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ON THE MEAN CURVATURE OF A DEVELOPABLE TIME-LIKE RULED SURFACE AT THE MOTION OF SPACE WITH ONE PARAMETER L/L' IN R_1^3

Abstract

In this study, the integral invariants of a time-like ruled surface are introduced. Furthermore, the mean of a developable time-like ruled surface in this space is calculated in terms of its development invariants.

1. Introduction.

A surface in the 3-dimensional Minkowski space $R_1^3 = \left(R_1^3, dx^2 + dy^2 - dz^2\right)$ is called a time-like surface if the induced metric on the surface is a Lorentz metric [1]. A ruled surface is a surface swept out by a straight line X moving along a curve α. The various positions of the generating line X are called the ruling of the surface. Such a surface, thus, has a parameterization in ruled form as follows,

$$\varphi(s,\mathrm{v}) = \alpha(s) + \mathrm{v}X(s). \tag{1.1}$$

We call α to be the base curve, and X to be director curve. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surface are called skew surfaces [6]. If there exists a common perpendicular to two preceding ruling in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction.

The time-like ruled surface M is given by the parameterization

$$\varphi : IxR \to R_1^3,$$
$$(s,\mathrm{v}) \to \varphi(s,\mathrm{v}) = \alpha(s) + \mathrm{v}X(s) \tag{1.2}$$

in R_1^3, where $\alpha : R \to R_1^3$ is a differentiable space-like curve parameterized by its arc-length in $R_1^3\left(<\alpha'(s),\alpha'(s)>=1\right)$ and $X(s)$ is the director vector of the director curve such that X is orthogonal to the tangent vector field T of the base curve α. $\{T,\overline{N},X\}$ is an orthogonal frame field along α in R_1^3, where $\overline{N}$ is the normal vector field of M along α. $\overline{N}$ and T are space-like and X is time-like

$$<T,T>=<\overline{N},\overline{N}>=1, \quad <X,X>=-1. \tag{1.3}$$

The curve of striction of a skew time-like surface is given by

$$\overline{\alpha}(s) = \alpha(s) - \frac{<\frac{d\alpha}{ds},\frac{dX}{ds}>}{<\frac{dX}{ds},\frac{dx}{ds}>}X(s) \tag{1.4}$$

$\overline{\alpha}(s)$ is a space-like curve [6]. Let P_x be distribution parameter of time-like ruled surface, then

$$P_x = -\frac{\det\left[\frac{d\alpha}{ds}, X, \frac{dX}{ds}\right]}{<\frac{dX}{ds},\frac{dX}{ds}>}. \tag{1.5}$$

Lemma 1. *The time-like ruled surface* $\varphi(s,v)$ *is developable iff the parameter of distribution* P_x *is equal zero. That is,*

$$\det\left[\frac{d\alpha}{ds}, X, \frac{dX}{ds}\right] = 0 \text{ , [3].} \tag{1.6}$$

2. The Frenet vectors, one parameter spatial motion and integral invariants.

If the principle vector field N of a space-like curve $\alpha(s)$ is space-like and the binormal vector field B is time-like, then we have following Frenet formula along $\alpha(s)$

$$\begin{aligned} D_T T &= k_1 N, \\ D_T N &= -k_1 T + k_2 B, \\ D_T B &= k_2 N \end{aligned} \tag{2.1}$$

[2, 6].

Let $\alpha : I \to R_1^3$ be a space-like curve and $\{T, N, B\}$ be Frenet vector, where T, N and B are tangent, principle normal and binormal vectors of the curve, respectively. B is time-like, T and N are space-like vectors.

The two coordinate systems $\{0; T, N, B\}$ and $\{O'; e_1, e_2, e_3\}$ are orthogonal coordinate systems in R_1^3 which represent the moving space L and the fixed space L', respectively. Let us express the displacements (L/L') of H with respect to L'. During the one parameter spatial motion H/H', each fixed line X of the moving space H, generates, in generally, a time-like ruled surface in the fixed space L'.

Definition 2.1. *The vector which is determined by the curvilinear integral along the given curve* α *of Pfaff vector* w *in the motion of* L/L' *as,*

$$D = \int w \tag{2.2}$$

is called the Streiner Rotation vector of motion L/L' *and the vector*

$$V = \int dx \tag{2.3}$$

is called the Steiner Transition Vector.

Definition 2.2. *The length of development* L_a *and the angle of development* λ_a *of the time-like ruled surface are calculated by the following equalities:*

$$L_a = \int ds = \int < d\alpha, X > \tag{2.4}$$

and

$$\lambda_a = < D, a > . \tag{2.5}$$

The Steiner Rotation Vector of motion, when L is chosen as $Sp\{T, N, B\}\alpha(s)$ in the moving space L/L' along the base curve of ruled surface given by (1.1) is obtained as

$$D = \int (k_2 T - k_1 B) ds \tag{2.6}$$

from (2.2). Where T and B are unit space-like tangent and time-like binormal vector fields at the base curve respectively k_1 and k_2 are also curvature functions of the base curve. The moving straight line X such that it is strictly connected to the system $\{T, N, B\}$ is represented uniquely with respect to this system as in the form

$$X = x_1 T + x_2 N + x_3 B \tag{2.7}$$

where the components x_i, $(i=1,2,3)$ are scalars in L. The angle of development and the length of the time-like ruled surface which is drawn by the straight line X in L' are obtained as in the form

$$\lambda_x = x_1\lambda_T + x_3\lambda_B \tag{2.8}$$

and

$$L_x = x_1 L_T, \tag{2.9}$$

where λ_T and λ_B are the angles of development in the direction of the tangent and binormal of ruled surface and L_T is also the length of development in the direction of the tangent vector. When this time-like ruled surface is developable. $P_x = 0$ by lemma 1. For this reason, we state the following theorem.

Theorem 2.1. *During the one-parameter motion L/L' the time-like ruled surface in the fixed space L' generated by fixed X of the moving space L is developable if and only if $\alpha(s)$ is a helix such that the harmonic curvature of the base curve $\alpha(s)$ of its development invariants satisfies the equality*

$$\frac{k_1}{k_2} = \frac{-\left(L_T^2 + L_x^2\right)\lambda B}{L_x\left(L_T\lambda_x - \lambda_T L_x\right)} \tag{2.10}$$

(1.3), (1.5), (2.1) and (2.7) equalities can obtained following theorem.

Theorem 2.2. *The parameters of distribution P_N and P_B of time-like ruled surface drawn in L' by the corresponding straight lines to the principle space-like normal $\vec{N}$ and time-like binormal B are obtained as,*

$$P_N = \frac{-k_2}{k_1^2 - k_2^2}, \quad P_B = -\frac{1}{k_2}, \tag{2.11}$$

respectively.

3. The mean curvature of developable time-like ruled surface.

If L is chosen as $Sp\{T,N,B\}$ in the motion of space L/L' with one parameter which draws the time-like ruled surface $\varphi(s,\mathrm{v})$ thus (1.1) is expressed as

$$\varphi(s,\mathrm{v}) = \alpha(s) + \mathrm{v}\left(x_1T + x_2N + x_3B\right). \tag{3.1}$$

Let us calculate the mean curvature of time-like ruled surface which is drawn by the straight line X in L' given by the following relation:

$$2H_a = \left(EN + GL - 2MF\right)/EG - F^2, \text{[4]} \tag{3.2}$$

where E,F,G and L,N,M are the coefficients of the first and second fundamental forms, respectively. These coefficients are obtained by partial differentiation of (3.1) according to s and v as in the form

$E = <\varphi_s,\varphi_s> = \mathrm{v}^2\left(x_3k_1 + x_1k_2\right)^2 + \mathrm{v}^2\left(k_2^2 - k_1^2\right) + 1 - 2\mathrm{v}x_2k_1,$

$F = <\varphi_s,\varphi_\mathrm{v}> = x_1,$

$G = <\varphi_\mathrm{v},\varphi_\mathrm{v}> = -1,$

$M = <\varphi_{s\mathrm{v}},n> = \dfrac{-k_2 - x_1\left(x_3k_1 + x_1k_2\right)}{\left\|\varphi_s\Lambda\varphi_\mathrm{v}\right\|},$

$N = <\varphi_{\mathrm{vv}},n> = 0,$

$L = <\varphi_{ss},n> = \{\left(x_3k_1 + x_1k_2\right)\left[-\mathrm{v}^2\left(x_1k_2 + x_3k_1\right)^2 + \mathrm{v}^2\left(k_1^2 - k_2^2\right) + 2\mathrm{v}x_2k_1\right] - x_3k_1 +$

$$+ \mathrm{v}k_2^1\left(1+x_1\left(x_3\frac{k_1}{k_2}+x_1\right)\right)-\mathrm{v}^2x_2\left(k_1'k_2-k_1k_2'\right)\}/\|\varphi_s\Lambda\varphi_\mathrm{v}\|,$$

$$n=\left(\varphi_s\Lambda\varphi_\mathrm{v}\right)/\|\varphi_s\Lambda\varphi_\mathrm{v}\| \tag{3.3}$$

is the normal of the time-like ruled surface. Let us substitute (3.3) in (3.2) and do the necessary calculations. If x_1 and x_3 are calculated by (2.8) and (2.9), k_1 and k_2 are calculated by (2.11) and k_1/k_2 is calculated by (2.10) and substituted in terms of development invariants of the time-like ruled surface in (3.2) mentioned above, then

$$H_a=-\frac{L_T}{2L_XP_B\|\varphi_s\Lambda\varphi_\mathrm{v}\|}. \tag{3.4}$$

Thus the mean curvature H_a of the time-like ruled surface was obtained in terms of development invariants of the surface. The following theorem may be given as consequence of this:

Theorem 3.1. *In the motion of space L/L' with one parameter defined along the space-like curve $\alpha: I\to R_1^3$ the mean curvature of the developable time-like ruled surface drawn in the fixed space L' by the fixed straight line X in the moving space L is*

$$H_a=-\frac{L_T}{2L_XP_B\|\varphi_s\Lambda\varphi_\mathrm{v}\|},$$

where L_T and L_X are length of development of the time-like ruled surface in the direction of the tangent line T and the straight line X, respectively. P_B and P_N are also parameters of distribution in the direction of the binormal and the normal.

Definition 3.1. *The surface is called minimal if its mean curvature is equal to zero [5].*

This definition implies that $L_T=0$ in (3.4). $L_T=\int ds$ by (2.4). $s=\text{constant}$ is obtained by the last two equalities. Since the striction curve is a point then the time-like ruled surface is a come. The following theorem may be given as a consequence of this.

Theorem 3.2. *In the motion of space L/L' which is defined along a space-like curve in R_1^3, the developable time-like ruled surface drawn in L' is minimal if it is a come.*

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