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THE PROBLEM OF EXACT CONTROLLABILITY FOR THE EQUATION OF OSCILLATION OF A STRING

Abstract

In the paper for the equation of oscillation of string for the given initial displacement and initial velocity it is searched such boundary-value controls that at the moment of time Y reduced the displacement and velocity to the functions identically equal to zero.

The problem of controllability for the partial differential equations is one of the interesting questions of control theory (see [1], [2]).

Recently such problems are more intensively studied by a number of mathematicians (see [3], [4], [5], [6] and etc.).

In the present paper is considered the one-dimensional wave equation describing the process of oscillation of a string for the time interval $n=r(t)$, whose ends are the points σ $_{r:}$ and $x=l$.

In each moment of the time σ_r the oscillations process is characterized by the displacement $u(t,x)$ of points of a string and by the velocity $u_t(t,x)$ of these points. For the fixed t a pair of functions $\{u(t,x),u_t(t,x)\}$ given on the segment $0\le x\le l$ is natural to call the state of oscillation system of the moment of time t.

Assume, that at the initial moment of time $t=0$ displacement and velocity of points of string are equal to $u(0,x)=u_0(x)$, $u_t(0,x)=u_1(x)$, and at the moment of time $t=T$ displacement and velocity of points of a string are equal to $u(T,x)=0$, $u_t(T,x)=0$ (this is called a rest-state).

Then there arises a problem of the existence of boundary controls at the ends of string and which provides the transition of oscillation process from the state $\{u_0(x),u_1(x)\}$ at $t=0$ to the rest-state at $t=T$.

It is found that the solution of this problem essentially depends on what relation are the length of the string l and the moment of time T.

Note that, the case $u(t,0)=\mu(t)$, $u(t,l)=v(t)$ has been studied by different methods (see [1]. [6]), however $\mu(t)\equiv v(t)$ in [1].

Let some controlled process be described by the one-dimensional wave equation:

$$u_{tt}-u_{xx}=0 \text{ in } Q=(0,T)\times(0,l) \tag{1}$$

with the initial

$$u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x) \text{ in } (0,l) \tag{2}$$

and with the boundary conditions

$$u(t,0)=\mu(t),\quad u_x(t,l)=v(t) \text{ in } (0,T). \tag{3}$$

Assume that $u_0\in L^2(0,l)$, $u_1\in\left(H^1(0,l)\right)'$, $\mu\in L^2(0,T)$, $v\in L^2[0,T]$. Let's denote $\mathrm{v}(t)=(\mu(t),v(t))$ and the weak solution of problem (1)-(3) which corresponds to the control $\mathrm{v}(t)$ by $u=u(v)\equiv u(t,x;v)$. Under the solution of problem (1)-(3) we understand the function u from $L^2(Q)$, such that for any $\eta(t,x)$ from $C^2(\overline{Q})$, $\eta(T,x)=0$, $\eta_t(T,x)=0$, $\eta(t,0)=0$, $\eta_x(t,l)=0$ *the integral identity*

$$\iint\limits_Q u(\eta_{tt}-\eta_{xx})dxdt+\int_0^l u_0(x)\eta_t(0,x)dx-\int_0^l u_1(x)\eta(0,x)dx-$$

$$-\int_0^T \mu(t)\eta_x(t,0)-\int_0^T v(t)\eta(t,l)dt=0.$$

is fulfilled.

As said above, the following problem is set: find such a control $\mathrm{v}(t)$ for which

$$u(T,x;\mathrm{v})=0,\ \ u_t(T,x;\mathrm{v})=0. \tag{4}$$

For the solution of this problem we'll consider the following boundary-value problem

$$\Phi_{tt}-\Phi_{xx}=0 \ \text{ in } \ Q, \tag{5}$$

$$\Phi(0,x)=\Phi^0(x),\ \ \Phi_t(0,x)=\Phi^1(x) \ \text{ in } \ (0,l), \tag{6}$$

$$\Phi(t,0)=0,\ \ \Phi_x(t,l)=0 \ \text{ in } \ (0,T), \tag{7}$$

As is known [7,8] problem (5)-(7) has a unique solution Φ in $H^1(Q)$ for the given $\Phi^0\in H^1(0,l)$, $\Phi^1\in L^2(0,l)$ moreover

$$\Phi\in C([0,T];H^1(0,l))\cap C([0,l];H^1(0,T)),$$

$$\Phi_t,\Phi_x\in C([0,T];L^2(0,l))\cap C([0,l];L^2(0,T)).$$

Further we'll consider the following problem

$$\psi_{tt}-\psi_{xx}=0 \ \text{ in } \ Q, \tag{8}$$

$$\psi(T,x)=0,\ \ \psi_t(T,x)=0 \ \text{ in } \ (0,l), \tag{9}$$

$$\psi(t,0)=\Phi_x(t,0),\ \ \psi_x(t,l)=\Phi(t,l) \ \text{ in } \ (0,T). \tag{10}$$

As problem (1)-(3) this problem also is a non homogeneous boundary problem. It is obvious that the solution of this problem depends on $\Phi^0(x)$ and $\Phi^1(x)$. As far as $\Phi_x(t,0)\in L^2[0,T],\Phi(t,l)\in H^1[0,T]$, problem (8)-(10) has a weak solution from $L^2(Q)$ and we uniquely determine the operator

$$L\{\Phi^0,\Phi^1\}=\{-\psi_t(0,x),\psi(0,x)\}. \tag{11}$$

Temporarily assume that, the operator L (which depends on T) is reversible (for sufficiently large T) in suitable Hilbert spaces. Then the considered problem is solvable. Indeed for the given $u_0(x), u_1(x)$ we solve the problem

$$L\{\Phi^0,\Phi^1\}=\{-u_1,u_0\} \tag{12}$$

and find $L\{\Phi^0,\Phi^1\}$.

Further we solve problem (5)-(7) and choose the control $\mathrm{v}(t)=(\mu(t),v(t))$ by the following way

$$\mu(t)=\Phi_x(t,0),\ \ v(t)=\Phi(t,l). \tag{13}$$

Then from (12), (13) and (8)-(10) it follows that $u(\mathrm{v})\equiv\psi$, consequently, (4) is true and we construct the control $\mathrm{v}(t)$ by (13) which reduces the system from the initial state (u_0,u_1) to the zero final state at time T.

Now we'll find sufficient conditions for reversibility of the operator L. Let's introduce the scalar product

$$\left\langle L\{\Phi^0,\Phi^1\},\{\Phi^0,\Phi^1\}\right\rangle=\int_0^l[\psi(0,x)\Phi^1(x)-\psi_t(0,x)\Phi^0(x)]dx.$$

Smoothing the initial functions $\Phi^0(x)$ and $\Phi^1(x)$ we obtain that the solution of problem (5)-(7) is sufficiently smooth. Further multiplying equation (8) by $\Phi(t,x)$ and integrating by part we've:

$$\left\langle L\{\Phi^0,\Phi^1\},\{\Phi^0,\Phi^1\}\right\rangle=\int_0^T\left[\Phi^2(t,l)+\Phi_x^2(t,0)\right]dt.$$

Then passing to the limit with respect to smoothing parameter we obtain that the last equality is true for the initial functions $\Phi^0(x)$ and $\Phi^1(x)$. This procedure we'll call as "smoothing procedure". Let's show that for the sufficiently large T

$$\left(\int_0^T\left[\Phi^2(t,l)+\Phi_x^2(t,0)\right]dt\right)^{1/2} \tag{14}$$

determines the norm on the set of the initial dates $\{\Phi^0,\Phi^1\}$ of problem (5)-(7) and norm (14) is equivalent to the ordinary norm of the space $H^1(0,l)\times L^2(0,l)$.

It is easy to prove the correctness of the following inequality: there exists such a constant c_1 that

$$\int_0^T\left[\Phi^2(t,l)+\Phi_x^2(t,0)\right]dt\le c_1\left[\left\|\Phi^0\right\|^2_{H^1(0,l)}+\left\|\Phi^1\right\|^2_{L^2(0,l)}\right]. \tag{15}$$

This follows from the results of paper [9]. Now our aim is to prove the correctness of the inequality

$$\int_0^T\left[\Phi^2(t,l)+\Phi_x^2(t,0)\right]dt\ge c_2(T-T_0)\left(\left\|\Phi^0\right\|^2_{H^1(0,l)}+\left\|\Phi^1\right\|^2_{L^2(0,l)}\right), \tag{16}$$

where c_2 and T_0 are some constants.

To prove (16) we'll apply again the «smoothing procedure». Considering this we'll multiply both sides of equation (5) by $m(x)\Phi_x(t,x)$ and integrate by parts, where $m(x)=x-l$.

We'll use the following denotations:

$$X=\left.\int_0^l\Phi_t(t,x)m(x)\Phi_x(t,x)dx\right|_0^T,$$

$$E(t)=\frac{1}{2}\int_0^l\left[\Phi_t^2(t,x)+\Phi_x^2(t,x)\right]dx,$$

where $E(t)$ is the energy integral of equation (5) at the moment of time t. From equation (5), it is easy to obtain that:

$$E(t)\equiv E_0=\frac{1}{2}\int_0^l\left[\left(\Phi_x^0(x)\right)^2+\left(\Phi^1(x)\right)^2\right]dx.$$

After multiplying equation (5) by $m(x)\Phi_x(t,x)$ and integrating by parts, we obtain

$$\int_0^T\int_0^l\left(\Phi_{tt}m(x)\Phi_x-\Phi_{xx}m(x)\Phi_x\right)dxdt=0$$

or

$$\left.\int_0^l\Phi_t m(x)\Phi_x dx\right|_0^T-\int_0^T\int_0^l m(x)\Phi_t\Phi_{tx}dxdt-\left.\int_0^T m(x)\Phi_x^2dt\right|_0^l+$$

$$\int_0^T\int_0^l \Phi_x\left(m(x)\Phi_x\right)_x dxdt = 0.$$

Hence, we have

$$X - \int_0^T\int_0^l \frac{m(x)}{2}\left(\Phi_t^2\right)_x dxdt - \int_0^T l\Phi_x^2(t,0)dt +$$
$$+\int_0^T\int_0^l \frac{m(x)}{2}\left(\Phi_x^2\right)_x dxdt + \int_0^T\int_0^l \Phi_x^2 dxdt = 0.$$

From the condition $\Phi(t,0)=0$ it follows that $\Phi_t(t,0)=0$. Then

$$X + \frac{1}{2}\int_0^T\int_0^l \Phi_t^2 dxdt - \int_0^T l\Phi_x^2(t,0)dt + \frac{l}{2}\int_0^T \Phi_x^2(t,0)dt -$$
$$-\frac{1}{2}\int_0^T\int_0^l \Phi_x^2(t,x)dxdt + \int_0^T\int_0^l \Phi_x^2 dxdt = 0.$$

Hence it follows that

$$X - \frac{l}{2}\int_0^T \Phi_x^2(t,0)dt + \frac{1}{2}\int_0^T\int_0^l \left[\Phi_t^2(t,x) + \Phi_x^2(t,x)\right]dxdt = 0.$$

From this equality passing to inequality we obtain

$$TE_0 \le |X| + \frac{l}{2}\int_0^T \Phi_x^2(t,0)dx. \qquad (17)$$

It is obvious, that

$$\left|\int_0^l \Phi_t(t,x)m(x)\Phi_x(t,x)dx\right| \le \frac{1}{2}\max_{0\le x\le l}|m(x)|\int_0^l\left(\Phi_t^2 + \Phi_x^2\right)dx = lE_0.$$

Therefore $|X| \le 2lE_0$. Then from inequality (7) we obtain

$$TE_0 \le 2lE_0 + \frac{l}{2}\int_0^T \Phi_x^2(t,0)dx,$$

and hence it follows that

$$\int_0^T \Phi_x^2(t,0)dx \ge \frac{2}{l}(T-2l)E_0.$$

Thus, we proved the inequality

$$\int_0^T \Phi_x^2(t,0)dx \ge \frac{l}{2}(T-2l)\int_0^l\left[\left(\Phi_x^0(x)\right)^2 + \left(\Phi^1(x)\right)^2\right]dx. \qquad (18)$$

Now, to prove inequality (16) it is sufficient to require that the solution of equation (5)-(7) fulfilled the following inequality

$$\int_0^T \Phi^2(t,l)dt \ge \frac{1}{l}(T-2l)\int_0^l\left(\Phi^0(x)\right)^2 dx. \qquad (19)$$

Then from (18) and (19) it follows inequality (16) where $c_2 = \frac{1}{l}$, $T_0 = 2l$.

Thus, we showed that for the sufficiently large T norm (14) and ordinary norm in $H^1(0,l)\times L^2(0,l)$ are equivalent. In turn this shows that the operator L is reversible, L is isomorphic from $H^1(0,l)\times L^2(0,l)$ to $\left(H^1(0,l)\right)' \times L^2(0,l)$. So the following theorem has been proved

Theorem. *Let $T > T_0 = 2l$ and inequality (19) be fulfilled for the solution Φ of problem (5)-(7). Then for each given pair (u_0, u_1) from $L^2(0,l)\times\left(H^1(0,l)\right)'$ one can find the control $v(t) = (\mu(t), \nu(t)) \in L_2(0,T)\times L_2(0,T)$, such that $v(t)$ reduces the system from the initial state (u_0, u_1) to the zero final state at time T.*

Note. One can cite an example, that for solution of problem (5)-(7) the inequality (19) is fulfilled. For example, let $\Phi(t,x)$ be a solution of the boundary-value problem

$$\Phi_{tt} - \Phi_{xx} = 0,$$

$$\Phi(0,x) = \sin\frac{\pi}{2l}x, \quad \Phi_t(0,x) = 0,$$

$$\Phi(t,0) = 0, \quad \Phi_x(t,l) = 0.$$

It is obvious that $\Phi(t,x) = \cos\frac{\pi}{2l}t\sin\frac{\pi}{2l}x$, and for this function the inequality (19) is automatically fulfilled.

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