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WEIGHTED IMBEDDING THEOREMS FOR ANISOTROPIC BANACH-VALUED SOBOLEV SPACES

Abstract

In this paper two-weighted L_p- estimate for singular integrals operator is obtained. On that ground weighted imbedding theorems on the anisotropic Banach-valued Sobolev spaces $W^l_{p,\omega}(R^n;B)$ are proved.

Let N be a set of natural numbers, $N_0 = N \bigcup \{0\}$, $n \in N$, $R_0^n = R^n \setminus \{0\}$, e^i be the base vector of standard basis in R^n, $x = (x_1,...,x_n) = \sum_1^n x_i e_i$, $y \in R^n$, $(x,y) = \sum_1^n x_i \cdot y_i$, $|x| = (x,x)^{1/2}$.

By $\alpha = (\alpha_1,...,\alpha_n)$, $k = (k_1,...,k_n)$ let's denote the multi-index with integer non-negative components $|\alpha| = \sum_1^n \alpha_i$, $(\alpha,k) = \sum_1^n \alpha_i \cdot k_i$.

Let B be a Banach space, the norm of element $a \in B$ let's denote by $\|a\|_B$. If $\Omega \subset R^n$ is a Borel set then we denote Lebesgue measure of the set Ω by $|\Omega|$ and by χ_Ω a characteristic function of the set Ω.

We'll say that the Banach space B is ζ - convex (or convex by Burkholder [6]); if symmetric function $\zeta(a,b)$ on $B \times B$ exists, convex by every variable and satisfying the conditions:

$$\zeta(0,0) > 0\,,\ \ \zeta(a,b) \le \|a+b\|_B \ \text{ for } \ \|a\|_B = \|b\|_B = 1\,.$$

Let ω be a positive, measurable function given in R^n and $1 \le p \le \infty$.

By $L_{p,\omega}(R^n,B)$ we'll denote the space of strongly measurable on R^n B-valued functions $f(x)$ for which the next norm is finite

$$\|f\|_{L_{p,\omega}(R^n,B)} = \left(\int_{R^n} \|f(x)\|_B^p dx \right)^{1/p} ,$$

(when $p = \infty$ the usual modification is supposed).

Let the vector $a = (a_1,...,a_n)$, $a_i > 0$ $(i = 1,...,n)$ be given, and the function $\rho(x)$ is a positive solution of the equation

$$\sum_{i=1}^n x_i^2 \rho^{-2a_i} = 1\,.$$

Let $K : R^n \setminus \{0\} \to R$ be an anisotropic singular kernel, satisfying the conditions:

1) $K(t^\alpha x) \equiv K(t^{\alpha_1} x_1,...,t^{\alpha_n} x_n) = t^{-|\alpha|} K(x)$ for any $t > 0$, $x \in R^n \setminus \{0\}$;

2) $\int_S K(x) \sum_{i=1}^n a_i x_i^2 d\sigma(x) = 0$;

3) $\int_0^1 \omega_K(t)\frac{dt}{t} < \infty$, where

$$\omega_K(t) = \sup\{|K(x) - K(y)| : x, y \in S, |x - y| \le t\},$$

S is a unique sphere in R^n, $d\sigma$ is a surface- measure on it .

Let $\varepsilon > 0$ be some number. Let's suppose for the vector-function $f : R^n \to B$

$$T_\varepsilon f(x) = \int_{\rho(x-y)>\varepsilon} K(x-y)f(y)dy,$$

where the integral is understood in the sense of Bochner-Lebesgue.

Let's consider the anisotropic singular integral operator

$$Tf(x) = \int_{R^n} K(x-y)f(y)dy := \lim_{\varepsilon \to 0+} T_\varepsilon f(x). \qquad (1)$$

In future we'll say that a pair of weights $(\omega, \omega_1) \in \widetilde{S}_{pq\alpha}(Q)$, [5], if

$$\sup_{t>0}\left(\int_t^\infty \omega_1(\tau)\tau^{(\alpha-Q)q+Q-1}d\tau\right)^{p/q}\left(\int_0^t \omega(\tau)^{1-p'}\tau^{Q-1}d\tau\right)^{p-1} < \infty,$$

$$\sup_{t>0}\left(\int_0^t \omega_1(\tau)\tau^{Q-1}d\tau\right)^{p/q}\left(\int_t^\infty \omega(\tau)^{1-p'}\tau^{(\alpha-Q)q+Q-1}d\tau\right)^{p-1} < \infty,$$

where $1 < p \le q < \infty$, $0 < \alpha < Q$.

Theorem 1. *Let B be a ζ- convex Banach lattice, $f \in L_{p,\omega(\rho(x))}(R^n; B)$ $1 < p < \infty$, the kernel $K(x)$ satisfy the conditions 1)-3) and $\omega(t), \omega_1(t)$ be positive functions on $(0,\infty)$.*

If

$$\sup_{t<\tau\le 8t} \omega_1(\tau) \le C \inf_{t<\tau\le 8t} \omega(\tau)$$

and $(\omega, \omega_1) \in \widetilde{S}_{pp0}(|a|)$. then the singular integral (1) exists almost at all $x \in R^n$ by norm B and it holds the inequality

$$\int_{R^n} \|Tf(x)\|_B^p \omega_1(\rho(x))dx \le C \int_{R^n} \|f(x)\|_B^p \omega(\rho(x))dx,$$

where the constant C doesn't depend on f.

Proof. Applying the Minkowski inequality we'll get

$$\left(\int_{R^n} \|Tf(x)\|_B^p \omega_1(\rho(x))dx\right)^{1/p} = \left(\sum_{k\in Z_{2^k<\rho(x)\le 2^{k+1}}} \|Tf(x)\|_B^p \omega_1(\rho(x))dx\right)^{1/p} \le$$

$$\le \left(\sum_{k\in Z_{2^k<\rho(x)\le 2^{k+1}}} \left\|T\left(f\chi_{\{\rho(y)\le 2^{k-1}\}}\right)(x)\right\|_B^p \omega_1(\rho(x))dx\right)^{1/p} +$$

$$+\left(\sum_{k\in Z_{2^k<\rho(x)\le 2^{k+1}}} \left\|T\left(f\chi_{\{2^{k-1}<\rho(y)\le 2^{k+2}\}}\right)(x)\right\|_B^p \omega_1(\rho(x))dx\right)^{1/p} +$$

$$+\left(\sum_{k\in Z_{2^k<\rho(x)\le 2^{k+1}}} \left\|T\left(f\chi_{\{\rho(y)>2^{k+2}\}}\right)(x)\right\|_B^p \omega_1(\rho(x))dx\right)^{1/p} = B_1 + B_2 + B_3.$$

Let's estimate B_1.

When $2^k < \rho(x) \le 2^{k+1}$, $\rho(y) \le 2^{k-1}$ it holds

$$\rho(y) \le \rho(x),\ \rho(x-y) \ge C\frac{1}{2}\rho(x).$$

By virtue of 3) $\omega_K(t) \to 0$, when $t \to 0$. That is why $K(x)$ is continuous on a unique sphere and by virtue of the condition 1) $K(x)$ is continuous on $R^n \setminus \{0\}$

$$B_1 \le C\left(\sum_k \int_{2^k \le \rho(x) \le 2^{k+1}} \omega_1(\rho(x))\rho(x)^{-|\alpha|p} \times\right.$$

$$\left. \times \int_{\rho(y) \le \rho(x)} \left(\|f(y)\|_B dy\right)^p dx\right)^{1/p} = C\left(\int_{R^n} \omega_1(\rho(x))\rho(x)^{-|\alpha|p} \times\right.$$

$$\left. \times \left(\int_0^{\rho(x)} \tau^{|\alpha|-1}\left(\int_S \|f(\delta_\tau \bar{y})\|_B d\sigma(\bar{y})\right)d\tau\right)^p dx\right)^{1/p} =$$

$$= C\left(\int_0^\infty \omega_1(t) t^{|\alpha|-|\alpha|p-1}\left(\int_0^t \tau^{|\alpha|-1}\left(\int_S \|f(\delta_\tau \bar{y})\|_B d\sigma(\bar{y})\right)d\tau\right)^p dt\right)^{1/p} \le$$

$$\le C\left(\int_0^\infty \omega(t) t^{|\alpha|-1}\left(\int_S \|f(\delta_\tau \bar{y})\|_B d\sigma(\bar{y})\right)^p dt\right)^{1/p} \le$$

$$\le C\left(\int_0^\infty \omega(t) t^{|\alpha|-1}\left(\int_S \|f(\delta_\tau \bar{y})\|_B^p d\sigma(\bar{y})\right)dt\right)^{1/p} = C\left(\int_{R^n} \|f(y)\|_B^p \omega(\rho(y))dy\|\right)^{1/p}.$$

Let's estimate B_3.

When $2^k < \rho(x) \le 2^{k+1}$, $\rho(y) \ge 2^{k+2}$ holds $\rho(x) \le \rho(y),\ \rho(x-y) \ge C\frac{1}{2}\rho(y)$.

Then

$$B_3 \le C\left(\sum_{k \in Z_{2^k < \rho(x) \le 2^{k+1}}} \omega_1(\rho(x))\left(\int_{\rho(y) \ge \rho(x)} \frac{\|f(y)\|_B}{\rho(y)^{|\alpha|}} dy\right)^p dx\right)^{1/p} =$$

$$= C\left(\int_{R^n} \omega_1(\rho(x))\left(\int_{\rho(x)}^\infty \tau^{-1}\left(\int_S \|f(\delta_\tau y)\|_B d\sigma(\bar{y})\right)^p d\tau\right)dx\right)^{1/p} =$$

$$= C\left(\int_0^\infty \omega_1(t) t^{|\alpha|-1}\left(\int_t^\infty \tau^{-1}\left(\int_S \|f(\delta_\tau \bar{y})\|_B d\sigma(\bar{y})\right)^p d\tau\right)dt\right)^{1/p} \le$$

$$\le C\left(\int_0^\infty \omega(t) t^{|\alpha|-1}\left(\int_S \|f(\delta_\tau \bar{y})\|_B d\sigma(\bar{y})\right)^p dt\right)^{1/p} \le C\left(\int_{R^n} \|f(y)\|_B^p \omega(\rho(y))dy\right)^{1/p}.$$

Now let's estimate B_2. Applying the theorem on L_p boundedness of B-valued anisotropic singular integral (1), ([4]), we'll get

$$B_2^p \le C\sum_{k\in Z}\left(\sup_{2^{k-1}<\rho(x)\le 2^{k+2}}\omega_1(\rho(x))\right)\int_{R^n}\left\|T\left(f\chi_{\{2^{k-1}<\rho(y)\le 2^{k+2}\}}\right)(x)\right\|_B^p dx \le$$

$$\le C\sum_{k\in Z}\left(\sup_{2^{k-1}<\rho(x)\le 2^{k+2}}\omega_1(\rho(x))\right)\int_{\{2^{k-1}<\rho(x)\le 2^{k+2}\}}\|f(x)\|_B^p dx \le$$

$$\le C\sum_{k\in Z}\int_{2^{k-1}<\rho(x)\le 2^{k+2}}\|f(x)\|_B^p\omega(\rho(x))dx \le C\int_{R^n}\|f(x)\|_B^p\omega(\rho(x))dx.$$

The theorem is proved.

Let's suppose $B_j=\frac{\partial}{\partial x_j}$, $D^k=D_1^{k_1}...D_n^{k_n}$. Let $C_0^\infty(R^n)$ be totality of all numerical infinitely differentiable finite in R^n functions. Let f and $D^k f$ are locally summable by Bochner functions on R^n. If for any function $\varphi\in C_0^\infty(R^n)$ the equality

$$\int_{R^n}\varphi(x)D^k f(x)dx=(-1)^{|k|}\int_{R^n}D^k\varphi(x)f(x)dx$$

is fulfilled, where $k=(k_1,...,k_n)$, ($k_i\ge 0$ are integers), then $D^k f$ is called generalized derivative of the function f of order k in R^n.

Let $L\in C_0^\infty(R^n)$

$$\int_{R^n}L(x)dx=1.$$

When $\forall f\in L_1^{loc}(R^n;B)$ as [1] the mean function

$$f_{\vartheta^\lambda}(x)=\vartheta^{-|\lambda|}\int_{R^n}L(y\vartheta^{-\lambda})f(x+y)dy,$$

$$\lambda=(\lambda_1,...,\lambda_n),\ \vartheta^{-\lambda}y=\left(\vartheta^{-\lambda_1}y_1,...,\vartheta^{-\lambda_n}y_n\right),\ \vartheta>0,\lambda_i>0$$

is defined.

Analogously [1] almost at all $x\in R^n$ we'll get

$$f(x)=f_{h^\lambda}(x)+\sum_{i=1}^n\lambda_i\int_0^h\vartheta^{-1-|\lambda|+\lambda_i l_i}d\vartheta\int_{R^n}L_i(\vartheta^{-\lambda}y)D_i^{l_i}f(x+y)dy, \tag{2}$$

where $0<\vartheta<h$, $f_{h^\lambda}(x)$ is a mean function, defined above, and the function $L_i\in C_0^\infty(R^n)$.

Let's consider the next integral operator, which plays an important role in imbedding theorems, obtained on the basis of integral representations (2)

$$K_{\varepsilon r}f(x)=\int_{R^n}K_{\varepsilon r}(x-y)f(y)dy,$$

where $K_{\varepsilon r}(x)=\int_\varepsilon^r L(\vartheta^{-\lambda}y)\vartheta^{-1-|\lambda|}d\vartheta,\ 0<\varepsilon<r\le\infty$.

Theorem 2. *Let* $L\in C_0^\infty(R^n)$, $\int_{R^n}L(x)dx=0$, $f\in L_{p,\omega}(R^n;B)$, $1<p<\infty$ *and the weight pair* $(\omega_1(\rho(x)))$, $\omega(\rho(x))$ *satisfy the conditions of theorem 1.*

Then $K_{\varepsilon r}f\in L_{p,\omega_1}(R^n;B)$ and when $\varepsilon\to 0$ $K_{\varepsilon r}f$ converges in $L_{p,\omega_1}(R^n;B)$ to some function $K_{0r}f$, moreover

$$\left\|K_{\varepsilon r} f\right\|_{L_{p,\omega_1}(R^n;B)} \le C_p \|f\|_{L_{p,\omega}(R^n;B)}, \tag{3}$$

where C_p doesn't depend on f, ε.

The proof of this theorem is conducted analogously to the proof of theorem 1.

Let ω_i, $i = 0,1,...,n$ be weight functions.

Let's define the weight Sobolev space $W^{l_1,...,l_n}_{p,\omega_0,\omega_1,...,\omega_n}(R^n;B)$, $l_i \ge 0$ are integers, $1 \le p \le \infty$ as a totality of strongly measurable B- valued functions $f(x)$, $x \in R^n$, which has a generalized derivatives $D_i^{l_i} f$ with finite norm

$$\left\|f, W^{l_1,...,l_n}_{p,\omega_0,\omega_1,...,\omega_n}(R^n;B)\right\| = \|f\|_{L_{p,\omega_0}(R^n;B)} + \sum_{i=1}^{n} \left\|D_i^{l_i} f\right\|_{L_{p,\omega_i}(R^n;B)}.$$

It holds

Theorem 3. *Let* $\frac{1}{\lambda} = l \in N^n$, $1 < p \le q < \infty, \text{æ} = \left|\left(\mathrm{k} + \frac{1}{\mathrm{p}} - \frac{1}{q}\right) : l\right| \le 1$ *and the weight pair* (ω_j, ω), $j = 0,1$ *depend only on* $\rho(x)$. *Let also the weight pair* (ω_j, ω), *belong to the class* $\widetilde{S}_{p,q,\text{æ}}(|1/l|)$.

Then when $\text{æ} < 1$ *for any Banach space* B *it holds the continuous imbedding*

$$D^k W^{l_1,...,l_n}_{p,\omega_0,\omega_1}(R^n;B) \subset L_{q,\omega}(R^n;B).$$

If also B *is a* ζ*-convex Banach lattice, then imbedding is also true when* $1 < p = q < \infty$, $\text{æ} = 1$.

In addition

$$\left\|D^k f\right\|_{L_{q,\omega}(R^n;B)} \le C \left\|f, W^{l_1,...,l_n}_{p,\omega_0,\omega_1}(R^n;B)\right\|,$$

where the constant C *doesn't depend on* f.

Proof. Differenting the equality

$$f_{\varepsilon^\lambda}(x) = f_{h^\lambda}(x) +$$
$$+ \sum_{i=1}^{n} \lambda_i \int_{\varepsilon}^{h} \vartheta^{|\lambda|} d\vartheta \int_{R^n} L_i\left(\vartheta^{-\lambda} y\right) D_i^{l_i} f(x+y) dy$$

and applying theorem 2 we'll get

$$\left\|\int_{\varepsilon}^{h} \vartheta^{|\lambda|-(k,\lambda)} d\vartheta \int_{R^n} L_i^{(k)}\left(\vartheta^{-\lambda} y\right) D_i^{l_i} f(x+y) dy\right\|_{L_{p,\omega}(R^n;B)} \le C \left\|D_i^{l_i} f\right\|_{L_{p,\omega_1}(R^n;B)}.$$

Besides

$$\left\|D^k f_{h^\lambda}\right\|_{L_{p,\omega}(R^n;B)} \le C \|f\|_{L_{p,\omega_0}(R^n;B)}.$$

Combining these estimates we get

$$\left\|D^k f_{\varepsilon^\lambda}\right\|_{L_{p,\omega}(R^n;B)} \le C \|f\|_{W^{l_1,...,l_n}_{p,\omega_0,\omega_1}(R^n;B)}.$$

For conclusion the proof of the theorem, we establish two facts; first of all the convergence of $D^k f_{\varepsilon^\lambda}$ to some element from $L_{p,\omega}(R^n;B)$ when $\varepsilon \to 0$ is proved; secondly, it is proved that this limit element is a generalized derivative $D^k f$ of the function f to which f_{ε^λ} converges when $\varepsilon \to 0$.

For proving the convergence $D^k f_{\varepsilon^\lambda}$ to some element from $L_{p,\omega}(R^n;B)$ when $\varepsilon \to 0$ we prove the fundamentalness of the sequence $\{D^k f_{\varepsilon^\lambda}\}$ in the norm $L_{p,\omega}(R^n;B)$. We have when $0<\varepsilon<\eta$

$$\left\|D^k f_{\varepsilon^\lambda} - D^k f_{\eta^\lambda}\right\|_{L_{p,\omega}(R^n;B)} \le C\sum_{i=1}^{n}\int_{\varepsilon}^{\eta} \upsilon^{-æ} d\upsilon \left\|M_i\right\|_{L_{1,\omega}(R^n)} \left\|D_i^{l_i} f\right\|_{L_{p,\omega}(R^n;B)} \le$$
$$\le C\eta^{1-æ}\left\|D_i^{l_i} f\right\|_{L_{p,\omega_1}(R^n;B)}.$$

Then by virtue of the Lebesgue theorem we conclude, that the sequence $\{D^k f_\varepsilon^\lambda\}$ is fundamental.

From here, in view of completeness of the space $L_{p,\omega}(R^n;B)$ it follows the convergence of $D^k f_{\varepsilon^\lambda}$ to some element g from $L_{p,\omega}(R^n;B)$ when $\varepsilon \to 0$. By definition the generalized Sobolev's derivative at every fixed ε for the derivative function $\psi \in C_0^\infty(R^n)$ it holds the equality

$$\int_{R^n} D^k \psi f_{\varepsilon^\lambda} dx = (-1)^{|k|} \int_{R^n} \psi D^k f_{\varepsilon^\lambda} dx .$$

Allowing for $f \in L_1^{loc}(R^n;B)$, and $f_{\varepsilon^\lambda} \to f(x)$ in $L_1^{loc}(R^n;B)$ passing to the limit when $\varepsilon \to 0$, we'll get:

$$\int_{R^n} D^k \psi f dx = (-1)^{|k|} \int_{R^n} \psi D^k g dx ,$$

from which it follows, that the limit element g of the sequence $\{D^k f_{\varepsilon^\lambda}$ is a generalized derivative of $D^k f$ of the function f.

The theorem is proved.

Corollary 1. *Let* $\frac{1}{\lambda} = l \in N^n$, $1 \le p \le q \le \infty$, $æ = 1 - \left|\left(k + \frac{1}{p} - \frac{1}{q}\right) \cdot l\right| > 0$ *and the weights pairs* (ω_j, ω), $j = 0,1$ *depend only on* $\rho(x)$. *Let the weight pairs* (ω_j, ω) *belong to the class* $\widetilde{S}_{p,q,æ}(\|1/l\|)$ *too.*

Then when

$$\left(k, \frac{1}{l}\right) = \sum_{i=1}^{n} \frac{k_i}{l_i} < 1, \quad \left(k + \frac{1}{p} - \frac{1}{q}, \frac{1}{l}\right) = 1, \; 1 < p < q < \infty$$

for the Banach space $B = l_\theta$, $1 \le \theta \le \infty$ it holds the continuous imbedding

$$D^k W_{p,\omega_0,\omega_1}^{l_1,\ldots,l_n}(R^n; l_\theta) \subset L_{q,\omega}(R^n; l_\theta).$$

If besides $1 < \theta < \infty$, then imbedding is true when $1 < p = q < \infty$, $\left(k, \frac{1}{l}\right) = 1$, and what is more the inequality

$$\left\|D^k f\right\|_{L_{q,\omega}(R^n;l^\theta)} \le C\left\|f, W_{p,\omega_0,\omega_1}^{l}(R^n; l^\theta)\right\|$$

with a constant in dependent of f, is true.

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