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COERCIVE SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH THE CONDITIONS OF CONJUGATIONS FOR THE SECOND ORDER DIFFERENTIAL OPERATOR EQUATIONS

Abstract

In the paper coercive solvability (by a spectral parameter and also by a space variable) of a boundary value problem for the second order differential-operator equations with a discontinuous (piecewise constant) coefficient for the second order derivative is investigated. At the discontinuous point conjugation condition is given.

In the present paper coercive solvability (by a spectral parameter and also by a space variable) of a boundary value problem for the second order differential operator equation with a discontinuous (piecewise constant) coefficient for the second order derivative is investigated.

Being investigated problem is reduced to the invetigation of coercive solvability of a boundary value problem with the conditions of conjugations for a system of the second order elliptic differential operator equations with constant coefficients.

The conjugation problem for a system of differential operator equations with constant coefficients in a finite segment is investigated in papers of V.K. Romanko [1, 2], V.I. Korzyuk, N.I. Yurchuk [3], S.G. Pyatkov [4] and so on.

The used method was previously applied for the study of two-point boundary value problems for elliptic differential opertor equations in papers of S.Ya. Yakubov, B.A. Aliyev [5], I.V. Aliyev, B.A. Aliyev [6], G.I. Laptev [7], S.S. Mirzoyev [8] and others.

Let H be a separable Hilbert space. By $L_p((0,1);H)$ denote a Banach space of the functions $x \to u(x):(0,1) \to H$ strongly measurable and summable in p-th power with the norm

$$\|u\|^p_{L_p((0,1);H)} = \int_0^1 \|u(x)\|^p_H \, dx < \infty , \quad p \in (1,+\infty).$$

Let A be a strongly positive operator in H, i.e. such that its domain of definition $D(A)$ is dense in H for some $\delta \in (0,\pi)$, all points of a complex plane from the angle $|\arg\lambda| < \delta$ belong to the resolvent set and the resolvent satisfies the estimation $\|(A-\lambda I)^{-1}\| \le c(1+|\lambda|)^{-1}$, where $c = const > 0$, I is a unique operator in H.

Let A be a strongly positive operator in H. Then

$$(H,H(A^n))_{\theta,p} = \Big\{ u : u \in H,\ \|u\|_{(H,H(A^n))_{\theta,p}} = = \int_0^{+\infty} t^{n(1-\theta)p-1} \|A^n e^{-tA} u\|^p_H \, dt < \infty,\ p>1,\ 0<\theta<1,\ n \in N \Big\}.$$

$(H,H(A^n))_{\theta,p}$ is an interpolational space between $H(A^n)$ and H ([9, p.109]).

By $W^n_p((0,1);H(A^n),H) = \{u : A^n u, u^{(n)} \in L_p((0,1);H)\}$ denote a space of vector functions with the norm

$$\|u\|_{W_p^n((0,1);H(A^n),H)} = \|A^n u\|_{L_p((0,1);H)} + \|u^{(n)}\|_{L_p((0,1);H)}.$$

It's known that [9, p.46] if $u \in W_p^n((0,1);H(A^n),H)$ then

$$u^{(j)}(\cdot) \in \left(H, H(A^n)\right)_{\frac{n-j-1/p}{n},p}, \qquad j = 0 \div (n-1).$$

Consider the following equation in H

$$-a(x)u''(x) + Au(x) - \lambda u(x) = f(x), \quad x \in [0,b) \cup (b,1] = \Omega \tag{1}$$

with the boundary conditions

$$\begin{aligned} L_1 u &\equiv \alpha_1 u^{(m_1)}(0) + \beta_1 u^{(m_1)}(b-0) = 0, \\ L_2 u &\equiv \alpha_2 u^{(m_2)}(b+0) + \beta_2 u^{(m_2)}(1) = 0, \end{aligned} \tag{2}$$

where A is a strongly positive operator in H, λ, is a complex parameter, α_i, β_i $(i=1,2)$ are complex numbers,

$$a(x) = \begin{cases} a_1 > 0, \ x \in [0,b), \ b \in (0,1), \\ a_2 > 0, \ x \in (b,1], \ a_1 \neq a_2 . \end{cases}$$

At the point $x = b$ we impose to the function $u(x)$ the additional conditions (the conjugation conditions)

$$\begin{aligned} L_3 u &\equiv \gamma_1 u^{(m_3)}(b-0) + \delta_1 u^{(m_3)}(b+0) = 0, \\ L_4 u &\equiv \gamma_2 u^{(m_4)}(b-0) + \delta_2 u^{(m_4)}(b+0) = 0, \end{aligned} \tag{3}$$

where γ_i, δ_i $(i=1,2)$ are complex numbers, $u^{(m_\nu)}(b-0)$, $u^{(m_\nu)}(b+0)$ are left and right hand limit values of $u^{(m_\nu)}(x)$ at the point $x = b$. $m_\nu = 0$ or 1 at each $\nu = 1 \div 4$.

In the space $L_p(\Omega;H)$ $(p>1)$ we put to the problem (1)-(3) accordingly the operator L determined by the equalities

$$D(L) = W_p^2\left(\Omega; H(A), H, L, L_\nu u\big|_{\nu=1}^{4} = 0\right) = \{u(x): \text{ for almost every } x \in \Omega \ u(x) \in D(A),$$
$$Au(x), u''(x) \in L_p(\Omega;H) \text{ and satisfies the conditions (2) and (3)}\},$$
$$Lu = -a(x)u''(x) + Au(x).$$

Theorem. *Assume that*

1. *A is a positive operator in H;*
2. $\gamma_1\delta_2\left(-\sqrt{a_2}\right)^{m_3-m_4} - \delta_1\gamma_2\left(\sqrt{a_1}\right)^{m_3-m_4} \neq 0, \quad \alpha_1, \beta_2 \neq 0.$

Then for some $r > 0$ all points of a complex plane from the angle $|\arg\lambda| > \delta$ by module more than r belong to a resolvent set of the operator L and for the solution of the equation $(L - \lambda I)u = f$ the estimation

$$\|a(x)u''(x)\|_{L_p(\Omega;H)} + \|Au\|_{L_p(\Omega;H)} + |\lambda|\|u\|_{L_p(\Omega;H)} \le c\|f\|_{L_p(\Omega;H)}$$

is valid.

Proof. Let's agree to write in the form of $\{u_1, u_2\}$ every function $u \in L_p(\Omega;H)$ whose contraction on $L_p((0,b);H)$ and $L_p((b,1);H)$ coincides with $u_1(x)$ and $u_2(x)$ respectively.

Then the equation (1) falls to a system of the second order differential operator equations in the direct sum $L_p((0,b);H) \oplus L_p((b,1);H)$

$$-a_1 u_1''(x) + Au_1(x) - \lambda u_1(x) = f_1(x), \ x \in (0,b), \tag{4}$$

$$-a_2 u_{21}''(x) + Au_2(x) - \lambda u_2(x) = f_2(x), \ x \in (b,1) \tag{5}$$

and the boundary conditions (2) and (3) take the form respectively

$$\begin{aligned}&\alpha_1 u_1^{(m_1)}(0)+\beta_1 u_1^{(m_1)}(b)=0,\\&\alpha_2 u_2^{(m_2)}(b)+\beta_2 u_2^{(m_2)}(1)=0;\end{aligned}\tag{6}$$

$$\begin{aligned}&\gamma_1 u_1^{(m_3)}(b)+\delta_1 u_2^{(m_3)}(b)=0,\\&\gamma_2 u_1^{(m_4)}(b)+\delta_2 u_2^{(m_4)}(b)=0,\end{aligned}\tag{7}$$

where $f_1(x), f_2(x)$ are contractions of $f(x)\in L_p(\Omega;H)$ on $L_p((0,b);H)$ and on $L_p((b,1);H)$ respectively.

It's known that $L_p((0,b);H)\oplus L_p((b,1);H)$ with the norm

$$\|\{u_1,u_2\}\|=\|u_1\|_{L_p((0,b);H)}+\|u_2\|_{L_p((b,1);H)}$$

is a Banach space.

We'll find a solution of the problem (4)-(7) from the direct sum

$$W_p^2((0,b);H(A),H)\oplus W_p^2((b,1);H(A),H).$$

We represent the solution of the problem (4)-(7) in the form of the sum

$$\{u_1,u_2\}=\{u_{11},u_{21}\}+\{u_{12},u_{22}\},$$

where $\{u_{11},u_{21}\}$ is a contraction on $[0,b]\times[b,1]$ of the solution of the system

$$\begin{aligned}&-a_1\widetilde{u}_{11}''(x)+(A-\lambda I)\widetilde{u}_{11}(x)=\widetilde{f}_1(x),\quad x\in(-\infty,+\infty),\\&-a_2\widetilde{u}_{21}''(x)+(A-\lambda I)\widetilde{u}_{21}(x)=\widetilde{f}_2(x),\quad x\in(-\infty,+\infty),\end{aligned}$$

where

$$\widetilde{f}_1(x)=\begin{cases}f_1(x), & x\in[0,b)\\ 0, & x\,\overline{\in}\,[0,b)\end{cases};\quad \widetilde{f}_2(x)=\begin{cases}f_2(x), & x\in(b,1]\\ 0, & x\,\overline{\in}\,(b,1]\end{cases}$$

and $\{u_{12},u_{22}\}$ is the solution of the system

$$\begin{aligned}&-a_1u_{12}''(x)+(A-\lambda I)u_{12}(x)=0,\quad x\in(0,b),\\&-a_2u_{22}''(x)+(A-\lambda I)u_{22}(x)=0,\quad x\in(b,1),\end{aligned}$$

with the boundary conditions

$$\begin{aligned}&\alpha_1 u_{12}^{(m_1)}(0)+\beta_1 u_{12}^{(m_1)}(b)=-\alpha_1 u_{11}^{(m_1)}(0)-\beta_1 u_{11}^{(m_1)}(b),\\&\alpha_2 u_{22}^{(m_2)}(b)+\beta_2 u_{22}^{(m_2)}(1)=-\alpha_2 u_{21}^{(m_2)}(b)-\beta_2 u_{21}^{(m_2)}(1)\end{aligned}$$

and with the conjugation conditions

$$\begin{aligned}&\gamma_1 u_{12}^{(m_3)}(b)+\delta_1 u_{22}^{(m_3)}(b)=-\gamma_1 u_{11}^{(m_3)}(b)-\delta_1 u_{21}^{(m_3)}(b),\\&\gamma_2 u_{12}^{(m_4)}(b)+\delta_2 u_{22}^{(m_4)}(b)=-\gamma_2 u_{11}^{(m_4)}(b)-\delta_2 u_{21}^{(m_4)}(b).\end{aligned}$$

Further proof of this theorem is analogous to theorem 1 of paper [6].

Note that the correct and univalent solvability of boundary value problems for the second order differential-operator equations with a discontinuous coefficient in Hilbert space is studied in [10].

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