MATHEMATICS

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ON THE CONVERGENCE OF THE FOURIER SERIES OF ORTHONORMAL POLYNOMIALS IN THE DOMAIN WITH PIECEWISE SMOOTH BOUNDARY

Abstract

Let $G \subset C$ be a finite Jordan region and $h(z)$ be a weight function defined on G; $\{K_n(z)\}_{n=0}^{\infty}$ is a system of orthonormal polynomials on the region G with $h(z)$. We denote the $H_2^1(h,G)$ as the class of analytic functions f in G and satisfying the following conditions:

$$\iint_G h(z)|f(z)|^2 d\sigma_z < \infty .$$

For each $f \in H_2^1(h,G)$ and $n = 0,1,2,...,$ as shown by $a_n(f)$ Fourier coefficients of the function f with

$$a_n := a_n(f) = \iint_G h(z) f(z) \overline{K_n(z)} d\sigma_z ,$$

and corresponds to f the series

$$\sum_{n=0}^{\infty} a_n K_n(z) . \qquad (*)$$

Let $S_n(f,z)$ be a partial sum of series (*) and $\varepsilon_n(z) := |f(z) - S_n(f,z)|$, $z \in G$. It is well known that $\varepsilon_n(z) \to 0$, $n \to \infty$. What is the speed of the approximation to zero of $\varepsilon_n(z)$? This problem has been studied in [2,5,7,15] when $L = \partial G$ is a K-quasiconformal, piecewise quasiconformal, piecewise smooth and belongs to the class $C(p,\alpha)$, $(p > 0, 0 < \alpha < 1)$ respectively.

In this study, the speed of $\varepsilon_n(z) \to 0$, $n \to \infty$, $z \in G$ in domains with piecewise-smooth boundary (exterior $\lambda\pi$, $0 < \lambda \le 2$, angles) depends on the properties of boundary arcs and the degree of their touch.

1. Introduction and the main result.

Let $G \subset C$ be a finite region bounded by a Jordan curve $L := \partial G$, $h(z) > 0$ be a weight function defined on G and $\{K_n(z)\}_{n=0}^{\infty}$ is a system of orthonormal polynomials on the region G with respect to the weight function $h(z)$, i.e.

$K_n(z) = a_n z^n + ... + a_0$, $\deg K_n = n$, $n = 0,1,2,...,$ $a_n > 0$ and

$$\iint_G h(z) K_n(z) \overline{K_m(z)} d\sigma_z = \begin{cases} 1, & n = m \\ 0, & n \ne m \end{cases} \qquad (1.1)$$

Let $H_2^1(h,G)$ be the class of analytic functions f in G such that

$$\|f\|_{H_2^1}^2 := \|f\|_{H_2^1(h,G)}^2 = \iint_G h(z)|f(z)|^2 d\sigma_z < \infty .$$

For each $f \in H_2^{'}(h,G)$ and $n = 0,1,2,...,$ defined by $a_n(f)$ Fourier coefficients of the function with

$$a_n := a_n(f) := \iint_G h(z)f(z)\overline{K_n(z)}d\sigma_z \tag{1.2}$$

and corresponds to f the following series

$$\sum_{n=0}^{\infty} a_n K_n(z). \tag{1.3}$$

It is well known that the series (1.3) being converged to f depends on the completeness of the set of polynomials in $H_2^{'}(h,G)$. Since $\{K_n(z)\}_{n=0}^{\infty}$ orthonormal polynomials system is defined with $h(z)$ and G, this also depends on the properties of the weight function $h(z)$ and the region G. We know that if the function $h(z)$ is uniformly bounded with positive constants from above and below i.e. there are constants c_1, c_2 such that

$$0 < c_1 \le h(z) \le c_2 < \infty, \tag{1.4}$$

then the set of polynomials is complete in $H_2^{'}(h,G)$.

In this work we assume the weight function

$$h(z) = |D(z)|^2, \tag{1.5}$$

where $D(z) \in A(\overline{G})$ and $D(z) \ne 0$, $\forall z \in \overline{G}$.

We denote the partial sum of the series (1.3) by $S_n(f;z)$ and let

$$\varepsilon_n(z) := |f(z) - S_n(f,z)|, \quad z \in G. \tag{1.6}$$

The function $h(z)$ defined in (1.5) satisfies condition (1.4), therefore, for every $z \in G$

$$\lim_{n\to\infty} \varepsilon_n(z) = 0 . \tag{1.7}$$

In many problems of approximation theory it is also interesting to find the speed of $\varepsilon_n(z) \to 0, \quad n \to \infty$.

First of all we give some definitions.

Definition 1.1. *The Jordan curve* $L \in C(p,\alpha)$, $p = 1,2,3,...,0 < \alpha < 1$, *if* $z = z(s)$ *is a natural representation of* L, *which* $z = z(s)$ p *times continuous differentiable and* $z^{(p)}(s) \in Lip\,\alpha$.

Definition 1.2. *We say that*

a) $L \in C_\theta$, *if* L *is smooth curve having continuously* $\theta(s) := \theta(z(s))$ *tangent line for every point* $z = z(s) \in L$;

b) $G \in C_\theta$, *if* $L = \partial G \in C_\theta$;

c) $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, *if* L *consists of the union of finite* C_θ*-arcs such that they have exterior angles* $\lambda_j \pi$ *at the corners with respect to* G *where two arc meet,* $0 < \lambda_j < 2$, $\min_j \lambda_j = \lambda$.

Definition1.3[12,p.97]. *We say that* L *is a* K*-quasiconformal* $(K \ge 1)$ *arc or curve, if there is a* K*-quasiconformal mapping* f *of a region* $D \supset L$ *such that* $f(L)$ *is a line segment or circle.*

Let $F(L)$ denote the set of all sense-preserving plane homeomorphisms f of regions $D \supset L$ such that $f(L)$ is a line segment or circle and

$$K_L = \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of such a mapping f. Then L is K – quasiconformal curve if and only if $K_L \le K < \infty$.

$D = C$ gives the global definition of K – quasiconformal curve and $D \subset C$ gives the local definition of K – quasiconformal curve (see, for example [1] ,[13]). Throughout this work, we consider the local definition because it is possible to determine the quasiconformality coefficient K for some simple regions. For instance, with the help of [13] we have

Corollary 1.1. *If L is a analytic curve, then $K = 1$.*

Corollary 1.2. *If $L \in C_\theta$, then $K = 1 + \varepsilon$ for all $\varepsilon > 0$.*

Definition 1.4. *We say that $G \in C_\theta(\lambda;\beta)$, $0 < \lambda < 2$, $\beta > 0$, if $L := \partial G$ is expressed as the union of the finite number C_θ-arcs connecting at points $z_0', z_1, ... z_m$, L is locally smooth at z_0' and the following conditions are satisfied :*

a) for every point z_j, $1 \le j \le p$, the domain G has $\lambda_j \pi$, ($0 < \lambda_j < 2$) exterior angles at the corners z_j, $\lambda = \min\limits_{1 \le j \le p}\{\lambda_j\}$;

b) for very point z_j, $p+1 \le j \le m$, in (x, y) local co-ordinate system with origin at z_j,

$$\{z = x + iy : c_1 x^{1+\beta} \le y \le c_2 x^{1+\beta} , 0 \le x \le \varepsilon_1\} \subset \overline{G},$$
$$\{z = x + iy : |y| \ge \varepsilon_2 x , 0 \le x \le \varepsilon_1\} \subset CG$$

for some certain constants $-\infty < c_1 < c_2 < +\infty$, $\varepsilon_i > 0$, $i = 1,2$.

It is obvious from Definition1.4 that each domain $G \in C_\theta(\lambda;\beta)$ may have $\lambda\pi$, $0 < \lambda < 2$, exterior angles at the z_j, $1 \le j \le p$, and interior zero angles at the points z_j, $p+1 \le j \le m$, which the boundary arcs touching $x^{1+\beta}$-speed. If $\beta = 0$ then domain G doesn't have interior zero angles and G belongs to $C_\theta(\lambda)$, $(0 < \lambda < 2)$, i.e. $C_\theta(\lambda) \equiv C_\theta(\lambda;0)$. If $\lambda = 1$, then the domain G has piecewise smooth boundary with only interior zero angles. We denote this class with $C_\theta(1;\beta), \beta > 0$.

P. K. Suetin in [15] proved that if $L \in C(p+1,\alpha)$ and $D(z) \in W^p H^\alpha(\overline{G})$, i.e. $D^{(p)}(z) \in A(\overline{G}) \cap Lip\alpha$, $p \ge 0$, $0 < \alpha < 1$, and $D(z) \ne 0$, $\forall z \in \overline{G}$ then

$$\varepsilon_n(z) \le c\,[dist(F,L)]^{-(p+3)} \frac{1}{n^{p+\alpha}} E_n(f, H_2'), \qquad z \in F \subset G, \tag{1.8}$$

where

$$E_n(f, H_2') := \min_{P_n}\left(\iint_G h(z)|f(z) - P_n(z)|^2 d\sigma_z\right)^{1/2}$$

is the best approximation in the class $H_2'(h,G)$ by polynomials $P_n(z)$ with $\deg P_n \le n$ and c is a constant independent of z and n. This result is seen that the speed of $\varepsilon_n(z) \to 0$, $n \to \infty$ is dependent on the properties of the weight function $h(z)$, boundary $L = \partial G$ and compact subset $F \subset G$. In the case when L is K-quasiconformal, piecewise quasiconformal curve and piecewise smooth curve, the estimation in the type (1.8) has been investigated in [2,5,7].

In this paper the same problem is investigated when $G \in C_\theta(\lambda;\beta)$ and the following is proved:

Theorem 1. *Let* $L \in C_\theta(\lambda;\beta)$, *for some* $0<\beta<1$, $0<\lambda<2$; $h(z)=|D(z)|^2$, $D(z)\in A(\overline{G})\cap Lip\alpha$, $0<\alpha\le 1$ *and* $D(z)\neq 0$, $\forall z\in\overline{G}$. *Then*

$$\varepsilon_n(z)\le cE_n(f,H_2')\Delta(z)\begin{cases} n^{-\mu'}, & if\ \alpha\le\dfrac{1}{2-\lambda}, 0<\lambda\le\dfrac{2-2\beta}{3+\beta}, \\ n^{-\mu'}, & or\ \alpha\le\dfrac{1-\beta}{2\lambda(1+\beta)}, \dfrac{2-2\beta}{3+\beta}<\lambda<2 \\ n^{-\mu}, & otherwise \end{cases} \tag{1.9}$$

for all $0<\mu<\min\left\{\dfrac{\lambda}{2-\lambda}, \dfrac{1-\beta}{2(1+\beta)}\right\}$, $0<\mu'<\alpha\lambda$ *and*

$$\Delta(z):=\begin{cases} \delta^{-\frac{5}{2}}(z_0), & 0<\lambda\le 1, \beta>0 \\ \delta^{-\frac{9-4\lambda}{2(2-\lambda)}}(z_0), & 1<\lambda<2, \beta=0 \end{cases}$$

where c *is a constant independent of* z *and* n.

Remark 1.1. If $\beta=0$, we obtain the same result in [7].

Remark 1.2. If $\beta=0$, $\lambda=1$, we get

$$\varepsilon_n(z)\le c\,E_n(f,H_2')\left[dist(z,L)\right]^{-5/2}\begin{cases} n^{\varepsilon-\alpha}, & 0<\alpha\le\dfrac{1}{2} \\ n^{\varepsilon-\frac{1}{2}}, & \dfrac{1}{2}<\alpha<1 \end{cases} \tag{1.10}$$

for arbitrary small $\varepsilon>0$. This show that if $0<\alpha<\dfrac{1}{2}$ then (1.10) is better than (1.8) in the case $p=0$ to within $\forall\varepsilon>0$.

2. Some auxiliary results.

We shall use the notation "$a\prec b$" for $a\le c\,b$, where c is independent a, b and "$a\approx b$" if simultaneously $a\prec b$ and $b\prec a$. For an arbitrary $z_0\in B\subset G$, let $w=\varphi(z,z_0)$ be the conformal mapping of G onto the unit disc normalized by $\varphi(z_0,z_0)=0, \varphi'(z_0,z_0)>0$. Whenever we write $w=\varphi(z)$, It will be understood that $w=\varphi(z,z_0)$ for a fixed z_0. Let $w=\Phi(z)$ conformal mapping of $\Omega=C\overline{G}$ onto $\widetilde{\Omega}=\{w:|w|>1\}$ normalized by $\Phi(\infty)=\infty$, $\lim\limits_{z\to\infty}\dfrac{\Phi(z)}{z}>0$.

For $t>0$, then $L_t:=\{z:|\varphi(z)|=t, if\ t<1;\ \ |\Phi(z)|=t, if\ t>1\}$; $G_t:=\operatorname{int}L_t$; $\Omega_t:=extL_t$.

Let L be a K-quasiconformal curve and $D\subset C$. Then the region D can be chosen to be the region G_{R_0}-G_{r_0}, for a certain number $1<R_0\le 2$ depending on φ,Φ,f and $r_0=R_0^{-1}$ [1,p.28]. In the case, it is shown that the function $\alpha(.)=f^{-1}\left\{\overline{f(.)}\right\}^{-1}$ is a K^2 – quasiconformal reflection across L as shown in [10,p.75], that is, $\alpha(.)$ is a K^2 – antiquasiconformal mapping leaving points on L fixed and satisfying the conditions

$$\alpha(G_{\widetilde{R}} - G) \subset G - \overline{G_{r_0}} \ , \ \alpha(G - G_{\widetilde{r}}) \subset G_{R_0} - \overline{G}$$

for some $1 < \widetilde{R} < R_0$, $r_0 < \widetilde{r} < 1$. By using the facts in [10,p.76] , [11,p.98] we can find a $C(K)$-quasiconformal reflection $\alpha^*(.)$ across L such that it satisfies the following:

$$\left|z_1 - \alpha^*(z)\right| \approx \left|z_1 - z\right|, \ \ z_1 \in L \, , \, z \in D \tag{2.1}$$

Lemma 2.1[5]. *Let L be a K-quasiconformal curve,*

$$z_1 \in L \, , \quad z_2, z_3 \in G \cap \left\{z : \left|z - z_1\right| \prec d(z_1, L_{R_0})\right\}, \quad w_j = \varphi(z_j), \, j = 1,2,3$$

$$\left(z_2, z_3 \in \Omega \cap \left\{z : \left|z - z_1\right| \prec d(z, L_{r_0})\right\}, \quad w_j = \Phi(z_j), \, j = 1,2,3\right).$$

If $\left|z_1 - z_2\right| \prec \left|z_1 - z_3\right|$ *then*

a) $\left|w_1 - w_2\right| \prec \left|w_1 - w_3\right|$;

b) $$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2}$$

and therefore, for any $z_3 \in L_{R_0} (z_3 \in L_{r_0})$

$$\left|w_1 - w_2\right|^{K^2} \prec \left|z_1 - z_2\right| \prec \left|w_1 - w_2\right|^{K^{-2}}. \tag{2.2}$$

Lemma 2.2[2],[8]. *Let L be a K-quasiconformal curve. Then, for every $z \in L$ and $z_0 \in G$ there exist an arc $\beta(z_0, z)$ in G joining z_0 to z and with the following properties:*

a) $d(\varsigma, L) \approx \left|\varsigma - z\right|$ *for every* $\varsigma \in \beta(z_0, z)$;

b) for every pair $\varsigma_1, \varsigma_2 \in \beta(z_0, z)$, *if* $\widetilde{\beta}(\varsigma_1, \varsigma_2)$ *is the subarc of* $\beta(z_0, z)$ *joining* ς_1 *to* ς_2, *then* $mes\widetilde{\beta}(\varsigma_1, \varsigma_2) \prec \left|\varsigma_1 - \varsigma_2\right|$.

Lemma 2.3[8]. *Let L be a K-quasiconformal curve. Then, for any rectifiable arc $\gamma \subset G$*

$$mes\gamma \approx mes\alpha^*(\gamma).$$

Lemma 2.4[2]. *Let L be a K-quasiconformal curve and* $G_\varepsilon :=$ $:= \left\{z : z \in G \, , d(z, L) < \varepsilon\right\}$. *Then*

$$mes \ \varphi(G_\varepsilon) \prec \varepsilon^{\frac{1}{\delta}} \quad , \forall \varepsilon > 0 \, , \quad where \ \ \delta = \min\left\{2; K^2\right\}.$$

Lemma 2.5[4]. *Let G be an arbitrary Jordan domain; $\gamma \in \Omega$ is rectifiable arc except for one endpoint $z' \in L$ and satisfies the following conditions:*

a) $mes\gamma(\varsigma_1, \varsigma_2) \prec \left|\varsigma_1 - \varsigma_2\right|$ *for all* $\varsigma_1, \varsigma_2 \in \gamma$;

b) there exist an monotonically increasing function $h(t)$ such that

$$d(\varsigma, L) \succ h(\left|\varsigma - z'\right|) \ \ for \ all \ \ \varsigma \in \gamma .$$

Let $f(z)$ be given measurable function on the arc γ and there exist an a monotonically increasing function $\nu(t), \nu(0) = 0$ such that

$$\left|f(\varsigma)\right| \prec \nu(\left|\varsigma - z'\right|) \ \ \text{for all} \ \ \varsigma \in \gamma .$$

Then for the functions

$$F_\gamma(z) := \int_\gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma \quad , z \notin \gamma$$

satisfied:

$$\|F'_\gamma\|^2_{H'_2} \prec \ell^{-2}\left[\int_0^\ell v(t)dt\right]^2 + \int_0^{c\ell} v^2(t)\left[\frac{1}{t}+\frac{1}{t^2}\int_0^t \frac{rdr}{h(r)}+\int_0^t \frac{dr}{rh(r)}\right]dt,$$

where $\ell := mes\gamma$.

Now suppose that a domain $G \in C_\theta(\lambda;\beta), 0<\lambda<2, \quad \beta>0$ is given. After that, for the sake of simplicity , we assume that $\beta>0$, $p=2, m=3$, $z_1=-1, z_2=1$;$(-1,1)\subset G$ and let local co-ordinate axis in Definitions 1.4 be parallel to OX and OY in the co-ordinate system;

$$L^1 := \{z \in L : \operatorname{Im} z \ge 0\}\ ,\ L^2 := \{z \in L : \operatorname{Im} z \le 0\}.$$

Then z'_0 is taken as an arbitrary point on L^2 (or on L^1 subject to the chosen direction).

We recall that the domain $G \in C_\theta(\lambda;\beta)$ has interior zero angles and $\lambda\pi$, $0<\lambda<2$ exterior angles at the nearest neighbourhood of each points $z_1=-1$ and $z_2=1$, respectively.

Each $L^i, i=1,2$ is a $(1+\varepsilon_i)$-quasiconformal arc from Corollary1.2. Then, there exist $\alpha_i^*(.)$ quasiconformal reflection across $L^i, i=1,2$ satisfying (2.1). We will employ the following notations at neighbourhood of $z_1=-1$:

$$\gamma^1 := \alpha_1^*\left\{z=x+iy \,/\quad y=\frac{2c_1+c_2}{3}(1+x)^{1+\beta}\right\},$$

$$\gamma^2 := \alpha_2^*\left\{z=x+iy \,/\quad y=\frac{c_1+2c_2}{3}(1+x)^{1+\beta}\right\},$$

where constants c_i, $i=1,2$ are taken from the Definition1.4 . According to Lemma 2.3 for any $\varsigma_1,\varsigma_2 \in \gamma^i$, $i=1,2$

$$mes\beta(\varsigma_1,\varsigma_2) \preceq |\varsigma_1-\varsigma_2|. \tag{2.3}$$

For a big enough $n>N(R_0)$ and an arbitrary small $\varepsilon<1$, let us choose $R=1+c.n^{\varepsilon-1}$ such that $1<R<R_0$. Let us choose points z^i, $i=1,2$ such that they are at the intersection of L_R and γ^i the first point in $\widetilde{L}^1_R := \{z \,/\, z\in L_R, \operatorname{Im} z \ge 0\}$ or in $\widetilde{L}^2_R = L_R - \widetilde{L}^1_R$ (according to motion on L_R). These points divide L_R into two parts:

$L^1_R := L^1_R(z^1,z^3,z^2)$ are connecting points z^1, z^3 and z^2 where $z^3 \in L_R$ and $\operatorname{Re} z^3 \ge 0$, $\quad L^2_R := L_R - L^1_R$, $\qquad L_R := \bigcup_{i=1}^{2} L^i_R$, $\quad \Gamma_R := \gamma^1 \cup \gamma^2 \cup L^1_R$, $\quad U := \operatorname{int}(\Gamma_R \cup L)$, $\gamma^i(R) := \Gamma_R \cap \gamma^i$, $i=1,2$. We extend the function $w=\varphi(z,z_0)$ to U as follows :

$$\widetilde{\varphi}(z,z_0) := \begin{cases} \varphi(z,z_0) & ,z\in\overline{G} \\ \dfrac{1}{\overline{\varphi(\alpha_j^*(z),z_0)}} & ,\quad z\in U \end{cases}$$

From the Cauchy-Pompeiu's formula [12,p.148], we get

$$\varphi(z,z_0) := \frac{1}{2\pi i}\int_{\Gamma_R}\frac{\widetilde{\varphi}(\varsigma,z_0)}{\varsigma-z}d\varsigma - \frac{1}{\pi}\iint_U \frac{\widetilde{\varphi}_{\bar\varsigma}(\varsigma,z_0)}{\varsigma-z}d\sigma_\varsigma.$$

Then using the above notations we obtain:

$$\varphi(z,z_0):=\frac{1}{2\pi i}\int_{L_R}\frac{f(\varsigma,z_0)}{\varsigma-z}d\varsigma+\sum_{i=1}^{2}\frac{1}{2\pi i}\int_{\gamma^i(R)}\frac{\widetilde{\varphi}(\varsigma,z_0)-\varphi(-1,z_0)}{\varsigma-z}d\varsigma-$$
$$-\frac{1}{\pi}\iint_{U}\frac{\widetilde{\varphi}_{\bar{\varsigma}}(\varsigma,z_0)}{\varsigma-z}d\sigma\varsigma\,, \tag{2.4}$$

where

$$f(\varsigma,z_0):=\begin{cases}\widetilde{\varphi}(\varsigma,z_0) & ,\varsigma\in L_R^1\\ \varphi(-1,z_0) & ,\varsigma\in L_R^2\end{cases}$$

Lemma 2.6. *Let* $G\in C_\theta(\lambda;\beta)$, $0<\lambda<2$, $\beta>0$, *and* $z_0\in G$. *Then, there exists the polynomial* $Q_n(z,z_0)$ *such that* $Q_n(z_0,z_0)=0$, $Q_n'(z_0,z_0)=\varphi'(z_0,z_0)$

$$\left\|\varphi'(.,z_0)-Q_n'(.,z_0)\right\|_{H_2'}\preceq\Delta_1(z_0)n^{-\mu} \tag{2.5}$$

for every $n\geq 2$, $\mu<\min\left\{\frac{\lambda}{2-\lambda},\frac{1-\beta}{2(1+\beta)}\right\}$ *and*

$$\Delta_1(z_0):=\begin{cases}\delta^{-\frac{3}{2}}(z_0), & 0<\lambda\leq 1,\beta>0\\ \delta^{-\frac{5-2\lambda}{2(2-\lambda)}}(z_0), & 1<\lambda<2,\beta=0\end{cases}$$

Proof. Since the first term in the integral representation (2.4) is analytic in $\overline{G}$, there exist a polynomial $P_n(z,z_0)$ of degree $\leq n$ [14,p.142] such that

$$\left|\frac{1}{2\pi i}\int_{L_R}\frac{f(\varsigma,z_0)}{(\varsigma-z)^2}d\varsigma-P_n'(z,z_0)\right|\prec\frac{1}{n}\qquad ,z\in\overline{G}\,, \tag{2.6}$$

where the right part is independent from z_0. So from (2.4) we get

$$\left\|\varphi'(.,z_0)-P_n'(.,z_0)\right\|_{H_2'}\prec\frac{1}{n}+\sum_{i=1}^{2}\left\|\int_{\gamma^i(R)}\frac{\widetilde{\varphi}(\varsigma,z_0)-\varphi(-1,z_0)}{(\varsigma-z)^2}d\varsigma\right\|_{H_2'}+$$
$$\left\|\iint_{U}\frac{\widetilde{\varphi}_{\bar{\varsigma}}(\varsigma,z_0)}{(\varsigma-z)^2}d\sigma_\varsigma\right\|_{H_2'}=:\frac{1}{n}+\sum_{k=1}^{3}J_k\,. \tag{2.7}$$

The term J_k, $k=1,2$ have identical properties at the neighbourhood of -1, therefore we will estimate any of J_k, $k=1,2$. For all $\varsigma\in\gamma^i(R)$, $i=1,2$ from (2.1) and use [2, Lemma 4]

$$\left|\widetilde{\varphi}(\varsigma,z_0)-\varphi(-1,z_0)\right|\prec\delta^{-\frac{1}{2}}(z_0)\left|\varsigma+1\right|^{\frac{1}{2}}.$$

Then using Lemma 2.5 we obtain

$$J_k:=\left\|\int_{\gamma^i(R)}\frac{\widetilde{\varphi}(\varsigma,z_0)-\varphi(-1,z_0)}{(\varsigma-z)^2}d\varsigma\right\|_{H_2'}\prec\delta^{-\frac{1}{2}}(z_0).\ell_i^{\frac{1-\beta}{2}}\,, \tag{2.8}$$

$k=1,2$, $\ell_i=mes\gamma^i(R)$, $i=1,2$. On the other hand, from (2.3), Lemma 2.3 and Lemma 2.1 we get for $i=1,2$

$$mes\gamma^i(R)\preceq\left|z^i+1\right|\preceq[d(z^i,L^i)]^{\frac{1}{1+\beta}}\preceq n^{\frac{\varepsilon-1}{(1+\beta)}} \tag{2.9}$$

and, for $k=1,2$

$$J_k \preceq \delta^{-\frac{1}{2}}(z_0)\, n^{\frac{\beta-1-\varepsilon}{2(1+\beta)}}, \quad \forall \varepsilon > 0, \quad k = 1,2\,. \tag{2.10}$$

Since the Hilbert Transformation

$$(Tf)(z) := -\frac{1}{\pi}\iint_{C}\frac{f(\varsigma)}{(\varsigma - z)^2}\, d\sigma_\varsigma$$

is a bounded linear operator from $H_2^\prime \to H_2^\prime$, considering

$$\iint_U \left|\widetilde{\varphi}_{\bar\varsigma}(\varsigma, z_0)\right|^2 d\sigma_\varsigma \approx \iint_U \left|\varphi'(\alpha_i^*(\varsigma), z_0)\right|^2 d\sigma_\varsigma \preceq \iint_{\alpha_i^*(U)} \left|\varphi'(\varsigma, z_0)\right|^2 d\sigma_\varsigma \preceq$$

$$\preceq mes\varphi(\alpha_i^*(U), z_0)\,,$$

by (2.1) we get

$$J_3 \preceq \left\|\widetilde{\varphi}\right\|_{H_2^\prime} \preceq \left(mes\varphi(\alpha_i^*(U), z_0)\right)^{\frac{1}{2}}. \tag{2.11}$$

Before calculating the (2.11), we divide U to there parts such that for sufficiently small $0 < \varepsilon_0 < \frac{1}{2}$, let $U_{\varepsilon_0}(\pm 1) := \left\{\varsigma : \left|\varsigma \pm 1\right| \le \varepsilon_0\right\}$ and indicate $V_1 := V_1^1 \cup V_1^2 := U \cap U_{\varepsilon_0}(-1)$, where $V_1^1 := \left\{z : z \in V^1, \operatorname{Im} z \ge 0\right\}$ and $V_1^2 := \left\{z : z \in V^1, \operatorname{Im} z \le 0\right.$, $V_2 := U_{\varepsilon_0}(+1)$, $V_3 := U - (V_1 \cup V_2)$. So, $J_3 := J_3^{(1)} + J_3^{(2)} + J_3^{(3)}$ where calculating on V_1, V_2, V_3 respectively. Using Lemma 2.4 and [2,Lemma 5] we get

$$mes\varphi(\alpha_i^*(V_1^i), z_0) \preceq \delta^{-\frac{1}{(1+\varepsilon)^2}}(z_0)\, n^{\frac{\varepsilon-1}{(1+\beta)}},$$

$$mes\varphi(\alpha_i^*(V_3), z_0) \prec \delta^{-1}(z_0)\, n^{\varepsilon-1},$$

$$J_3^{(1)} \prec \delta^{-\frac{1}{2}}(z_0)\, n^{\frac{\varepsilon-1}{2(1+\beta)}} \quad , \quad J_3^{(3)} \prec \delta^{-\frac{1}{2}}(z_0)\, n^{\frac{\varepsilon-1}{2}}. \tag{2.12}$$

Let $G^* \in C_\theta(\lambda)$, $(0 < \lambda < 2)$, and F be a conformal mapping of G^* onto the unit disc and $\partial G^* = \partial G$ in the V^2. We know that how to estimate $mesF(\alpha_i^*(V^2), z_0)$ in [7,lemma3]. Conformal mappings φ and F have identical properties at the sufficiently small neighbourhood of $z_2 = +1$. According the same method in [7, Lemma3] we get

$$mes\varphi(\alpha_i^*(V_2), z_0) \prec \delta^{-\frac{1}{2-\lambda}}(z_0)\, n^{-2\eta}, \quad \eta < \min\left\{\frac{\lambda}{2-\lambda}, \frac{1}{2}\right\}$$

and

$$J_3^{(2)} \prec \delta^{-\frac{1}{2(2-\lambda)}}(z_0)\, n^{-\eta}, \tag{2.13}$$

we obtain

$$J_3 \prec \delta^{-\frac{1}{2(2-\lambda)}}(z_0)\, n^{-\eta} \tag{2.14}$$

from (2.12) and (2.13). So, according to (2.7),(2.10) and (2.14) we obtain

$$\left\|\varphi'(\cdot, z_0) - P_n'(\cdot, z_0)\right\|_{H_2^\prime} \preceq \frac{1}{n} + \delta^{-\frac{1}{2}}(z_0)\, n^{\frac{\beta-1-\varepsilon}{2(1+\beta)}} + \delta^{-\frac{1}{2(2-\lambda)}}(z_0)\, n^{-\eta}. \tag{2.15}$$

Now, let us $Q_n(z, z_0)$ is defined

$$Q_n(z, z_0) := \begin{cases} P_n(z, z_0) - P_n(z_0, z_0) + (z - z_0)[\varphi'(z_0, z_0) - P_n'(z_0, z_0)], & n > N(R_0) \\ (z - z_0)\varphi'(z_0, z_0), & n \le N(R_0) \end{cases}$$

It is easy to check (2.15) is satisfied and $Q_n(z_0,z_0)=0$, $Q_n'(z_0,z_0)=\varphi'(z_0,z_0)$ and according means value theorem we get

$$\|\varphi'(.,z_0)-Q_n'(.,z_0)\|_{H_2'} \le \|\varphi'(.,z_0)-P_n'(.,z_0)\|_{H_2'} + \|\varphi'(z_0,z_0)-P_n(z_0,z_0)\|_{H_2'} \prec$$
$$\prec (1+\delta^{-1}(z_0))\|\varphi'(.,z_0)-P_n'(.,z_0)\|_{H_2'} \qquad (2.16)$$

complete the proof from (2.15) and (2.16).

Lemma 2.7. *Let $G\in C_\theta(\lambda;\beta)$; $h(z)$ be defined by (1.5) and $D(z)\in Lip\alpha$, $0<\alpha\le 1$. Then for any $n\ge 2$ there exist a polynomial $T_n(z,z_0)$ such that $T_n(z_0,z_0)=\dfrac{\varphi'(z_0,z_0)}{D(z_0)}$ and*

$$\left\|\frac{\varphi'(.,z_0)}{D}-T_n(.,z_0)\right\|_{H_2'} \prec \Delta_1(z_0).\begin{cases} n^{-\mu'} & \mu'\le\mu \\ n^{-\mu} & \mu'>\mu \end{cases}$$

for all $0<\mu<\min\left\{\dfrac{\lambda}{2-\lambda},\dfrac{1-\beta}{2(1+\beta)}\right\}$, $0<\mu'<\alpha\lambda$, $\Delta_1(z_0)$ as it is in Lemma 2.6.

Proof. By assumption $\dfrac{1}{D(z)}\in Lip\alpha$, $z\in\overline{G}$. Hence, since $L=\partial G$ consist of the union of finite C_θ-arcs with $(1+\varepsilon_i)$-quasiconformal coefficient from Corollary1.2 and does not have exterior zero angles, by [11,theorem3] there are polynomials $\widetilde{Q}_m(z)$ such that

$$\left\|\frac{1}{D}-\widetilde{Q}_m\right\|_{C(\overline{G})} \prec d^{\alpha}(z,L_{1+\frac{1}{m}}). \qquad (2.17)$$

Let $Q_m(z,z_0)=\widetilde{Q}_m(z)-\widetilde{Q}_m(z_0)+\dfrac{1}{D(z_0)}$. From (2.17) we get

$$\left\|\frac{1}{D}-Q_m\right\|_{C(\overline{G})} \prec m^{-\mu'} \qquad (2.18)$$

for all $0<\mu'<\alpha\lambda$. Then, using (2.18) and Lemma 2.6 we obtain

$$\left\|\frac{\varphi'(.,z_0)}{D}-P_l'(.,z_0)Q_m(.,z_0)\right\|_{H_2'} \le \left\|\frac{1}{D}\right\|_{C(\overline{G})}\|\varphi'(.,z_0)-P_l'(.,z_0)\|_{H_2'}+$$
$$+\left\|\frac{1}{D}-Q_m\right\|_{C(\overline{G})}\|\varphi'(.,z_0)-P_l'(.,z_0)\|_{H_2'}+\|\varphi'(.,z_0)\|_{H_2'}\left\|\frac{1}{D}-Q_m\right\|_{C(\overline{G})} \prec$$
$$\prec \Delta_1(z_0).(l^{-\mu}+l^{-\mu}m^{-\mu'})+m^{-\mu'}$$

for all $0<\mu<\min\left\{\dfrac{\lambda}{2-\lambda},\dfrac{1-\beta}{2(1+\beta)}\right\}$, $0<\mu'<\alpha\lambda$. By defining $m:=n$ if $\mu'\le\mu$, $m:=\left[n^{\frac{\mu}{\mu'}}\right]+1$ if $\mu'>\mu$ and $T_n:=P_l'Q_m$ we complete the proof.

Lemma 2.8. *Let G be a Jordan domain, such that there exist a polynomials $Q_n(z,z_0)$, $Q_n(z_0,z_0)=\dfrac{\varphi'(z_0,z_0)}{D(z_0)}$ satisfied the following properties:*

$$\left\| \frac{\varphi'(.,z_0)}{D} - T_n(.,z_0) \right\|^2_{H_2^1} \preceq \sigma_n(z_0) \qquad (2.19)$$

for some $\{\sigma_n(z_0)\}_{n=0}^{\infty}$ *with* $\sigma_n(z_0) \to 0, \quad n \to \infty$, *for every* $z_0 \in G$. *Then*

$$\sum_{i=n}^{\infty} |K_i(z_0)|^2 = O\left(\delta^{-2}(z_0).\sigma_n(z_0)\right). \qquad (2.20)$$

Proof. It is well known that the function minimizing the integral

$$J(.) := \iint_G h(z)|.|^2 d\sigma_z \qquad (2.21)$$

in the class of functions f analytic in G, square integrable over G normalised by $f(z_0) = \lambda_0 := \varphi'(z_0, z_0) / D(z_0)$ is $f_0(z, z_0) := \varphi'(z, z_0) / D(z)$ [13].

On the other hand, the polynomial minimizing (2.21) in the class of polynomials $Q_{n-1}(z)$, $\deg Q_{n-1}(z) \le n-1$, by normalised $Q_{n-1}(z_0) = \lambda_0$ is

$$\widetilde{Q}_{n-1}(z) := \lambda_0 \frac{\sum_{i=0}^{n-1} \overline{K_i(z_0)}\, K_i(z)}{\sum_{i=0}^{n-1} |K_i(z_0)|^2}, \quad \text{and} \quad J(\widetilde{Q}_{n-1}) = \frac{|\lambda_0|^2}{\sum_{i=0}^{n-1} |K_i(z_0)|^2}. \qquad (2.22)$$

Also,

$$\pi \le \iint_G h(z) \left|\widetilde{Q}_{n-1}(z, z_0)\right|^2 d\sigma_z = \pi + \iint_G h(z) \left|f_0(z, z_0) - \widetilde{Q}_{n-1}(z, z_0)\right|^2 d\sigma_z \le$$

$$\le \pi + \iint_G h(z) |f_0(z, z_0) - T_{n-1}(z, z_0)|^2 d\sigma_z \le \pi + c \iint_G |f_0(z, z_0) - T_{n-1}(z, z_0)|^2 d\sigma_z, \qquad (2.23)$$

where T_{n-1} is an arbitrary polynomial with $T_{n-1}(z_0, z_0) = \lambda_0$. So, we get

$$\frac{|\lambda_0|^2}{\sum_{i=0}^{n-1} |K_i(z_0)|^2} = \pi + O(\sigma_n(z_0))$$

from (2.21),(2.23),(2.19) and consequently

$$\sum_{i=0}^{n-1} |K_i(z_0)|^2 = \frac{|\lambda_0|^2}{\pi} - O\left(|\lambda_0|^2.\sigma_n(z_0)\right). \qquad (2.24)$$

Let $m > n$. From (2.24) we have

$$\sum_{i=n}^{m} |K_i(z_0)|^2 = O\left(|\lambda_0|^2.\sigma_n(z_0)\right) - O\left(|\lambda_0|^2.\sigma_m(z_0)\right)$$

and taking limit as $m \to \infty$

$$\sum_{i=n}^{\infty} |K_i(z_0)|^2 = O\left(|\lambda_0|^2 \sigma_n(z_0)\right).$$

By considering $|\lambda_0| \approx \delta^{-1}(z_0)$ [1,Lemma 3], we complete the proof.

Proof of Theorem 1. Using the Minkowski's inequality we obtained

$$\varepsilon_n(z) = |f(z) - S_n(f, z)| = \left| \sum_{k=0}^{\infty} a_k K_k(z) - \sum_{k=0}^{n} a_k K_k(z) \right| = \left| \sum_{k=n+1}^{\infty} a_k K_k(z) \right| \le$$

$$\le \left(\sum_{k=n+1}^{\infty} |a_k|^2 \right)^{1/2} \cdot \left(\sum_{k=n+1}^{\infty} |K_k(z)|^2 \right)^{1/2}. \qquad (2.25)$$

It is well known that $E_n(f, H_2^{\prime}) = \left(\sum_{k=n+1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}$ in $H_2^{\prime}$ and using Lemma 2.7-2.8 we complete the proof.

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