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SOME REMARKS ON THE WILSON THEOREM AND THE LEGENDRE HYPOTHESIS

Abstract

It is established that for each prime $p \geq 3$*, the modulo* p *equality* $(x-1)!\,(p-x)! \equiv -1 \pmod p$ *holds if* $x \in \{1,...,p\}$ *is odd and* $(y-1)!(p-y)! \equiv 1 \pmod p$ *if* $y \in \{2,...,p-1\}$ *is even; this generalizes the «if part» of the Wilson theorem. Further, some remarks are given concerning the Legendre conjecture about the difference* $p_{n+1} - p_n$ *for large values of* n.

The statements are presented without proofs, since the reasonings used in them are elementary (although often not quite obvious); the brief explanations will be given only for theorems 1, 3.

Let $\mathbf{E}$ (respectively, $\mathbf{O}$ and $\mathbf{P}$) be the set of all even (respectively, odd and prime) numbers, where $1 \notin \mathbf{P}$ by definition, $\widetilde{\mathbf{P}} := \{p \in \mathbf{P} | p \neq 2\}$ and for the natural n $x \overset{n}{=} y \overset{def}{\Leftrightarrow} x \equiv y \pmod n$. As well known, for $n \geq 2$ the equivalencies $n \in \mathbf{P} \Leftrightarrow (n-1)! \overset{n}{=} -1$ (the Wilson theorem) and resulting from it $n \in \mathbf{P} \Leftrightarrow (n-2)! \overset{n}{=} 1$ (the Leibnitz theorem) are true. For $p \in \widetilde{\mathbf{P}}$ we consider the predicate $A_p(x) \overset{def}{\Leftrightarrow} (x-1)!(p-x)! \overset{p}{=} -1$ on the set $\{1,...,p\} \cap \mathbf{O}$ and the predicate $B_p(x) \overset{def}{\Leftrightarrow} (x-1)!(p-x)! \overset{p}{=} 1$ on the set $\{2,...,p-1\} \cap \mathbf{E}$; for $p = 2$ we set analogously $A_2(1) \overset{def}{\Leftrightarrow} (1-1)!\,(2-1)! \overset{2}{=} -1$, $B_2(2) \overset{def}{\Leftrightarrow} (2-1)!\,(2-2)! \overset{2}{=} 1$.

Then the propositions $A_2(1), B_2(2)$ are trivially true, and for $p \in \widetilde{\mathbf{P}}$ we've true $A_p(1), B_p(2)$. More precisely, for $p \in \widetilde{\mathbf{P}}$ we've

$$A_p(1) \Leftrightarrow A_p(p) \Leftrightarrow (p-1)! \overset{p}{=} -1,$$

$$B_p(2) \Leftrightarrow B_p(p-1) \Leftrightarrow (p-2)! \overset{p}{=} 1.$$

In this connection for $p \in \widetilde{\mathbf{P}}$ the question is of interest, what are the truth values of the predicate $A_p(x)$ for other $x \in \{1,...,p\} \cap \mathbf{O}$ and, correspondingly, of the predicate $B_p(x)$ for other $x \in \{2,...,p-1\} \cap \mathbf{E}$.

Theorem 1. *For* $p \in \widetilde{\mathbf{P}}$ $A_p(x)$ *is true for all* $x \in \{1,...,p\} \cap \mathbf{O}$ *and* $B_p(x)$ *is true for all* $x \in \{2,...,p-1\} \cap \mathbf{E}$.

Since truth of $A_p(1), B_p(2)$ gives base for application of induction method, and $A_p(p) \Leftrightarrow A_p(1)$ and $B_p(p) \Leftrightarrow B_p(2)$, it is sufficient to verify the truth of the implication

$A_p(x) \Rightarrow B_p(x+1)$ for $x \in \{1,\dots,p-2\} \cap \mathbf{O}$ and the implication $B_p(y) \Rightarrow A_p(y+1)$ for $y \in \{2,\dots,p-1\} \cap \mathbf{E}$. This is easily realized using properties of the relation $\overset{p}{=}$.

Remark 1. For $p \in \widetilde{\mathbf{P}}$ considering $A_p([p/2]+1)$ in case of odd $[p/2]+1$ and $B_p([p/2]+1)$ in case of its evenness we obtain: $([p/2]!)^2 \overset{p}{=} -1$ in even case and $([p/2]!)^2 \overset{p}{=} 1$ in odd case.

Legendre said the following hypothesis on neighboring prime numbers: if p_n means the n-th prime number, then for all sufficiently large numbers n

$$p_{n+1} - p_n < \sqrt{p_n} \quad (*)$$

It is neither proved nor disproved up to now.

Below we'll state some facts concerning this hypothesis.

Statement 1. *If (*) is true then* $p_{n+1} < \sqrt{2}p_n$.

Corollary. *If (*) is true then*

$$\sqrt{p_{n+1}p_n} < \frac{p_{n+1}+p_n}{2} < \sqrt{p_{n+1}p_n} + \frac{1}{8}.$$

Statement 2. *If (*) is true, then*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{2}$$

Statement 3. *If the natural numbers* m,n *satisfy the inequalities*

$$p_{n+1} \le \left(1+\frac{1}{m}\right)p_n \text{ and } p_n < m^2,$$

then for such n *the Legendre hypothesis is true.*

Statement 4. *If the natural numbers* m *and* n *satisfy the inequalities*

$$m^2 < p_n < p_{n+1} < m^2 + m,$$

then for such n *the Legendre hypothesis is true.*

Theorem 2. *Let* $p_{n+1} - p_n = 2m$ $(n \ge 3)$. *Then for all* $m \le \sqrt{2p_n} + 1$ *the equality*

$$p_{n+1} + p_n = 2\left[\sqrt{p_{n+1}p_n}\right] + 2$$

is true.

Now let's show, what follows from the simultaneous satisfaction of the inequalities

$$p_{n+1}^2 > p_n p_{n+2} \tag{1}$$

and

$$p_n + p_{n+2} > 2p_{n+1} \tag{2}$$

for some $n \in N$.

Theorem 3. *From the simultaneous satisfaction of (1) and (2) for some* $n \in N$ *it follows the breach of Legendre's hypothesis for such* n.

Remark 2. From Sierpinsky hypothesis («between two consecutive squares of natural numbers lie at least two prime numbers»), it follows:

$$\left[\sqrt{p_{n+1}}\right] - \left[\sqrt{p_n}\right] = \begin{cases} 0, & if \quad m^2 < p_n < p_{n+1} < (m+1)^2, \\ 1, & if \quad m^2 < p_n < (m+1)^2 < p_{n+1} < (m+2)^2. \end{cases}$$

Remark 3. The immediate checking shows that the pairs

$$(3;5),\ (7;11),\ (13;17),\ (23,29),\ (31,37),\ (113,127)$$

don't satisfy the inequality (*); on the other hand, the computer checking shows that there are no other pairs of neighbouring prime numbers distorting the inequality (*) among numbers $p < 6000$.

Remark 4. In connection with theorem 3 we note that the inequalities

$$p_{n+1}^2 < p_n p_{n+2} \quad \text{and} \quad p_{n+2} + p_n \leq 2p_{n+1}$$

can't be fulfilled simultaneously.

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