the Hamilton variational principle we obtain the functional

$$\overline{\Pi} = \frac{1}{2}\iint\limits_{\Omega-L_0}\left[\bar{\sigma}_{ij}\bar{\varepsilon}_{ij} - \rho\omega^2\bar{u}_i\bar{u}_i\right]d\Omega \tag{6}$$

and from $\delta\overline{\Pi} = 0$ we derive the equations (4) and boundary conditions for forces in (5).

The formulated problem is solved by employing FEM and for this purpose the region $\Omega - L_0$ is divided into rectangular finite elements Ω_i with nine nodes and Lagrange family shape functions of the second order [3] are used. In this case taking the symmetry of the problem with respect to $x_1 = l/2$ into account we consider only the part $\{0 \le x_1 \le l/2; -h/2 \le x_2 \le +h/2\}$ and take eight element in the direction of Ox_2 axis and 40 element in the direction of the Ox_1 axis. The parameters of the finite elements are shown in Fig.2. In Fig.2 ξ and η are the local normalized coordinates for Ω_i and

$$\xi = \frac{x_1 - x_{10}}{\beta},\ \ \eta = \frac{x_2 - x_{20}}{\alpha}. \tag{7}$$

For each finite element Ω_i the shape function are as follows

$$N_1 = \frac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta),\ \ N_2 = \frac{1}{4}(\xi^2 + \xi)(\eta^2 - \eta),\ \ N_3 = \frac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta),$$

$$N_4 = \frac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta),\ \ N_5 = \frac{1}{4}(\xi^2 - 1)(\eta^2 - \eta),\ \ N_6 = -\frac{1}{2}(\xi^2 + \xi)(\eta^2 - 1),$$

$$N_7 = -\frac{1}{2}(\xi^2 - 1)(\eta^2 + \eta),\ \ N_8 = -\frac{1}{2}(\xi^2 - \xi)(\eta^2 - 1),\ \ N_9 = (\xi^2 - 1)(\eta^2 - 1). \tag{8}$$

The displacement in Ω_i are approximated by

$$\bar{u}_i \approx \tilde{\bar{u}} = N^i\bar{a}^i, \tag{9}$$

where

$$\left(\bar{a}^i\right)^T = \{\bar{u}^i_{11}, \bar{u}^i_{21}, \bar{u}^i_{12}, \bar{u}^i_{22}, \ldots, \bar{u}^i_{18}, \bar{u}^i_{28}, \bar{u}^i_{19}, \bar{u}^i_{29}\},\ \ N^i = \begin{Bmatrix} N^i_1 & 0 & \ldots & N^i_9 & 0 \\ 0 & N^i_1 & \ldots & 0 & N^i_9 \end{Bmatrix}. \tag{10}$$

In (10) the components of the vector a^i are the values of the displacements at the nodes of Ω_i. The second index of these components shows the number of the node. Moreover in the present investigation for keep of the order of the singularities of the stresses and strains at the crack tips the spatial type finite elements [4] are used. According to [4], for these elements we use also the shape functions (8), however, in these cases instead of (7) the relations

$$\frac{x_1 - x_{10}}{\beta} = \sum_{i=1}^{9} N_i x_{1i},\ \ \ \frac{x_2 - x_{20}}{\alpha} = \sum_{i=1}^{9} N_i x_{2i}, \tag{11}$$

are used. In (11) x_{1i} and x_{2i} are global coordinates of the nodes of the finite elements containing the crack tips (Fig.3). As an example, we consider the finite element denoted by I in Fig.3 and for simplicity of the explanation assume that $x_{10} = \beta$, $x_{20} = \alpha$. In the framework of these assumptions for the element I we obtain the following relation from the (11), (8) and Fig.3.

$$\frac{x_1}{\beta} = \xi^2 + \xi + \frac{1}{2}\left(\xi^2 - 1\right)\left(\frac{1}{2}\left(\eta^2 + \eta\right) - 3\right), \frac{x_2}{\alpha} = \eta^2 + \eta - \frac{1}{2}\left(\eta^2 - 1\right)\left(\frac{1}{2}\left(\xi^2 - \xi\right) + 1\right). \quad (12)$$

Consider the strain ε_{11}^I which, according to (4) and (9), is calculated by expression

$$\bar{\varepsilon}_{11}^I = \frac{\partial \bar{u}_1^I}{\partial x_1} = \sum_{i=1}^{9}\left(\frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x_1} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x_1}\right)\bar{u}_{1i}^I. \quad (13)$$

From (12) we obtain that at $\eta = -1$

$$\xi = 1 + \sqrt{4 - 2\frac{x_1}{\beta}}, \quad \frac{\partial \xi}{\partial x_1} = \frac{-1}{\beta}\frac{1}{\sqrt{4 - 2x_1/\beta}}. \quad (14)$$

It follows from Fig.3 that the crack tip in the element I corresponds to $x_1 = 2\beta$ and according to (13) and (14) under $x_1 \to 2\beta$ or under $r \to 0$, where $r = 2 - x_1/\beta$, $\bar{\varepsilon}_{11}^1 \sim 1/\sqrt{r}$. By the similar manner it is proved that in the finite elements containing the crack tips and shown in Fig.3 the needed singularity order of the stresses and strains are kept.

Thus, in the framework of t he above described solution procedure from

$$\frac{\partial \overline{\Pi}}{\partial \bar{a}}\delta \bar{a} = 0, \quad (\bar{a})^T = \{\bar{a}^1, \bar{a}^2, ..., \bar{a}^L\} \quad (15)$$

we obtain the equation

$$\left(\mathbf{K} - \rho\omega^2\mathbf{M}\right)\bar{a} = 0, \quad (16)$$

where

$$\mathbf{K} = \sum_{k=1}^{L}\mathbf{K}^k, \ \mathbf{K}^k = \{\mathbf{K}_{ij}^k\}, \quad \mathbf{M} = \sum_{k=1}^{L}\mathbf{M}^k, \quad i,j = 1,2,...,9,$$

$$\mathbf{K}_{ij}^k = \iint_{\Omega_k}\left(\mathbf{B}_j^k\right)^T \mathbf{D}^k \mathbf{B}_i^k d\Omega_k, \ \mathbf{M}^k = \rho\iint_{\Omega_k}\left(N^k\right)^T N^k d\Omega_k. \quad (17)$$

In (17) L is a number of finite elements and moreover the following notation is used

$$\mathbf{D}^k = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{pmatrix}, \quad \mathbf{B}_i^k = \begin{pmatrix} \frac{\partial N_1^k}{\partial x_1} & 0 \\ 0 & \frac{\partial N_2^k}{\partial x_2} \\ \frac{\partial N_1^k}{\partial x_2} & \frac{\partial N_2^k}{\partial x_1} \end{pmatrix}. \quad (18)$$

For the obtaining of the numerical results we assume that the plate is a multilayered composite consisting of the alternating layers of two isotropic homogeneous materials. The reinforcing layers will be assumed to be located in planes which are parallel to the plane Ox_1x_3 (Fig.1). Young moduli and Poisson coefficient of these materials we denote by E_k and ν_k $(k = 1,2)$ respectively. It is known that in the continuum approach the above layered composite material is taken as transversally isotropic material with normalized mechanical properties whose isotropy axis lies on the Ox_2 axis. Moreover it

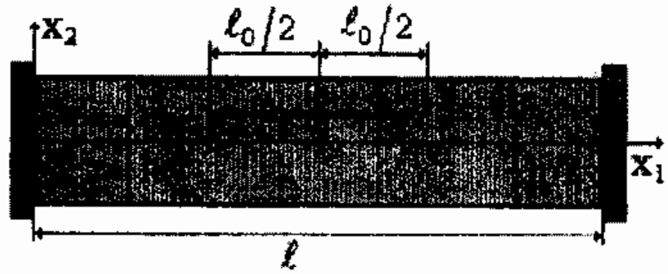

Fig.1

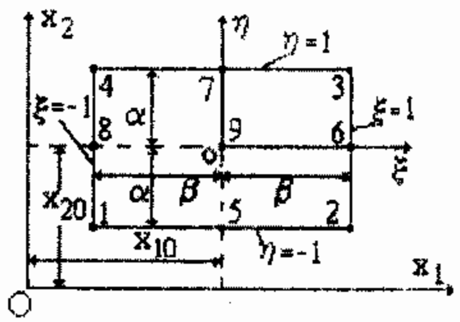

Fig.2

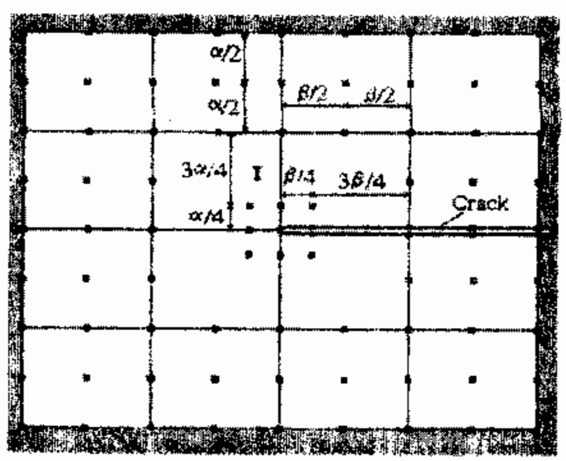

Fig.3

Table 3.

h_u	l_0/l				
	0.9	0.7	0.5	0.3	0.1
$h/2$	1.66	1.93	2.12	2.19	2.20
$3h/8$	1.34	1.84	2.11	2.19	2.20
$h/4$	0.85	1.45	2.06	2.19	2.20

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