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THE GENERALIZED CHARACTERISTIC POLYNOMIAL OF A SIMPLE GRAPH

Abstract

In this article we introduce the notion of the generalized characteristic polynomial of a simple graph and we study some of it's important properties and then we calculate these polynomials for cyclic, dihedral and dicyclic groups.

Keywords: generalized matrix function, characteristic polynomial, simple graph.

1. Introduction.

Let G be a subgroup of S_m, the symmetric group of degree m. If χ is an irreducible character of G over C, we define

$$d_\chi^G(A) = \sum_{\sigma\in G}\chi(\sigma)\prod_{i=1}^{m} a_{i,\sigma(i)},$$

where A is an $m\times m$ matrix over C. The function d_χ^G is called the generalized matrix function associated with G and χ. When $G=S_m$ and $\chi=e$, the alternative character of S_m, then d_χ^G is the determinant function. If $G=S_m$ and χ is the principle character, then d_χ^G is the permanent function

$$per(A) = \sum_{\sigma\in S_m}\prod_{i=1}^{m} a_{i,\sigma(i)}.$$

Now let $\Gamma=(V,E)$ be a simple graph (a finite, nondirected graph without loops or multiple edge), with the vertex set V, edge set E and $|V|=m$. If A is the adjacency matrix of Γ we call the polynomial

$$p_\chi^G(A,x) = d_\chi^G(xI-A)$$

the generalized characteristic polynomial of Γ. When d_χ^G is determinant function, we get the ordinary characteristic polynomial of Γ. The properties of this polynomial could be found in every standard graph theory book. In the case that d_χ^G is permanent function, we get the permanental polynomial of a graph which have been studied by many mathematicians as well as R. Merris, K.R.Rebman, and W.Watkins in [3]. One of the important results obtained in [3] is as follows.

For

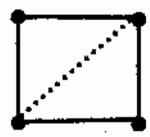
$$per(xI-A) = x^m - c_1x^{m-1} + c_2x^{m-2} - \ldots + (-1)^m c_m$$

it is easy to show that c_k is the sum of k by k principle subpermanents of A (cf.[4]). In particular $c_1 = tr(A)$ and $c_m = per(A)$. For a graph Γ with the adjacency matrix A we have

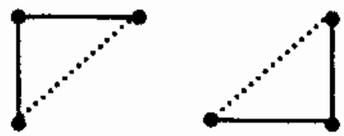
$$c_i = \sum_H 2^{k(H)},\ \ 1\le i\le m,$$

where the sum is over all subgraphs H on i vertices whose components are single edges or circuits and $k(H)$ is the number of circuits.

1.2. Example. Let Γ be the following graph



then $c_1 = 0$ and $c_2 = (number\ of\ edges) = 5$. To compute c_3 we note that there are two subgraphs of order 3 whose components are circuit, c_3:



so we have

$$c_3 = 2^1 + 2^1 = 4.$$

There is one subgraph of order 4 which is a circuit and there are two subgraps of order 4 whose components are single edges, c_4

so

$$c_4 = 2^1 + 2^0 + 2^0 = 4.$$

Thus, $per(xI - A) = x^4 + 5x^2 - 4x + 4$, (cf.[3]).

2. Generalities.

Now we return to the generalized characteristic polynomial $p_\chi^G(A, x)$. For better understanding of this polynomial we define the set C_G as follow;

$$C_G = \{g \in G \mid g(i) \neq i \Rightarrow (i, g(i)) \in E \quad for \ \ all \ i\}.$$

Indeed C_G is the set of all elements of G which do not transfer any vertex to it's non-adjecent vertex. It is trivial that $1 \in C_G$ and if $g \in C_G$ then $g^{-1} \in C_G$. It should be noticed that in general C_G is not a subgroup of G. Suppose that for every $g \in G$, $Fix(g)$ is the number of fixed points of g. We put

$$X_i = \{g \mid Fix(g) = i\}.$$

It is trivial that $X_m = \{1\}$, $X_{m-1} = \phi$ and G is disjoint union of $X_i's$. By using the above notations we calculate $p_\chi^G(A, x)$. First consider the following simple lemma.

2.1. Lemma. $\prod_{g(i) \neq i} a_{i,g(i)} = 1$ *if and only if* $g \in C_G$.

Proof:

$$\begin{aligned}\prod_{g(i) \neq i} a_{i,g(i)} = 1 &\Leftrightarrow \left(\forall i, g(i) \neq i \Rightarrow a_{i,g(i)} = 1\right) \\ &\Leftrightarrow \left(\forall i, g(i) \neq i \Rightarrow (i, g(i)) \in E\right) \\ &\Leftrightarrow g \in C_G.\end{aligned}$$

□

2.2. Theorem.

$$p_\chi^G(A,x)=\sum_{g\in C_G}(-1)^{m-Fix(g)}\chi(g)\chi^{Fix(g)}$$

Proof:

$$p_\chi^G(A,x)=d_\chi^G(xI-A)=\sum_{g\in G}\chi(g)\prod_{i=1}^m\left(x\delta_{i,g(i)}-a_{i,g(i)}\right)$$
$$=\sum_{g\in G}\chi(g)\left(\prod_{i=g(i)}x\right)\left(\prod_{g(i)\neq i}\left(-a_{i,g(i)}\right)\right)$$
$$=\sum_{g\in G}(-1)^{m-Fix(g)}\chi(g)\left(\prod_{i\neq g(i)}a_{i,g(i)}\right)x^{Fix(g)}$$

$=\sum_{g\in C_G}(-1)^{m-Fix(g)}\chi(g)x^{Fix(g)}$, where $\delta_{i,k}$ is Kronecker's symbol.

From this theorem we get the following results about C_G.

2.3. Corollary

a) $d_\chi^G(A)=\sum_{g\in\chi_0\cap C_G}\chi(g)$.

b) $d_\chi^G(I+A)=\sum_{g\in C_G}\chi(g)$.

c) *In a graph* Γ, *the number of permutations of vertices which transfer each vertex to itself or to its adjecent vertex is* per $(I+A)$. □

Since $G=\bigcup_{i=0}^m X_i$, *therefore* $C_G=\bigcup_{i=0}^m(C_G\cap X_i)$, *and hence theorem 2.2 can be restated as following.*

2.4. Corollary. *If*

$$p_\chi^G(A,x)=c_0x^m-c_1x^{m-1}+\ldots+(-1)^m c_m$$

then we have

$$c_i=\sum_{g\in X_{m-i}\cap C_G}\chi(g)$$

in particular

$$c_0=\chi(1),\ and\ c_m=d_\chi^G(A).$$ □

Now we consider the relation between C_G and the group of automorphisms of graph Γ. Let $H=Aut(\Gamma)$.

2.5. Lemma. H *normalizes the subset* C_G *of* G.

Proof. Let $\sigma\in H$ and $g\in C_G$. If $(\sigma g\sigma^{-1})(i)\neq i$, we prove that $(i,\sigma g\sigma^{-1})\in E$. Since $g\sigma^{-1}\neq\sigma^{-1}(i)$ therefore $(\sigma^{-1}(i),g\sigma^{-1}(i))\in E$ and $(i,\sigma g\sigma^{-1}(i))\in E$, because σ is an automorphism. □

According to this lemma, H acts on C_G. If $g\in C_G$ then the stabilizer subgroup of g is $C_H(g)$. Let $orb(g_0),\ldots,orb(g_k)$ be the distinct orbits of this action then

$$C_G=\bigcup_{i=0}^K(orb(g_i))$$

and

$$p_\chi^G(A,x)=\sum_{i=0}^k\sum_{g\in orb_H(g_i)}(-1)^{m-Fix(g_i)}\chi(g_i)x^{Fix(g_i)}$$
$$=\sum_{i=0}^k|orb_H(g_i)|(-1)^{m-Fix(g_i)}\chi(g_i)x^{Fix(g_i)}$$
$$=\sum_{i=0}^k(-1)^{m-Fix(g_i)}[H:C_H(g_i)]\chi(g_i)x^{Fix(g_i)}.$$

For $x=1$ we have

$$p_\chi^G(A,1) = \chi(1) + \sum_{i=1}^{k} (-1)^{m-Fix(g_i)} \chi(g_i)[H : C_H(g_i)]$$

which yields

$$\sum_{i=0}^{k} (-1)^{m-Fix(g_i)} \frac{\chi(g_i)}{|C_H(g_i)|} = \frac{d_\chi^G(I-A)}{|H|}.$$

The special case of above relation can be stated as follows.

2.6. Corollary. *Let* Γ *be a graph of order* m *and* C *be the set of all permutations of vertices of* Γ *which transfer each vertex to itself or to its adjacent vertex. Let also* $g_0, g_1, \ldots, g_k$ *be the representatives of orbits of* C *under the action of* $Aut(\Gamma)$.

Then we have

$$\sum_{i=0}^{k} \frac{(-1)^{m-Fix(g_i)}}{|C_{Aut(\Gamma)}(g_i)|} = \frac{1}{|Aut(\Gamma)|} per(I-A).$$

3. Examples. In this section we consider a simple graph $\Gamma = (V,E)$ and calculate its generalized characteristic polynomial in the following three cases.

a) G is a cyclic group

b) G is a dihedral group

c) G is a dicyclic group.

Case a. Let $g = (12\ldots m) \in S_m$ be a cycle with length m, and $G = < g >$ be the cyclic group generated by g. Let

$$\chi : G \to C$$

be the irreducible character of G be defined by

$$\chi(g) = e^{\frac{2\pi i}{m}}.$$

Since every irreducible character of G is power of χ, therefore it is sufficient to calculate $p_\chi^G(A,x)$. Note that A is an adjacency matrix of Γ, but calculation shows that it can be any arbitrary $m \times m$ matrix.

$$p_\chi^G(A,x) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} \left(x\delta_{i,\sigma(i)} - a_{i,\sigma(i)} \right).$$

From $i = \sigma(i)$ and $\sigma = g^r$ we have $i = g^r(i)$, thus $r + i \equiv i \pmod{m}$ and $r \equiv i \pmod{m}$. So

$$p_\chi^G(A,x) = \chi(1) \prod_{i=1}^{m} (x - a_{i,i}) + \sum_{\sigma \in G^*} \chi(\sigma) \prod_{i=1}^{m} \left(-a_{i,\sigma(i)} \right)$$

where $G^* = G - \{1\}$. But we have

$$\prod_{i=1}^{m} (x - a_{i,i}) = \sum_{i=0}^{m} E_i(a_{1,1}, a_{2,2}, \ldots, a_{m,m})(-1)^i x^{m-i},$$

where E_k is the k-th elementary symmetric function. Therefore we have

$$p_\chi^G(A,x) = \sum_{i=0}^{m-1} (-1)^i E_i(a_{1,1}, a_{2,2}, \ldots, a_{m,m}) x^{m-i} + (-1)^m d_\chi^G(A).$$

In particular when A is the adjacency matrix we have

$$p_\chi^G(A,x) = x^m + (-1)^m d_\chi^G(A).$$

Case b. Now let $G = D_{2m}$ be the dihedral group which is generated by two permutations:

$$a = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ 2 & 3 & \dots & m & 1 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ 1 & m & \dots & 3 & 2 \end{pmatrix}.$$

The generators a and b satisfy in relations

$$a^m = b^2 = 1 \qquad and \qquad ab = ba^{-1}$$

and we have

$$D_{2m} = \{1, a, a^2, \dots, a^{m-1}, b, ab, \dots, a^{m-1}b\}.$$

To study $p_\chi^G(A,x)$ we consider two cases.

Case b-1. *m is an odd integer.*

In this case the character table of D_{2m} is as follows

g	1	a^r	b
$\lvert C_G(g)\rvert$	2m	m	2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$2\cos\left(\frac{2\pi rj}{m}\right)$	0

$$1 \le r,\ j \le \frac{m-1}{2}.$$

Note that for this group the values of $Fix(g)$ are:

$$Fix(1) = m,\ \ Fix(a^r) = 0,\ \ and \quad Fix(a^r b) = 1$$

By applying theorem 2.2 we have

$$p_\chi^G(A,x) = x^m + \lambda(\Gamma)x - d_{\chi 1}^G(A),$$

where

$$\lambda(\Gamma) = \sum_{g \in X_1 \cap K_G} \chi_i(g) \quad \text{for} \quad 1 \le i \le 2.$$

Finally for $p_{\psi_j}^G(A,x)$, since on conjugacy classes of b, ψ_j is 0, therefore by theorem 2.2 of we have

$$p_{\psi_j}^G(A,x) = 2x^m - d_{\psi_j}^G(A)$$

$$1 \le j \le l-1.$$

Case b-2. *m is an even integer.*

In this case the character table of $G = D_{2m}$ is:

g	1	a^l	a^r	b	a b
$\lvert C_G(g)\rvert$	2m	2m	m	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^l$	$(-1)^r$	1	-1
χ_4	1	$(-1)^l$	$(-1)^r$	1	1
ψ_j	2	$2(-1)^j$	$2\cos\frac{2\pi rj}{m}$	0	0

$$1 \le r, j \le l-1,\ and\ \ l = \frac{m}{2}.$$

An easy calculation shows that

$$Fix(a^r) = 0 \text{ when } 1 \le r \le m-1$$
$$Fix(a^r b) = 2 \text{ when } r \text{ is an even integer}$$
$$Fix(a^r b) = 0 \text{ when } r \text{ is an odd integer.}$$

The generalized characteristic polynomial for this group will be:

$$p_{\chi_i}^G(A,x) = x^m + \lambda(\Gamma)x^2 + d_{\chi_i}^G(A)$$
$$p_{\psi_j}^G(A,x) = 2x^m + d_{\psi_j}^G(A), \text{ where } \lambda(\Gamma) = \sum_{g \in X_2 \cap C_G} \chi_i(g) \quad \text{for} \quad 1 \le i \le 4.$$
$$1 \le j \le l-1.$$

Case c. Let $G = T_{4m}$ be the dicyclic group which is generated by the following permutations

$$a = (12\ldots 2m)(2m+1\ 2m+2\ldots 4m)$$
$$b = (1\ 2m+1\ m+1\ 3m+1)(2\ 4m\ m+2\ 3m)(3\ 4m-1\ m+3\ 3m-1)\ldots$$
$$(m-1\ 3m+3\ 2m-1\ 2m+3)(m\ 3m+2\ 2m\ 2m+2).$$

Note that

$$T_{4m} = \langle a,b \rangle \le S_{4m}$$

and

$$a^m = b^2 = (ab)^2.$$

In particular if m is a power of 2 then we get the generalized quaternion group. By considering the structure of a and b we see these permutations have no fixed points, so for every $1 \ne g \in T_{4m}$ we have

$$Fix(g) = 0$$

Again we consider two cases.

Case c-1. *m is odd.*

The character table of $G = T_{4m}$ is

g	1	a^m	a^r	b	a b
$\lvert C_{T_{4m}}(g)\rvert$	4m	4m	2m	4	4
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^r$	i	-i
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^r$	-i	i
ψ_j	2	$2(-1)^j$	$2\cos\frac{\pi r j}{m}$	0	0

$$1 \le r,\ j \le m-1$$

The corresponding polynomials are

$$p_{\chi_j}^G(A,x) = x^{4m} + d_{\chi_j}^G(A), \quad 1 \le j \le 4.$$
$$p_{\psi_j}^G(A,x) = 2x^{4m} + d_{\psi_j}^G(A), \quad 1 \le j \le m-1.$$

Case c-2. *m is even.*

The character table of $T = 4m$ is

g	1	a^m	a^r	b	a b
$\lvert C_{T_{4m}}(g)\rvert$	4m	4m	2m	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
	1	1	$(-1)^r$	1	-1

χ_3	1	1	$(-1)^r$	-1	1
χ_4	2	$2(-1)^j$	$2\cos\frac{\pi r j}{m}$	0	0
ψ_j					

$$1 \le r,\ j \le m-1$$

In this case the characteristic polynomials are the same as in case c-1.

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