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**GADJIEV T.S.**

# SOME SHARP ESTIMATES OF SOLUTIONS OF MIXED BOUNDARY PROBLEMS FOR NONLINEAR EQUATIONS IN NONSMOOTH DOMAINS

**Abstract**

*In this paper we study the behavior of solution near of the conical points. We obtain sharp estimate of speed decay of solution and their derivatives near the boundary.*

A mixed value problem is considered in the bounded domain $\Omega \subset R^n$, $n \geq 2$

$$\frac{d}{dx_i} a_i(x,u,u_x) - a(x,u,u_x) = 0, \tag{1}$$

$$u\big|_{\Gamma_1} = 0, \quad a_i(x,u,u_x)\cos(n,x_i)\big|_{\Gamma_2} = 0, \tag{2}$$

where $\partial\Omega$ is a boundary of the domain $\Omega$ and $d\Omega = \Gamma_1 \cup \Gamma_2$. With respect to the domain $\Omega$ we demand the fulfillment of isoperimetric inequalities [1].

Denote $\Omega_a^b = \Omega \cap \{(r,\omega) | 0 \leq a < r < b; \omega \in G\}$ is a layer in $R^n; \Omega_b = \Omega \setminus \Omega_0^b$, $\forall b > 0, \Gamma_a^b = \partial\Omega \cap \{(r,\omega)/ 0 \leq a < r < b, \omega \in \partial G\}$ is a lateral surface of the layer $\Omega_a^b$; $G_\rho = \Omega_0^d \cap \{|x| = \rho\}; 0 < \rho \leq d$.

We assume that the coefficients $a_i(x,u,p)$ are continuously differentiable, $a(x,u,p)$ are continuous in totality of arguments, and the conditions

$$\partial |p|^{m-2} \xi^2 \leq \frac{\partial a_i(x,u,p)}{\partial p} \xi_i \xi_j \leq \mu |p|^{m-2} \xi^2, \ \forall \xi \in R^n \tag{3}$$

with positive constants $v$, $\mu > 0$

$$a_i(0,0,p) = |p|^{m-2} p_i, \ i = 1,..., \ \forall p \in R^n, \tag{4}$$

are fulfilled, there exist nonnegative constants $c_0, c_1, c_2$ and the functions $g(x), \varphi_1(x), \varphi_2(x) \in L_{q|(m-1)}$, $f(x) \in L_{q|m}$, $q > n$ that

$$\left( \sum_{i,j=1}^{n} \left| \frac{\partial (a_i(x,u,p) - a_i(0,0,p))}{\partial p_j} \right|^2 \right)^{1/2} \leq c_0 |x|^\alpha |z|^{m-2} + \varphi_1(x), \tag{5}$$

$\forall \alpha \in (0,1)$,

$$\left( \sum_{i=1}^{n} |a_i(x,0,0)|^2 \right)^{1/2} \leq g(x), \tag{6}$$

$$\left( \sum_{i=1}^{n} \left| \frac{\partial a_i(x,u,p)}{\partial u} \right|^2 \right)^{1/2} \leq c_1 |p|^{m-2} + \varphi_2(x), \tag{7}$$

$$|a(x,u,p)| + \left| \sum_{i=1}^{n} \frac{\partial a_i(x,u,p)}{\partial x_i} \right| \leq c_2 |p|^{m-1} + f(x) \tag{8}$$

$$a(x,u,p) \operatorname{sgn} u \geq -\left( C_2 |p|^{m-1} + f(x) \right). \tag{9}$$

Let's consider an eigen-value problem

$$div_\omega\left(\left(\lambda^2\psi^2+|\nabla_\omega\psi|^2\right)^{(m-2)/2}\nabla_\omega\psi\right)+\lambda(\lambda(m-1)+n-m)\left(\lambda^2\psi^2+|\nabla_\omega\psi|^2\right)^{(m-2)/2}=0,$$

$$\omega\in G; \tag{10}$$

$$\psi(\omega)/_{\gamma_0}=0,\ \frac{\partial\psi(\omega)}{\partial N}/_{\gamma_1}=0,\ \omega\in\partial G,$$

where $\partial G=\gamma_0\cup\gamma_1, N$ is an external normal to $\partial G$. Suppose that there exist members $d>0, k_0\geq 0, k_1\geq 0, k_2\geq 0$ and $\beta\geq\lambda(m-1)+\alpha/2-m,\ \delta\geq\alpha m/(2(m-1)+(m-2)(\lambda-1))$ where $\lambda$ are eigen values of problem (10), such that in the domain $\Omega_0^d$

$$f(x)\leq k_0|x|^\beta,\ \varphi_1(x)\leq k_1|x|^\delta,\ \varphi_2(x)\leq k_2|x|^{\delta-1}. \tag{11}$$

By applying the Hölder and Cauchy problems we get from the conditions (5)-(9)

$$a_i(x,u,p)p_i\geq(\nu/(2(m-1)))|p|^m-C(\nu,m)g^{m'}(x), \tag{12}$$

where $1/m+1/m'=1$.

Regularity of generalized solutions for the equations (1) and various boundary value problems for these equations and its solvability in the smooth domain is well known. Our goal in this paper is the obtaining of some exact estimates of behavior of the solutions of mixed boundary value problems (1), (2). For the case of Dirichlet's problem for different classes of equations of type (1) such estimates were obtained in papers by Mierzemann E. [4], Tolksdorf P. [5], Borsuk M. [6]. In V.Kondratyev's and O.Oleynik's papers the exact estimates were obtained for linear equations. In the case of Diriclet's problem for one class of equations of type (1) in smooth domains for $m=2$ the exact estimates were obtained in paper [8].

In paper [9] under conditions (3), (6) and (9) it was obtained the boundedness of solution of the problem (1), (2) and an apriori estimation

$$\sup_{x\in\Omega}|u(x)|\leq C_3\left(\|f\|_{q/m,\Omega}+\|\varphi_2\|_{q/(m-1),\Omega}\right),$$

where $C_3$ depends on $n,m,q,C_2,mes\Omega$. Besides, in paper [9] under conditions (3), (6) and (8) the Hölder property of the solution in $\overline{\Omega}$ of bounded generalized solutions of the problem (1), (2) and an apriori estimations

$$|u(x)|\leq C_4|x|^\gamma,\ x\in\overline{\Omega},\ \gamma\in(0,1), \tag{13}$$

where $\gamma, c_4$ depends on $\max_\Omega|u(x)|, n,m,q,\nu,\mu,mes\,\Omega,\|f\|_{q/m,\Omega},\|\varphi_2\|_{q/(m-1),\Omega}$ are obtained.

By $W^1_{m,0}(\Omega)$ we denote a Banach space obtained by the closure of the space of functions from $C^1(\Omega)$ becoming zero near $\Gamma_1$ in the norm $W^1_m(\Omega)$.

The function $u(x)\in W^1_{m,0}(\Omega)$ we shall call the generalized solution of the problem (1), (2) if it satisfies the following integral identity

$$J(u,\eta)=\int_\Omega\left[a_i(x,u,u_x)\eta_{x_i}+a(x,u,u_x)\eta\right]dx=0, \tag{14}$$

for any non-negative function $\eta(x)\in W^1_{m,0}(\Omega)$.

Denote $L_m u=div\left(|\nabla u|^{m-2}\nabla u\right),\ m>1$. Cite some subsidiary results.

**Lemma 1.** *There exists a positive function $\psi$ being the solution of the problem (10) and corresponding to $\lambda>\max\left(0;\frac{m-n}{m-1}\right)$ and $\psi^2+|\nabla_\omega\psi|^2>0$ in $G$.*

**Lemma 2.** *If $\lambda>1$, then $\Omega_0\subset\{x_n\geq 0\}$.*

**Lemma 3.** *For $\forall\varepsilon>0$ there exists the function $\psi_\varepsilon(\omega)$ and numbers $\delta_\varepsilon>0,\ \lambda_\varepsilon\in[\lambda-\varepsilon,\lambda+\varepsilon]$ such that the barrier function $\omega_\varepsilon(x)=r^{\lambda_\varepsilon}\psi_\varepsilon(\omega)$ satisfies the inequalities:*

1) $-L_m\omega_\varepsilon(x)\ge\delta_\varepsilon r^{(\lambda+\varepsilon-1)(m-1)-1},\ x\in\Omega_0^1$,

2) $\delta_\varepsilon r^{\lambda+\varepsilon}\le\omega_\varepsilon(x)\le\delta_\varepsilon^{-1}r^{\lambda-\varepsilon},\ x\in\overline{\Omega}_0^1$,

3) $\delta_\varepsilon r^{\lambda+\varepsilon-1}\le|\nabla\omega_\varepsilon(x)|\le\delta_\varepsilon^{-1}r^{\lambda-\varepsilon-1},\ x\in\overline{\Omega}_0^1$,

4) $|\nabla^2\omega_\varepsilon(x)|\le\delta_\varepsilon^{-1}r^{\lambda-\varepsilon-2},\ x\in\Omega_0^1$.

The proof of these lemmas are carried out similar to the corresponding lemmas from [10] with natural measurements . In case when $m=2$ Lemma 1 has been proved also by A.F.Filippov [11].

Let $\eta(x)\in W_{m,0}^1(\Omega_0^d)$ be an arbitrary non-negative function and $\eta(x)=0,\ \forall x\in\Omega_d$. For any $A>0$, $\omega_\varepsilon(x)$ is a barrier function from lemma 3. We have:

$$J(A\omega_\varepsilon,\eta)=\int_{\Omega_0^d}\left[a_i(x,A\omega_\varepsilon(x),A\nabla\omega_\varepsilon(x))\eta_{x_i}+a(x,A\omega_\varepsilon(x),A\nabla\omega_\varepsilon(x))\eta(x)\right]dx.$$

Hence taking into account the conditions on $a_i(x,u,p)$, $a(x,u,p)$ and condition (4) we get

$$J(A\omega_\varepsilon,\eta)=\int_{\Omega_0^d}\eta(x)\left[a(x,A\omega_\varepsilon(x),A\nabla\omega_\varepsilon(x))-\frac{da_i(x,A\omega_\varepsilon(x),A\nabla\omega_\varepsilon(x))}{dx_i}\right]dx=$$

$$=-\int_{\Omega_0^d}\eta(x)\left[A\omega_{\varepsilon_{x_i x_j}}-\frac{d\left(a_i(x,A\omega_\varepsilon,A\nabla\omega_\varepsilon)-a_i(0,0A\nabla\omega_k)\right)}{\partial\left(A\omega_{\varepsilon_{x_j}}\right)}+A^{m-1}L_m\omega_\varepsilon(x)\right.$$

$$\left.+A\omega_{\varepsilon_{x_i}}\frac{da_i(x,A\omega_\varepsilon,A\nabla\omega_\varepsilon)}{\partial(A\omega_\varepsilon)}-a(x,A\omega_\varepsilon,A\nabla\omega_\varepsilon)+\frac{da_i(x,A\omega_\varepsilon,A\nabla\omega_\varepsilon)}{\partial x_i}\right]dx.$$

Taking into account conditions (5), (7), (8) we get

$$J(A\omega_\varepsilon,\eta)\ge-\int_{\Omega_0^d}\eta(x)\left[A^{m-1}L_m\omega_\varepsilon(x)+c_0A^{m-1}r^\alpha|\nabla\omega_\varepsilon|^{m-2}|\omega_{\varepsilon_{xr}}|+A\varphi_1(x)|\omega_{\varepsilon_\alpha}|+\right.$$

$$\left.+(c_1+c_2)A^{m-1}|\nabla\omega_\varepsilon|^{m-1}+A\varphi_2(x)|\nabla\omega_\varepsilon|+f(x)\right]dx.$$

By lemma 3 from the properties of the barrier function $\omega_\varepsilon(x)$ we have

$$J(A\omega_\varepsilon,\eta)\ge\int_{\Omega_0^d}\eta(x)r^{(\lambda+\varepsilon)(m-1)-m}\left[A^{m-1}\left(\delta_\varepsilon-(c_0+c_1+c_2)\delta_\varepsilon^{1-m}r^{\alpha-2\varepsilon(m-1)}\right)-\right.$$

$$\left.-A\delta_\varepsilon^{-1}r^{\lambda-\varepsilon-2}(\varphi_1(x)+r\varphi_2(x))-f(x)\right]dx.$$

We get from condition (11)

$$J(A\omega_\varepsilon,\eta)\ge\left[A^{m-1}\left(\delta_\varepsilon-(c_0+c_1+c_2)\delta_\varepsilon^{1-m}r^{\alpha-2\varepsilon(m-1)}\right)-k_0d^{\alpha/2-\varepsilon(m-1)}-\right.$$
$$\left.-A(k_1+k_2)\delta_\varepsilon^{-1}d^{m(\alpha-2\varepsilon(m-1))/2(m-1)}\right]\int_{\Omega_0^d}\eta(x)r^{(\lambda+\varepsilon)(m-1)-m}dx. \tag{15}$$

Further, we choose $A,d,\varepsilon$ so that

$$J(A\omega_\varepsilon,\eta)\ge0,\ \forall\eta\ge0. \tag{16}$$

So, if we choose $\varepsilon=\alpha/4(m-1)$ and the inequality

$$(c_0+c_1+c_2)\delta_\varepsilon^{1-m}d^{\alpha/2}\le\delta_\varepsilon/2,$$
$$A(k_1+k_2)\delta_\varepsilon^{-1}d^{m\alpha/4(m-1)}\le\delta_\varepsilon A^{m-1}/8,\ \ k_0d^{\alpha/4}\le\delta_\varepsilon A^{m-1}/8, \tag{17}$$

is fulfilled, then by lemma 3 there will be found $\delta_\varepsilon$ and a barrier function $\omega_\varepsilon(x)$ with peculiarities 1)-4). It is easy to see that the equalities (17) will be fulfilled, if $d>0$ is sufficiently small, and $A>0$ is sufficiently large, for example, if

$$0<d\le \delta_\varepsilon^{2m/\alpha}\left(2(c_0+c_1+c_2)\right)^{-2/\alpha};\ A\ge \max\Big[(8k_0)^{1/(m-1)}\delta_\varepsilon^{(m-2)/2(m-1)}\left(2(c_0+c_1+c_2)\right)^{1/2(1-m)};$$
$$(8(k_1+k_2))^{1/(m-2)}\delta_\varepsilon^{(m-2)/2(m-1)}\left(2(c_0+c_1+c_2)\right)^{m/2(1-m)(m-2)}\Big].$$

Such a choice of parameter provides the fulfillment of the inequality (16), namely

$$J(A\omega_\varepsilon,\eta)\ge \frac{1}{4}\delta_\varepsilon A^{m-1}\int_{\Omega_0^d}\eta(x)r^{(\lambda+\varepsilon)(m-1)-m}dx\ge 0,\ \forall\,\eta(x)\ge 0.$$

**Theorem 1.** *Let $u(x)$ be a generalized solution of the problem (1)-(2), conditions (3)-(11) with respect to the coefficients, and isoperimetricity condition with respect to the domain be fulfilled. Besides, assume that the function $a(x,u,p)$ is continuously differentiable with respect to $u,p$ and under fixed $x,p$ it doesn't increase with respect to $u$, and one of the following conditions is fulfilled*

1) $a_i(x,u,p)$, $i=\overline{1,n}$ *doesn't depend on* $u$;
2) $a(x,u,p)$ *doesn't depend on* $p$;
3) *the matrix* $\begin{pmatrix} D_{p_j}a_i(x,u,p) & D_{p_i}a(x,u,p)\\ D_{ii}a_i(x,u,p) & D_{ii}a_i(x,u,p)\end{pmatrix}$ *is non-positively-definite.*

*Then there exist such numbers $d>0$, $k_0>0$ that*

$$|u(x)|\le K_0|x|^{\lambda-\varepsilon},\ \forall\varepsilon>0,\ x\in\Omega_0^d, \tag{18}$$

*where $K_0$ is determined only by the parameters of the problem, and the number $d$ is defined first from the inequalities (17).*

**Proof.** Assume $\omega(x)=\omega_\varepsilon(x)$, $\varepsilon=\alpha/4(m-1)$ is a barrier function, $\delta_\varepsilon$ is a number existing in correspondence with lemma 3. $d>0$ is a number from the inequality (17). Then from (16)

$$J(A\omega,\eta)\ge 0=J(u,\eta)\quad \text{in}\quad \Omega_0^d. \tag{19}$$

Taking into account lemma 3 we have

$$0=u(x)\le A\omega(x),\ \ \forall x\in\Gamma_0^d. \tag{20}$$

Again from lemma 3 by virtue of 2) we have

$$A\omega(x)\ge A\delta_\varepsilon d^{\lambda+\varepsilon}\qquad \forall x\in G_d. \tag{21}$$

Then we get from (13) and (21), that

$$u(x)\le c_4d^{\gamma}\le A\delta_\varepsilon d^{\lambda+\varepsilon}\le A\omega(x),\ \ \forall x\in G_d \tag{22}$$

if

$$A\ge c_4d^{\gamma-\lambda-\varepsilon}\delta_\varepsilon^{-1} \tag{23}$$

is satisfied.

Taking into account the inequalities (19), (20), (21) and assumptions of the theorem we get by the comparison principle [3]

$$u(x)\le A\omega(x),\quad x\in\overline{\Omega}_0^d. \tag{24}$$

The inequality

$$u(x)\ge -A\omega(x),\quad x\in\overline{\Omega}_0^d$$

is proved in an analogous way.

Thus, if $A$ satisfies the inequalities (17), (23), then by lemma 3 by virtue of 2) we get the required estimate (18).

Theorem 1 is proved.

**Theorem 2.** *Assume that all conditions of theorem 1 are fulfilled, besides, there exist the numbers* $c_5, c_6 \geq 0$ *such that*

$$\left(\sum_{i=1}^{n}\left|a_i(x,u,p)-a_i(y,v,p)\right|^2\right)^{1/2} \leq c_5(1+|p|)^{m-1}\left(|x-y|^{\alpha}+|u-v|^{\alpha}\right), \qquad (25)$$

$\forall \alpha \in (0,1]$,

$$|a(x,u,p)| \leq c_6(1+|p|)^m \qquad (26)$$

*for* $\forall (x,u,p) \in \partial\Omega \times [-k_0,k_0] \times R^n$ *and* $\forall (y,v) \in \Omega \times [-k_0,k_0]$, *where* $k_0 = const$ *is a constant from the estimate (18).*

*Then there exist the constants* $K_1, d > 0$ *that are independent of* $u(x)$ *and definable only by the known parameters of the problem and the domain* $\Omega$, *such that*

$$|\nabla u(x)| \leq K_1|x|^{\lambda-1-\varepsilon}, \quad \forall \varepsilon \in (0,\lambda-1), \quad x \in \Omega_0^d. \qquad (27)$$

**Proof.** For the proof we use some estimates of smooth domains and Kondratyev's layer method. We consider the function $v(x') = \rho^{-\lambda+\varepsilon}u(\rho x')$ in the layer $\Omega_{1/2}^1$. It is a generalized solution in $\Omega_{1/2}^1$ of the equation

$$\frac{\partial \tilde{a}_i(x',v,v_{x'})}{\partial x_i'} - \tilde{a}(x',v,v_{x'}) = 0, \quad x \in \Omega_{1/2}^1, \qquad (28)$$

where $\tilde{a}_i(x',v,v_{x'}) = a_i(\rho x', \rho^{\lambda-\varepsilon}v, \rho^{\lambda-\varepsilon-1}v_{x'})$, $\tilde{a}(x',v,v_{x'}) = \rho a(\rho x', \rho^{\lambda-\varepsilon}v, \rho^{\lambda-\varepsilon-1}v_{x'})$.
Under the conditions of our theorem the condition of theorem 1 [12] on the boundness of the gradient module of the solution interior to domain and near the smooth piece of the boundary

$$\sup_{\Omega_{1/2}^1}|\nabla' v| \leq M_1', \qquad (29)$$

is fulfilled. Here $M_1' > 0$ is determined only by the quantities $\lambda, \alpha, \mu\nu^{-1}, m, k_0, n$ and the domain $\Omega$, where $k_0$ is a constant from the estimate (18). Coming back to the function $u(x)$, we get from (29)

$$|\nabla u(x)| \leq M_1\rho^{\lambda-1-\varepsilon}, \quad x \in \Omega_{\rho/2}^{\rho}.$$

Assuming $|x| = 2\rho/3$, hence we arrive at the desired estimate (27). Theorem 2 is proved.

**Remark 1.** In the case $m=2, n>2$ we can relax some assumptions with resect to coefficients and obtain the estimates (18), (27) for $|u(x)|$ and $|\nabla u(x)|$ in the neighborhood of the boundary point with $\varepsilon = 0$.

The proof of this fact will be given in the other paper.

**Remark 2.** In the case $m=2$, $n=2$ in the plane domain with the size of the angle $0<\omega<\pi$ under some assumptions with respect to coefficients we can obtain the estimates

$$|u(x)| \leq c_1|x|^{\frac{\pi}{2\omega}}, \quad |x|<d$$

$$|\nabla u(x)| \leq c_2|x|^{\frac{\pi}{2\omega}-1}, \quad |x|<d.$$

The proof of them fact will be in the other paper.

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**Gadjiev T.S.**
Institute of Mathematics & Mechanics of NAS of Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20(off.).
E-mail: tgadjiev@mail.az

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