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CONTROLLING TRAFFIC LIGHTS AT A BOTTLENECK WITH RENEWAL ARRIVAL PROCESSES

Abstract

In [3] and [4] the control of traffic lights at a bottleneck with a stochastic volume of traffic is discussed. The present paper generalizes the developed theory to the case of arrival processes being renewal processes. The finiteness of the asymptotic expected queue length is proved by a domination principle. Computer experimentation shows, that the optimal time of open passage does not only depend on the traffic intensity but also on the distribution of interarrival times, which means that a precise traffic control requires to estimate this distribution.

1. Introduction.

In [3] and [4] the asymptotic behavior of the queueing process at a bottleneck controlled by traffic lights is discussed assuming that the arrivals of vehicles form a homogeneous Poisson process. moreover, a concept of optimal control of traffic lights at a bottleneck with the time of open passage as control variable (in the hand of the administrator of the installation) is introduced there. A time of open passage is called optimal, if it minimizes the first moment of the limiting distribution of the queueing processes.

In [5] it is argued that the Poisson assumption is justified in the case of a low traffic intensity. For the purpose of modelling higher intensities one sometimes prefers to use independent interarrival times following a bunched exponential distribution or an Erlang distribution (cf. [5]), which leads to a renewal arrival process.

The aim of the present paper is to extend the concept developed in [3], [4] to the more general class of renewal processes. To this end we show the weak convergence of the queueing process by a self-contained proof (section 3) and prove the finiteness of the asymptotic expected queue length by a domination principle.

In section 5 we introduce the notion of the optimal time of open passage basing on the convergence results in the previous sections.

Usually, in traffic control only intensities are measured. Our computer experimentation (section 6) shows, that this only allows to handle the worst case situation. Precise traffic control requires in addition to estimate the distribution of the interarrival times.

2. Renewal Arrival Processes.

In this section we shortly describe the stationary renewal arrival process at the bottleneck. Renewal processes are of course well-known in queueing theory, for details we refer the reader to [6].

Let the arrivals of vehicles at the bottleneck described by a sequence of random variables $T_1, T_2, \ldots$ defined over some probability space (Ω, A, P). It is now supposed that the sequence (T_n) satisfies the condition that the interarrival times

$$(T_n - T_{n-1})_{n\geq 2} \tag{2.1}$$

form a sequence of independent and identically distributed square-integrable random variables. The distribution function F of $T_{n+1} - T_n$ is supposed to have the property that

$$F(0) = 0, \tag{2.2}$$

from this follows

$$E(T_{n+1} - T_n) > 0\,. \tag{2.3}$$

Furthermore, the arrival of the first vehicle T_1 at the bottleneck is assumed to be independent of the sequence $(T_n - T_{n-1})_{n\geq 2}$ and to satisfy

$$P(T_1 > u) = \frac{\int\limits_u^\infty (1 - F(t))dt}{\int\limits_0^\infty (1 - F(t))dt}\,. \tag{2.4}$$

We then define the renewal arrival process $(A_t)_{t\geq 0}$ associated with the sequence of arrivals (T_n) by

$$A_t := \max\{n : T_n \leq t\}.$$

The «starting» condition (2.4) ensures the arrival process to be stationary in the sense that the distribution of the random variable $A_t - A_s$, which counts the number of arrivals in the time interval $(s;t]$ depends only on $t - s$.

The intensity of the renewal process is defined by the expectation

$$I := E(A_1) = E(A_{t+1} - A_t) \tag{2.5}$$

and satisfies the equation

$$I = (E(T_n - T_{n-1}))^{-1}.$$

(2.6)

Thus, I is the expected number of vehicles arriving at the bottleneck in a time interval of length 1.

3. Model Description and Queueing Process.

Following [3] and [4] the technical part of a bottleneck controlled by trafic lights (symmetric case) is described by

$$\Delta, t_R \tag{3.1}$$

Δ in [veh/s] is the passage capacity (for both sides) of the bottleneck. t_R in [s] denotes the clearance times (both sides).

The arrival process $A = (A_t)_{t\geq 0}$ for an arbitrary direction is assumed to be a renewal process on the probability space (Ω, A, P) *in the sense of the previous section* with the sequence (T_n) of arrival time and intensity I being the traffic intensity in [veh/s] in a traffic-theoretic interpretation.

It is sensible to assume the times of open passage (signalized by GREEN and afterwards by YELLOW) to be the same for both sides. the time $t_F > 0$ of open passage is the control variable in the hand of the administrator of the installation.

The duration of the closed passage (for both sides) is given by

$$t_C := 2t_R + t_F \tag{3.2}$$

while

$$t_U := 2(t_R + t_F) \tag{3.3}$$

represents the length of a full control period. The function $\overline{\alpha} : \mathbf{R}_+ \to \mathbf{Z}_+$, defined by

$$\overline{\alpha}(t) := \begin{cases} 0, & 0 < t < t_C \\ [(t - t_C)\Delta], & t_C \leq t \leq t_U \end{cases} \tag{3.4}$$

and the condition that $\bar{\alpha}$ is periodic with the period t_U on $\mathbf{R}_+$, represents the maximal number of vehicles, which can pass the bottleneck from the beginning of a control period until the time t of a control period.
(Notice that $[a]$ means the greatest integer number less than or equal to a).

The number

$$\alpha(t_F) := [t_F \cdot \Delta] = \bar{\alpha}(t_U) \qquad (3.5)$$

denotes the maximal number of vehicles, which may pass the bottleneck during one control period respectively during one phase free passage.

Let

$$N_j(t_F) := A_{(j+1)t_U} - A_{jt_U} \qquad (3.6)$$

denote the number of arriving vehicles in the $(j+1)$ -th control period that depends on t_F by t_U. We assume the queue length to the time 0 to be $L_0 := 0$. The sequence of random variables (L_j) that describe the process of queue lengths (of vehicles) at the end of the time of free passage is defined recursively by

$$L_{j+1} := \left(L_j + N_j(t_F) - \alpha(t_F)\right)^+ \qquad (3.7)$$

Recursions of the type (3.7) are in fact well-known in queue theory, the asymptotic behaviour depends on the value $\alpha(t_F)$ and expectation

$$\lambda(t_F) := E(N_j(t_F)). \qquad (3.8)$$

By the definition of the intensity I of the arrival process we have

$$\lambda(t_F) = I \cdot t_U = 2I(t_F + t_R), \qquad (3.9)$$

$\lambda(t_F)$ is the expectation of the number of vehicles arriving at the bottleneck during a control period of length t_U.

Now it is possible to prove the following: If the expected number of arrivals given by $\lambda(t_F)$ is greater than the number of vehicles $\alpha(t_F)$ that may pass the bottleneck during one control period then the system collapses, if $\lambda(t_F) < \alpha(t_F)$ then the process of queue lengths is stabilizing.

To prove this convergence statement we base on a straightforward self-contained proof rather than showing that the Lindley recursion (3.7) defines a woarkload process in a system $D/G/1$, which would mean to clear a lot of technical details.

If U_j is the random variable defined by

$$U_j := N_j(t_F) - \alpha(t_F). \qquad (3.10)$$

then it can be shown by induction and by the stationarity of the arrival process that

$$P(L_j = n) = P\left(\max_{0 \le i \le j} \sum_{k=1}^{i} U_k = n\right). \qquad (3.11)$$

With the help of the strong law of large number for the sequence (U_j) one obtains that

$$E(U_j) > 0 \Rightarrow \max_{0 \le i \le j} \sum_{k=1}^{i} U_k \to \infty \quad P-a.s. \qquad (3.12)$$

and

$$E(U_j) < 0 \Rightarrow \max_{0 \le i \le j} \sum_{k=1}^{i} U_k \text{ converges to a random variable } Y : \Omega \to \mathbf{Z}_+ P-a.s. \qquad (3.13)$$

Having (3.10), (3.11) in mind this implies

$$\lambda(t_F) - \alpha(t_F) > 0 \Rightarrow P(L_j < \infty) \to 0, \ j \to \infty \qquad (3.14)$$

(case of a traffic collapse), and

$$\lambda(t_F)-\alpha(t_F)<0 \Rightarrow P(L_j \in A) \rightarrow P(L \in A),\ j \rightarrow \infty \quad (A \in \mathbf{Z}_+) \qquad (3.15)$$

(case of a stabilizing queue), where L is a limiting random variable $L:\Omega \rightarrow \mathbf{Z}_+$, which describes the asymptotic queue length at the bottleneck in a probabilistic sense. For the purpose of establishing an objective function it is now of interest, whether the expetation $E(L)$ of the asymptotic queue is finite in t he case $\lambda(t_F)-\alpha(t_F)<0$.

4. Finiteness of the asymptotic expectation.

In this section we assume $\lambda(t_F)-\alpha(t_F)<0$. According to (3.7) we may express the queue length L_{j+1} in terms of arrivals and departures by

$$L_{j+1} = L_j + N_j(t_F) - D_j(t_F), \qquad (4.1)$$

where

$$D_j(t_F) := \min\{L_j + N_j(t_F), \alpha(t_F)\} \qquad (4.2)$$

equals the number of vehicles leaving the bottleneck in the interval $(jt_U; (j+1)t_U]$ of the $(j+1)$ -th control period.

We will now define a queueing system of type $GI/D/1$ with arrivals (T_n) and constant service times $t_U \cdot \alpha(t_F)^{-1}$ and let $\widetilde{L}_j$ describe the queue length in this system in to the time jt_U, $j = 0,1,\ldots$ with initial condition $\widetilde{L}_0 = 0$ (for a detailed description of systems $GI/GI/1$ we refer the reader to [1], chapter 11).

By the definition of the service times, we obtain for the number of costumers being served in the interval $(jt_U; (j+1)t_U]$, denoted by $\widetilde{D}_j(t_F)$, the inequality

$$\widetilde{D}_j(t_F) \leq \min\{\widetilde{L}_j + N_j(t_F), \alpha(t_F)\}. \qquad (4.3)$$

This is of course due to the fact, that by definition of the service times in this system $GI/D/1$ maximal $\alpha(t_F)$ services of length $t_U \cdot \alpha(t_F)^{-1}$ can take place in a time interval of length t_U. Similar to the recursion (4.1) we are able to express the queue length in this system $GI/D/1$ in terms of departures and arrivals by the recursion

$$\widetilde{L}_0 = 0, \quad \widetilde{L}_{j+1} = \widetilde{L}_j + N_j(t_F) - \widetilde{D}_j(t_F). \qquad (4.4)$$

We now compare the bottleneck process (L_j) with the process $(\widetilde{L}_j)$.

4.5. Lemma. *The inequality* $\widetilde{L}_j \geq L_j$ *holds for every* $j \in \mathbf{Z}_+$.

Proof. We prove the statement by induction.
For $j = 0$ it is trivially true.
Let the assertion be true for $j \in \mathbf{Z}_+$. Then (4.1)-(4.4) imply

$$\widetilde{L}_{j+1} - L_{j+1} \geq \widetilde{L}_j - L_j - \min\{\widetilde{L}_j + N_j(t_F), \alpha(t_F)\} + \min\{L_j + N_j(t_F), \alpha(t_F)\}. \qquad (4.5)$$

Because $\widetilde{L}_j \geq L_j$ was assumed to be valid, an evaluation of the minima leads in any case to the conclusion $\widetilde{L}_{j+1} \geq L_{j+1}$. □

The Lemma states that the system $GI/D/1$ is a dominating queueing system for the bottleneck process. From the condition $\lambda(t_F)-\alpha(t_F)<0$ we now get that in the system $GI/D/1$ the expectation of service times $t_U \cdot \alpha(t_F)^{-1}$ strictly less than the expectation of the square integrable interarrival times given by

$$E(T_{n+1}-T_n)=I^{-1}.$$

In this case it is known that the system $GI/D/1$ is stable and that the asymptotic expectation is finite (see [1], 11.1.5 and 11.4.2), i.e. there exists a random variale $\widetilde{L}:\Omega\to\mathbf{Z}_+$ with the property that

$$P(\widetilde{L}_j\in A)\to P(L\in A),\ j\to\infty\quad (A\subset\mathbf{Z}_+) \tag{4.6}$$

and

$$E(\widetilde{L})<\infty. \tag{4.7}$$

This fact together with 4.5 implies the following result.

4.8. Corollary. *The expectation of the asymptotic queue length* $E(L)$ *for the bottleneck process is finite.*

Proof. from basic probability theory the expectation $E(L)$ and $E(\widetilde{L})$ can be expressed by

$$E(L)=\sum_{n=0}^{\infty}P(L>n),$$

$$E(\widetilde{L})=\sum_{n=0}^{\infty}P(\widetilde{L}>n),$$

From Lemma 4.5, (4.6) and (3.15) we get

$$P(L>k)=\lim_{j\to\infty}P(L_j>k)\le\lim_{j\to\infty}P(\widetilde{L}_j>k)=P(\widetilde{L}>k).$$

From this and the above expressions for the expectations it follows

$$E(L)\le E(\widetilde{L})<\infty.$$

As the expectation of the limiting random variable for the bottleneck process is finite, one is able to apply the strong law of large numbers that follows from the ergodic theorem (see [2]).

4.9. Corollary. *For the process of queue lengths it follows*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{j=0}^{m}L_j=E(L)$$

P - almost sure.

This gives the opportunity to approximate the expectation $E(L)$ in a computer experiment.

5. Optimality Concept.

As mentioned in section 2, the time of open passage acts the control variable for the bottleneck process. Generally, in traffic control one is interested

- in maximizing the efficiency of a traffic installation in the sense of optimal throughput per time unit and
- in minimizing the mean individual waiting time.

Having in mind that the queue length goes to infinity with probability 1 in the case $\lambda(t_F)-\alpha(t_F)>0$, it is sensible to choose a time of open passage t_F that satisfies $\lambda(t_F)-\alpha(t_F)>0$ if possible, in which case we call t_F *ergodic*.

In order to define an objective function the results of the previous section give the opprtunity to work with the asymptotic expectations in the case that the bottleneck process converges to equilibrium. Let us first consider the efficiency.

As a definition of an ergodic time of open passage t_F we take the expected number of vehicles leaving the bottleneck per time unit in equilibrium. Define

$$D(t_F) := \min\{L + N(t_F), \alpha(t_F)\} \tag{5.1}$$

with $N(t_F)$ being a random variable following the distribution of $A_{(j+1)t_U} - A_{jt_U}$ and being independent of the asymptotic queue L. $D(t_F)$ can be interpreted as the asymptotic number of vehicles leaving the bottleneck during one control period. Then

$$e(t_F) := t_U^{-1} \cdot E(D(t_F)) \tag{5.2}$$

may stand for the efficiency. But in equilibrium we have by recursion (4.1)

$$E(L) = E(L) + E(N(t_F)) - E(D(t_F)), \tag{5.3}$$

from which it follows

$$e(t_F) = t_U^{-1} \cdot E(N(t_F)) = I, \tag{5.4}$$

i.e., the efficiency is the same for all ergodic t_F. This, of course, is just the intensity-conservation principle for the stable bottleneck process.

Consequently, one is led to the minimization of the waiting time over the set of all ergodic t_F. As computer experiments have shown, the expected queue length at the end of the time of closed passage divided by the traffic intensity is a good estimator for the time a newly arriving vehicle has to wait until it has the possibility to pass the bottleneck. As $E(L)$ is the asymptotic expectation of the queue length at the end of the time of open passage, the number

$$E(L) + I \cdot t_C \tag{5.5}$$

equals the asymptotic expectation of the queue length at the end of the end of closed passage because the expected number of arrivals during the time of closed passage of length t_C is $I \cdot t_C$. We therefore take

$$J(t_F) := \frac{E(L) + I \cdot t_C}{I} \tag{5.6}$$

as an estimator for the asymptotic waiting time. Note thereby, that $E(L)$ depends on the special choice of t_F. With the help of the function $J(.)$ we are in the situation to establish the notion of an optimal time of open passage.

5.7. Definition. *An erdonic time of open passage t_F^* is called optimal, iff $J(.)$ defined by (5.6) has a minimum in t_F^*.*

Thus, the optimality concept remains the same as in the Poisson case, but in the case of renemal arrival processes the value of the objective function $J(.)$ does not only depend on the traffic intensity I but also on the distribution of the interarrival times.

6. Experimental Results.

In order to determine the optimal time of open passage in a computer experiment, we calculate the asymptotic expected queue length basing on the strong law of large numbers (see 4.9) for a class of Erlang distributions. A justification for the Erlang Distribution is given in [5]. The expermintation, cp. Fig.1, shows that the optimal time of open passage does not only depend on the traffic intensity but also on the distribution of interarrival times, which means that a precise traffic control requires to estimate this distribution.

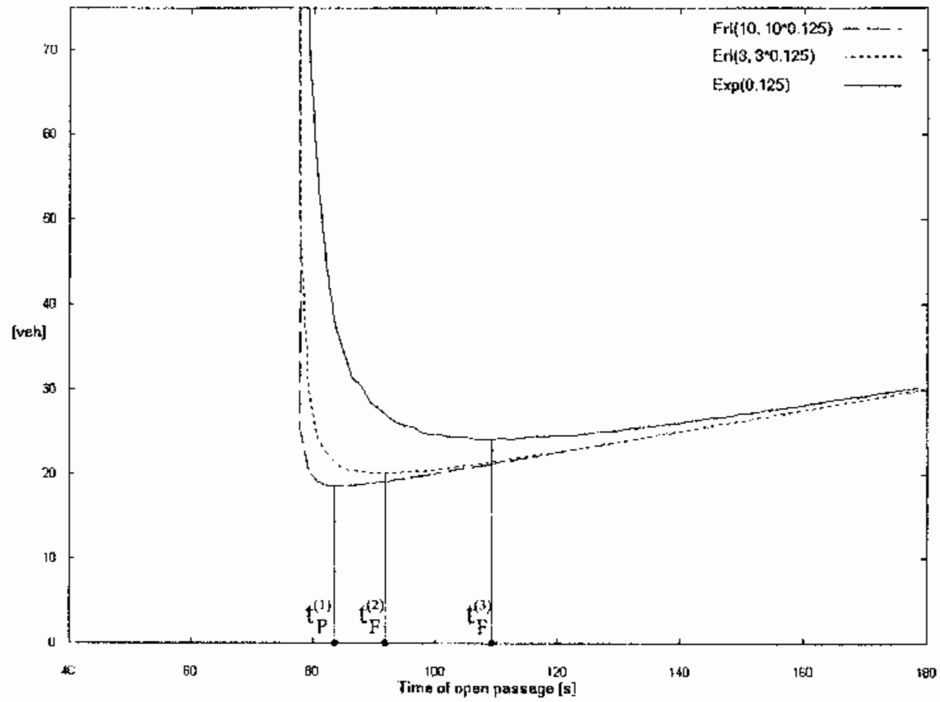

Fig.1.

In Fig.1 the dependence is demonstrated by plotting the asymptotic expected queue length for three different Erlang distributions $Erl(n,\alpha)$ choosing $n=1,3,10$ and $\alpha=0.125, 3\cdot 0.125, 10\cdot 0.125$ for varying time of open passage. The corresponding renewal processes all have intensity 0.125. Note that $Erl(1,0.125)$ is just the Exponential distribution $Exp(0.125)$ with parameter 0.125, so that this distribution of interarrival times corresponds to the Poisson arrival process. In Fig.1 $t_P^{(1)}, t_P^{(2)}, t_P^{(3)}$ denote the optimal times of open passage associated with the arrival processes having $Erl(10,10\cdot 0.125)$, $Erl(3,3\cdot 0.125)$, $Exp(0.125)$ as distribution of interarrival times, respectively.

The experimentation shows, that the Poisson arrival process may serve to describe a worst case situation, requiring only to known the traffic intensity.

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