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# STATISTICAL DISTRIBUTION OF GAUSSIAN CURVATURE OF SEA SURFACE AT THE MIRROR REFLECTION POINTS


## Abstract

*The statistical distribution of Gaussian (complete) curvature at the points of mirror reflection is discussed. Theoretical expression of this distribution and its asymptotics are compared with the experimentally determined distributions. Behavior of the distribution for small and large values of radius of the curvature, and also the power law "minus three", are investigated. It's shown that the distribution acquires sufficiently simple form for the large values of radii in magnitude of curvature.*


**1. Introduction.** As is known [1,2] the statistical characteristics of brightness of the light reflected from the troubled look sea surface are determined by means of the statistical characteristics of radii of curvature at the points of mirror reflection and by means of number of mirror points. Therefore the investigation of the statistical characteristics of mirror points has a great significance for a distant optical sensing of sea surface state.

**2. Main formulas.** The statistical distribution of Gaussian (complete) curvature $\omega=\zeta_{xx}\zeta_{yy}-\zeta_{xy}^2$ for the homogeneous Gaussian random surface $z=\zeta(x,y)$ at the points of mirror reflection with gradients $\zeta_x(x,y)\equiv\gamma_x=0,\ \zeta_y(x,y)\equiv\gamma_y=0$ has been obtained by Longuet-Higgins [3]. The density of the distribution $w(\omega)$ derived in [3] was expressed by contour integral in a complex plane. Comparatively simple integral representation for the same density of distribution, convenient for practical computations was found by Gardashov [4]. This representation written for the dimensionless curvature $\overline{\omega}=\dfrac{\omega}{(3H)^{\frac{1}{2}}}$ has the following form:

$$w_-(\overline{\omega})=\frac{1}{\Phi(t)}\frac{\left(t^2-t+1\right)^{5/4}}{\sqrt{t(1-t)}}(-\overline{\omega})e^{-\overline{\omega}\sqrt{t^2-t+1}}\int_0^{\pi/2}\frac{e^{\overline{\omega}m(\alpha)}}{\sqrt{m(\alpha)}}d\alpha \tag{1}$$

for $\overline{\omega}<0$, and

$$w_+(\overline{\omega})=\frac{1}{\Phi(t)}\frac{\left(t^2-t+1\right)^{5/4}}{\sqrt{t(1-t)}}\overline{\omega}e^{-\overline{\omega}\sqrt{t^2-t+1}}\int_0^{\pi/2}\frac{e^{\overline{\omega}m(\alpha)}}{\sqrt{m(\alpha)}}\left[1-F\left(\sqrt{\overline{\omega}m(\alpha)}\right)\right]d\alpha \tag{2}$$

for $\overline{\omega}>0$, where $\Phi(t)$ is expressed by the elliptic integrals [5]

$$K(k)=\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\alpha}}d\alpha \text{ and } E(k)\int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\,d\alpha$$

by the formula:

$$\Phi(t)\sqrt{t(1-t)}\left[\sqrt{\frac{1+t}{t}}E(k)-\sqrt{\frac{t}{1+t}}K(k)\right],\text{ where } k=\sqrt{\frac{1-2t}{1-t^2}}.$$

Actually, $\Phi(t)$ is a monotonically decreasing function, correspondingly with maximum and minimum values: $\Phi(0)=1$ and $\Phi\left(\frac{1}{2}\right)=\frac{\pi}{2\sqrt{3}}\approx 0.907$. We denote by $F(x)$

the function of errors $F(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, and $m(\alpha)$ is determined as $m(\alpha)=\frac{(t+1)\sqrt{t^2-t+1}}{t}\left(1-k^2\sin^2\alpha\right)$. The dimensionless parameter $t$ characterizes the structure of sea-way and can receive values from the interval $\left[0,\frac{1}{2}\right]$. The lower value $t=0$ corresponds to two system of waves, cut across at an angle, and the upper value $t=\frac{1}{2}$ may hold in the different cases, for example, for the isotropic surface, or in the case when the angular distribution of energy has the specific "peak" [3]. The parameter $t$ is determined by means of the energy spectrum $E(k_x,k_y)$ of sea-way in the following way. At first moments of the energy spectrum

$$m_{pq}=\int\int_{-\infty}^{\infty} E(k_x,k_y)k_x^p k_y^q dk_x dk_y \tag{3}$$

and then the quantities

$$H=\frac{1}{3}\left(m_{40}m_{04}-4m_{31}m_{13}+3m_{22}^2\right) \tag{4}$$

and

$$\Delta=\det\begin{pmatrix} m_{40} & m_{31} & m_{22} \\ m_{31} & m_{22} & m_{13} \\ m_{22} & m_{13} & m_{04}\end{pmatrix} \tag{5}$$

are calculated.

Further the roots $l_1, l_2, l_3$ of the cubic equation

$$4l^3-3Hl-\Delta=0 \tag{6}$$

are found.

One can show that all roots are real and satisfy the following conditions [3]

$$\left.\begin{aligned} & l_1+l_2+l_3=0 \\ & l_2l_3+l_3l_1+l_1l_2=-\frac{3}{4}H\le 0 \\ & l_1l_2l_3=\frac{1}{4}\Delta\ge 0 \\ & l_3\le l_2\le 0\le l_1 \end{aligned}\right\} \tag{7}$$

The parameter $t$ is represented as $t=-\frac{l_2}{l_1}$.

Note that, the quantity $H$ is related with the average $\omega^2$ by: $3H=average\{\omega^2\}$, where the averaging is taken in all points of the surface $z=\zeta(x,y)$ (in contrast to the averaging only in mirror points).

The asymptotical expression for the distribution $w(\overline{\omega})$ also has been obtained [4]. It has the following form:

$$w_-(\overline{\omega})\approx\frac{1}{2\Phi(t)}\sqrt{\frac{\pi(1-t)}{(1-2t)(2-t)}}\left(t^2-t+1\right)^{3/4}\sqrt{-\overline{\omega}}\,e^{\overline{\omega}\frac{\sqrt{t^2-t+1}}{1-t}}, \quad \overline{\omega}\to-\infty, \tag{8}$$

$$w_+(\overline{\omega}) \approx \frac{1}{2\Phi(t)}\sqrt{\frac{\pi}{(1+t)(2-t)}}\left(t^2-t+1\right)^{3/4}\sqrt{\overline{\omega}}\,e^{-\overline{\omega}\sqrt{t^2-t+1}}, \quad \overline{\omega}\to+\infty . \tag{9}$$

For the expansion $w(\overline{\omega})$ in the neighborhood $\overline{\omega}=0$ we obtain

$$w_-(\overline{\omega}) \approx \frac{1}{\Phi(t)}\frac{\left(t^2-t+1\right)^{5/4}}{\sqrt{t(1-t)}}\overline{\omega}\left(\sqrt{\frac{t}{(t+1)\sqrt{t^2-t+1}}}K(k)\right), \quad \overline{\omega}\to 0^- , \tag{10}$$

$$w_+(\overline{\omega}) \approx \frac{1}{\Phi(t)}\frac{\left(t^2-t+1\right)^{5/4}}{\sqrt{t(1-t)}}\overline{\omega}\left(\sqrt{\frac{t}{(t+1)\sqrt{t^2-t+1}}}K(k)-\sqrt{\pi\overline{\omega}}\right), \quad \overline{\omega}\to 0^+ . \tag{11}$$

In the problems of reflection of light from the disturbed look sea-surface, the learning of distribution of the quantity $\overline{\rho}=\frac{1}{|\overline{\omega}|}$ ("the square of curvature radius"), which may be obtained from $w(\overline{\omega})$ immediately has a great significance

$$W(\overline{\rho})=\frac{1}{\overline{\rho}^2}\left[w_-\left(-\frac{1}{|\overline{\rho}|}\right)+w_+\left(\frac{1}{|\overline{\rho}|}\right)\right]. \tag{12}$$

With the help of (10) and (11) for the asymptotics $W(\overline{\rho})$ at $\overline{\rho}\to+\infty$ we can find:

$$W(\overline{\rho}) \approx \frac{1}{\Phi(t)}\frac{\left(t^2-t+1\right)^{5/4}}{\sqrt{t(1-t)}}\frac{1}{\overline{\rho}^3}\left(2\sqrt{\frac{t}{(t+1)\sqrt{t^2-t+1}}}K(k)-\sqrt{\frac{\pi}{\overline{\rho}}}\right). \tag{13}$$

The density of the distribution $w(\overline{\omega})$ can be expressed by the known functions [4] in the following two limiting cases.

**Case 1$^0$:** Let $t=0$. Then for $w(\overline{\omega})$ we've:

$$w(\overline{\omega})=\frac{1}{2}|\overline{\omega}|K_0(|\overline{\omega}|), \tag{14}$$

where $K_0(x)=\int_1^{+\infty}\frac{e^{-xt}}{\sqrt{t^2-1}}dt$ is the modified Bessel function of an imaginary argument. In this connection the asymptotic expression for the density $w(\overline{\omega})$ and its expansion in the neighborhood $\overline{\omega}=0$ correspondingly has the form:

$$w(\overline{\omega}) \approx \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{|\overline{\omega}|}e^{-|\overline{\omega}|}, \quad \overline{\omega}\to\infty , \tag{15}$$

$$w(\overline{\omega}) \approx \frac{1}{2}|\overline{\omega}|\ln\frac{1}{|\overline{\omega}|}, \quad |\overline{\omega}|\to 0 . \tag{16}$$

Then for the density of the distribution $W(\overline{\rho})$ and its asymptotes we obtain:

$$W(\overline{\rho})=\frac{1}{\overline{\rho}^3}K_0\left(\frac{1}{\overline{\rho}}\right), \tag{17}$$

$$W(\overline{\rho}) \approx \frac{\ln\overline{\rho}}{\overline{\rho}^3}, \quad \overline{\rho}\to+\infty . \tag{18}$$

**Case 2$^0$:** Let $t=\frac{1}{2}$. Then the density of the distribution gets the form:

$$w_-(\overline{\omega})=\frac{3}{2}(-\overline{\omega})e^{\sqrt{3}\overline{\omega}}, \quad \overline{\omega}<0 , \tag{19}$$

$$w_+(\overline{\omega})=\frac{3}{2}\overline{\omega}e^{\sqrt{3\overline{\omega}}}\left[1-F\left(\sqrt{\frac{3\sqrt{3}}{2}\overline{\omega}}\right)\right],\quad \overline{\omega}>0, \tag{20}$$

which is asymmetric and has the asymptotics:

$$w_-(\overline{\omega})\approx\frac{3}{2}(-\overline{\omega})e^{\sqrt{3\overline{\omega}}},\quad \overline{\omega}\to-\infty, \tag{21}$$

$$w_+(\overline{\omega})\approx\sqrt{\frac{\sqrt{3}}{2\pi}}\sqrt{\overline{\omega}}\,e^{-\frac{\sqrt{3}}{2}\overline{\omega}},\quad \overline{\omega}\to+\infty \tag{22}$$

and also the expansion in the neighborhood $\overline{\omega}=0$

$$w_-(\overline{\omega})\approx\frac{3}{2}(-\overline{\omega})\left(1-\sqrt{3}\overline{\omega}\right),\quad \overline{\omega}\to0^-, \tag{23}$$

$$w_+(\overline{\omega})\approx\frac{3}{2}\overline{\omega}\left(1-2\sqrt{\frac{3\sqrt{3}}{2\pi}}\sqrt{\overline{\omega}}+\sqrt{3}\overline{\omega}\right),\quad \overline{\omega}\to0^+. \tag{24}$$

Then for the asymptotics of density of the distribution $W(\overline{\rho})$ we obtain:

$$W(\overline{\rho})\approx\frac{3}{\overline{\rho}^3}\left(1-\sqrt{\frac{3\sqrt{3}}{2\pi}}\frac{1}{\sqrt{\overline{\rho}}}\right),\quad \overline{\rho}\to+\infty. \tag{25}$$

Finally, since the statistical distribution of the quantity $\overline{\omega}$ has already been found, the distribution of the dimensionless curvature $\overline{\Omega}=\dfrac{\Omega}{(3H)^{\frac{1}{2}}}$ and the quantities $\overline{R}=\dfrac{1}{|\overline{\Omega}|}$ at the points of mirror reflection with gradients $\zeta_x(x,y)\equiv\gamma_x$, $\zeta_y(x,y)\equiv\gamma_y$ can be immediately determined:

$$W_q(\overline{\Omega})=qw(q\overline{\Omega}), \tag{26}$$

$$W_*(\overline{R})=\frac{1}{\overline{R}^2}W_q(\overline{\Omega}), \tag{27}$$

where $\overline{\Omega}=\dfrac{\overline{\omega}}{q}$ and $q=\left(1+\gamma_x^2+\gamma_y^2\right)^2$.

The average value of the quantity $\overline{R}$ is:

$$\langle\overline{R}\rangle=\int_0^{+\infty}\overline{R}W_*(\overline{R})d\overline{R}=\frac{\pi q\sqrt{t^2-t+1}}{2\Phi(t)}. \tag{28}$$

The second and higher moments of density of the distribution $W_*(\overline{R})$ don`t exist.

Note that the average number of mirror points in the unit of area $<N>$ with gradients $\overline{\zeta}_x(x,y)=\gamma_k$, $\overline{\zeta}_y(x,y)=\gamma_y$, is determined by the relation:

$$\langle N\rangle=\frac{2}{\pi}\sqrt{\frac{3H}{t^2-t+1}}\Phi(t)w_2(\gamma_x,\gamma_y), \tag{29}$$

where $w_2(\gamma_x,\gamma_y)$ is a density of distribution of surface gradients.

From (27) and (28) we see that the product of averages of "the square f curvature radius" $\overline{R}$ and the number of mirror points $N$, is proportional to the density of distribution of surface gradients

$$\langle N\rangle\langle\overline{R}\rangle = q\sqrt{3H}\,w_2(\gamma_x,\gamma_y). \tag{30}$$

For the two-dimensional (cylindrical) random homogeneous surface $z=\zeta(x)$, the density of distribution of the curvature radii $r=\frac{(1+\gamma^2)^{\frac{3}{2}}}{|\zeta''(x)|}$ at the points of the mirror reflection $\zeta'(x)=\gamma$, has been obtained in article [5].

This density written for dimensionless radius of the curvature $X=\sqrt{\pi}\frac{r}{\langle r\rangle}$, where $\langle r\rangle=\sqrt{\frac{\pi}{2}}\frac{(1+\gamma^2)^{\frac{3}{2}}}{\sigma_2}$, $\sigma_2^2=\langle\zeta''^2(x)\rangle$ has the very simple form:

$$W_0(X)=\frac{2}{X^3}\exp\left(-\frac{1}{X^2}\right) \tag{31}$$

for Gaussian surface.

As is obvious, the density of the distribution $W_0(X)$ doesn't contain any parameter, i.e. it is universal for all two-dimensional Gaussian surfaces. From (31) it is seen that the asymptotics of the density $W_0(X)$ for $x\to+\infty$ is

$$W_0(X)\approx\frac{2}{X^3}. \tag{32}$$

Correctness of the obtained formulas for the density of the distribution $w(\omega)$ has been verified with the help of numerical experiments containing different configurations of irradiation observation and wide range of wave situations. The results of numerical experiments are well conformed to the formulas (1) and (2). As statistic of number of mirror points increases and the accuracy of estimations increase, the difference vanishes. Besides, the estimations by our formulas exactly coincide with estimations by Longuet-Higgins formulas. For example, the density of the distribution $W(\overline{\rho})$, calculated by (12) for two values of the parameter $t$ has been represented in fig. 1. Comparison of these curves with the analogous curves in fig. 2 from the Longuet-Higgins paper [3] shows their complete coincidence.

As is obvious from fig.1, the curves $0\le\overline{\rho}\le 5$ are deformed very little for all possible values of the parameter $t$ in the interval $W(\overline{\rho})$. However, as is obvious from fig. 2, as $\overline{\rho}$ increases, the curves begin to diverge, since they have different asymptotics described correspondingly by formulas (17), (13) and (25). As follows from these formulas the density $W(\overline{\rho})$ has asymptotics $\sim\frac{1}{\overline{\rho}^3}$ for the values $0<t\le\frac{1}{2}$, and the asymptotics $\sim\frac{\ln\overline{\rho}}{\overline{\rho}^3}$ for $t=0$.

In a number of articles, for example in [6, 7, 8] the statistic characteristics of mirror points are studied experimentally. Generalization of results of serial measuring realized both in a tank and in open ocean, are given in Nosov and Pashin's paper [7]. There it is stated that density of the distribution $W(\overline{\rho})$ can be approximated by the power law $\sim\frac{1}{\overline{\rho}^\alpha}$, where $\alpha$ linearly depends on velocity of wind: $\alpha=0.95+o.08v$. In these experiments the quantity $\alpha$ was found in limits from 1.1 up to 1.8 depending on meteorological conditions. In paper [7] for the quantity $\alpha$ it is found the values from 2 up to 2.5. But the theoretical formulas (13), (17) and (25) show that in remainder of the curve $W(\overline{\rho})$ must be $\alpha = 3$. We attempted to explain causes of this divergence. The matter in this that the different parts of theoretical curve given by formulas (1) and (2) represented in fig. 1 and 2 can be approximated by the power law $\approx \frac{A}{\overline{\rho}^{\alpha}}$ with corresponding values of parameters $A$ and $\alpha$. We give the values of parameters $A$ and $\alpha$ found by the least square method for the different parts of theoretical curve (i.e. for the different intervals $(a,b)$ the variation of dimensionless "square of radius of curvature" $\overline{\rho}$) in table 1.

**Table 1.**

The values of parameter $A$ and $\alpha$ for the different intervals $(a,b)$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a | 0.5 | 1.0 | 2.0 | 3.0 | 3.0 | 3.0 | 5.0 |
| b | 5.0 | 5.0 | 5.0 | 5.0 | 10.0 | 20.0 | 30.0 |
| $\alpha$ | 1.37 | 1.92 | 2.31 | 2.46 | 2.52 | 2.52 | 2.68 |
| A | 0.37 | 0.45 | 0.61 | 0.74 | 0.80 | 0.80 | 1.0 |

As is obvious from table 1 as deleting the considered part of curve to its remainder, the exponent $\alpha$ increases tending to its own theoretical limit $\alpha = 3$.

The obtained experimental dependence of the exponent $\alpha = 0.95 + 0.08v$ on velocity of wind $v$ has been stipulated just on passing to dimensional "square radius of curvature" one and the same part of the curve corresponds to different velocities of wind. Indeed, the dimensionless $\overline{\rho}$ and dimensional $\rho$ "square of radii of curvature" are related with ratii: $\rho = \frac{\overline{\rho}}{(3H)^{\frac{1}{2}}}$, where the quantity $H$ definable by (4) depends on moments of energy spectrum and by the same taken on velocity of wind. The calculated values of quantity $H$ for the different velocities of wind $v$ by using the spectrum of sea-way from [9] are led in table 2. For adequacy of comparison with experimental dates from [7] we cut off the spectrum length $\lambda_{min} = 3\,\text{mm}$; since shorter waves were lying beyond the limits of sensibility of receiving mounting. The boundary values for the dimensional "square of radius of curvature" $\rho = \rho_a$ corresponding to the values of the dimensionless "square of radius of curvature" $\overline{\rho} = 10$, beginning from this one can consider that the theoretical curves in fig. 2 are get out to asymptotics, i.e. the formula (13) is correct, also is given in table

**Table 2.**

The density of quantities $H$ and $\rho_a$ on velocity of wind $v$

| | | | | |
|---|---|---|---|---|
| v (m/s) | 1 | 2 | 3 | 4 |
| H ($m^4$) | 0.1557+3 | 0.1778+3 | 0.2129+7 | 0.1890+8 |
| $\rho_a$ ($mm^2$) | 465000 | 21500 | 3300 | 1300 |
| v (m/s) | 5 | 6 | 7 | 8 |
| H ($m^4$) | 0.8321+8 | 0.2335+9 | 0.4853+9 | 0.8261+9 |
| $\rho_a$ ($mm^2$) | 650 | 375 | 260 | 200 |
| v (m/s) | 9 | 10 | 11 | 12 |
| H ($m^4$) | 0.1232+10 | 0.1688+10 | 0.2192+10 | 0.2751+10 |

| $\rho_o$ (mm$^2$) | 165 | 75 | 47 | 11 |
|---|---|---|---|---|
| v (m/s) | 13 | 14 | 15 | 16 |
| H (m$^4$) | 0.3378+10 | 0.4086+10 | 0.4887+10 | 0.5791+10 |
| $\rho_a$ (mm$^2$) | 10 | 9 | 8 | 7.5 |

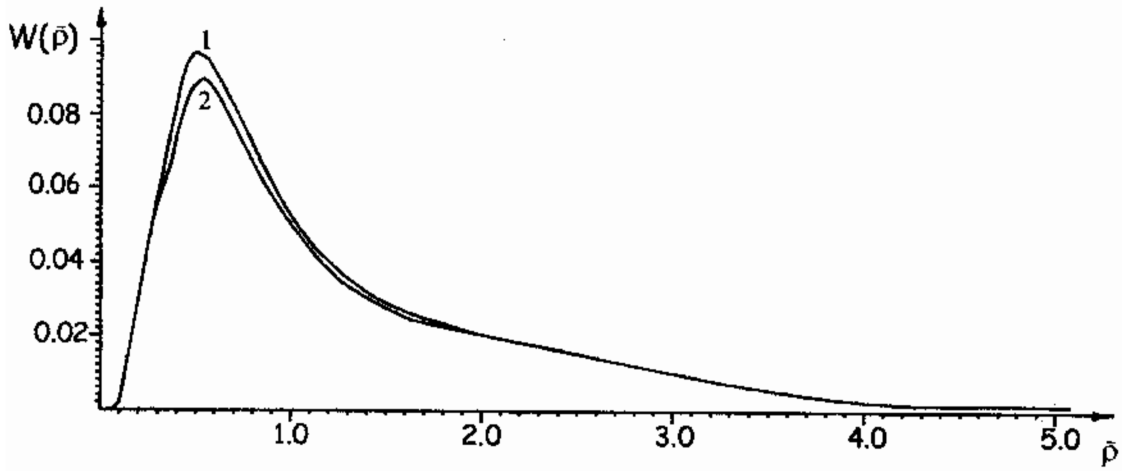


**Fig. 1.** The density of distribution $W(\bar{\rho})$ calculated by (12) by means of the formulas (1) and (2), for two values of parameter $t$; curve 1 at $t=0$; curve 2 at $t=\frac{1}{2}$.

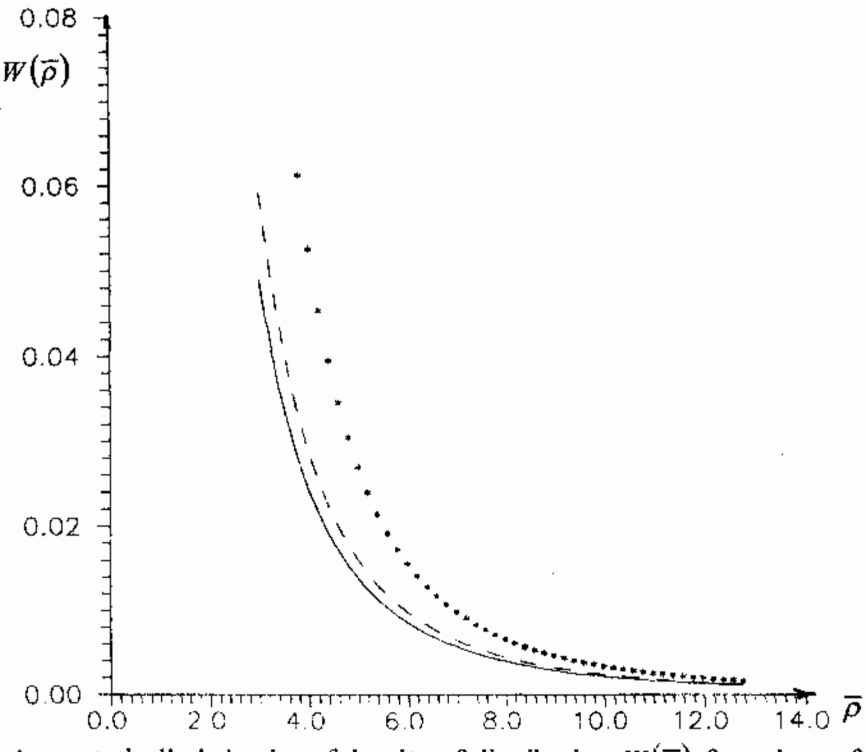


**Fig. 2.** Asymptotically behavior of density of distribution $W(\bar{\rho})$ for values of parameter $t=0.3$; continuous curve- by (12); dotted line by (25); asterisks- by $W(\bar{\rho})=\dfrac{3}{\bar{\rho}^3}$.

**3. Conclusion.** Abstracting the stated above we can state that the power law of distribution of radii of curvature with the exponent depending on velocity of wind may be of satisfactory approximation only for the specific interval of radii of curvature but not for the very large radii of curvature, as the strict asymptotics always has the form $\sim \frac{1}{\overline{\rho}^{\alpha}}$ for $\overline{\rho} \to +\infty$. The another cause of that the bound in experiment exponent is less than 3, apparently that is, the real sea-way differ from Gaussian one dimensional surface.

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