2001 VOL. XIV(XXII)

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ON GLOBAL BEHAVIOR OF SOLUTIONS OF A FOURTH ORDER NON-LINEAR EQUATION ON AN UNBOUNDED DOMAIN

Abstract

It's considered the fourth order parabolic type non-linear equation on an unbounded domain and proved the existence, uniqueness of solution of initial-boundary value problem for the equation. It's also proved the existence of the minimal, global B-attractor for the dynamical system generated by this problem.

There are a lot of papers devoted to the study of questions on global behavior of solutions of non-linear evolution equations considered in bounded domains. In these papers, depending on a character of dissipation, the semi-groups corresponding to considered initial-boundary value problems are divided into compact and asymptotically compact semi-groups.

The compact semi-groups correspond to the problems for parabolic equations and asymptotically compact semi-groups correspond to the problems for hyperbolic type equations ([1]).

Beginning from 90-th years it's appeared the papers devoted to the studying of the existence of attractor of dynamic systems corresponding to non-linear evolution equations in unbounded domains ([2]-[4]). For such domains the main difficulty is that even for parabolic equation considered in these domains the operators of corresponding semi-group aren't compact. Consequently, the proving scheme of existence of the attractor for such problems differs from that, which is explicitly described in [1] and in other papers. Mentioned difficulty create problem of proving the existence of attractor for dynamic systems of corresponding hyperbolic evolution equations in unbounded domains as well.

The present paper is devoted to the study of global behavior of solutions of a fourth order non-linear parabolic equation in the unbounded domain like strip.

Thus consider the following problem

$$u_t + \Delta^2 u - \Delta f(u) = g(x), \ \ c \in \Omega, t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \ \ x \in \Omega, \tag{2}$$

$$u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0, \ \ t > 0, \tag{3}$$

where $\Omega = \mathbf{R}^n \times (0,d)$; $n = 1,2$; $d > 0$; u_0, $f(\cdot)$ are some given functions, the conditions for which will be mentioned below, $g \in L_2(\Omega)$.

The equations as (1) describe different processes encountered in hydrodynamics and the combustion theory. For example, for $f(u) = -u + \alpha_1 u^2 + \alpha_2 u^3$, where α_i are some constants, $\alpha_2 > 0$, the equation (1) is called Cahn-Hillard equation and it describes the process of formulation of flames in the phase changes ([5]). In the case $f(u) = \alpha u - \frac{1}{\alpha} u^2$, where $\alpha > 0$, (1) is called Civashinsky equation, as was shown in [6], it describes the process of cell instability at solidification of diluted binary mixtures. Paper [7] is devoted to such equations in the case of the bounded domain, where is given the detailed analysis of the previous papers. Unlike (1)-(3) the problem considered in [7] corresponds a compact semi-group. Therefore the scheme of proving the existence of minimal global B- attractor is analogous to the scheme from paper [1] for such semi-groups. The considered problem (1)-(3) as it'll be shown below, generates an asymptotic compact semi-group. Later for proving theorem on the existence of minimal global B-attractor of a semi-group, corresponding to problem (1)-(3), the terminology, definitions and some auxiliary results from papers [1], [8] will be used.

As it follows from proof of theorem 1 in [7]

$$\Omega_1^R = \Omega_R \cap \Omega, \ \text{where}\ \Omega_R = \{x \in \mathbf{R}^{n+1}, |x| < R\},\ \Omega = \mathbf{R}^n \times (0,d)$$

the problem (1)-(3) is identically solvable in $C\left(\mathbf{R}^+, \overset{\circ}{W}{}_2^1(\Omega^R)\right)$ under the condition

$$f(u)u - \mathcal{F}(u) \geq c_1 u^2,\ \mathcal{F}(u) = \int_0^u f(s)ds, \tag{4}$$

$$\mathcal{F}(u) \geq -c_2 u^2, f'(u) \geq -c_3, \tag{5}$$

$$f''(u) \leq c_4 \left(1 + |u|^p\right), \tag{6}$$

where c_i will be refined below, p is arbitrary for $n = 1$; $p \leq 1$, for $n = 2$.

In this case the corresponding decisive operators $V_t, t \in R^+$ create a continuous semi-group in the phase space $\overset{\circ}{W}{}_2^1(\Omega^R)$. We'll establish below bounded dissipativity of the semi-group $V_t : \overset{\circ}{W}{}_2^1(\Omega^R) \to \overset{\circ}{W}{}_2^1(\Omega^R)$, $t \in R^+$. In proving, the apriori estimation for the solutions independent of R will be obtained. Tending $R \to \infty$ will be proved one-valued solvability and an analogous estimation for a solution of the initial problem (1)-(3). Note that as at paper [7] for $n = 1$ for one-valued solvability of (1)-(3) it is sufficient the fulfillment of conditions (4)-(5). Not changing denotation for u and V_t, $t \in \mathbf{R}^+$ we give the next result.

Theorem 1. *Let the conditions (4)-(5), where* $2C_i d^2 < 1, (i = 1,2)$, $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$, $g \in L_2(\Omega)$ *be satisfied. Then the semi-group* $V_t : \overset{\circ}{W}{}_2^1(\Omega) \to \overset{\circ}{W}{}_2^1(\Omega)$, $t \in \mathbf{R}^+$ *corresponding to the problem (1)-(3) is boundedly dissipative.*

Proof. In the beginning we note that for $\forall u \in \overset{\circ}{W}{}_2^1(\Omega)$ in Ω the Poincare inequality

$$\left(\int_\Omega u^2(x)dx\right)^{\frac{1}{2}} \leq d\left(\int_\Omega |u_x(x)|^2 dx\right)^{\frac{1}{2}},$$

is satisfied, where d is some positive constant. According to this inequality we can determine the norm in the space $\overset{0}{W}{}_2^1(\Omega)$ as

$$\|u\|_{\overset{\circ}{W}{}_2^1(\Omega)} = \|\nabla u\|_{L_2(\Omega)}.$$

It's known that the Laplacian operator Δ is positive defined in $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and has in $L_2(\Omega)$ the bounded inverse operator P. Starting from this, we can rewrite the problem (1)-(3) in the equivalent form:

$$Pu_t - \Delta u + f(u) = Pg, \tag{1`}$$

$$u(x,0) = u_0(x), \tag{2`}$$

$$u\big|_{\partial\Omega} = 0. \tag{3`}$$

Consider the next functional

$$\Phi_\gamma(u)=\frac{1}{2}\|\nabla u\|^2+(\mathcal{F}(u),1)+\frac{\gamma}{2}(Pu,u)-(Pg,u). \tag{7}$$

Later on in the expression (7) $\|\cdot\|,(\cdot,\cdot)$ are a norm and a scalar product in the space $L_2(\Omega)$ respectively. Differentiate the functional $\Phi_\gamma(u)$ and use (1'). We obtain

$$\begin{gathered}\frac{d}{dt}\Phi_\gamma(u)=-(u_t,\Delta u)+(f(u),u_t)+\gamma(Pu_t,u)-(Pg,u_t)=\\=-(Pu_t,u_t)-\gamma\|\nabla u\|^2-\gamma(f(u),u_t)+\gamma(Pg,u).\end{gathered} \tag{8}$$

We multiply both sides of (7) by some $\delta>0$ and summing it (8)

$$\begin{gathered}\frac{d}{dt}\Phi_\gamma(u)+\delta\Phi_\gamma(u)=-\left\|P^{\frac{1}{2}}u_t\right\|^2-\gamma\|\nabla u\|^2-\gamma(f(u),u)+\gamma(Pg,u)+\\+\frac{\delta}{2}\|\nabla u\|^2+\delta(\mathcal{F}(u),1)+\frac{\gamma\delta}{2}\left\|P^{\frac{1}{2}}u\right\|^2-\delta(Pg,u)\le\gamma\frac{\delta}{2}\left\|P^{\frac{1}{2}}u\right\|^2+\\+\left(-\gamma+\frac{\delta}{2}\right)\|\nabla u\|^2+\gamma((\mathcal{F}(u),1)-(f(u),u))-(\gamma-\delta)(\mathcal{F}(u),u)+(\gamma-\delta)(Pg,u).\end{gathered} \tag{9}$$

Taking into account the conditions (4), (5) and choosing γ,δ such that $0<\delta<\gamma<1$ from(9) we get

$$\begin{gathered}\frac{d}{dt}\Phi_\gamma(u)+\delta\,\Phi_\gamma(u)\le\left(-\gamma+\frac{\delta}{2}\right)\|\nabla u\|^2+\frac{\gamma\delta}{2}\left\|P^{\frac{1}{2}}u\right\|^2+\gamma(c_1+c_2)\|u\|^2+\\+\frac{\gamma}{2\varepsilon_1}\|Pg\|^2+\frac{\gamma\varepsilon_1}{2}\|u\|^2\le\left(-\gamma+\frac{\delta}{2}\right)\|\nabla u\|^2+\frac{\gamma\delta d^2}{2}\left\|P^{\frac{1}{2}}\right\|^2\|\nabla u\|^2+\\+\gamma d^2(c_1+c_2)\|\nabla u\|^2+\frac{\gamma\varepsilon_1 d^2}{2}\|\nabla u\|^2+\frac{\gamma}{2\varepsilon_1}\|Pg\|^2\le\\\le\left(-\gamma+\frac{\delta}{2}+\frac{\gamma\,\delta\,d^2}{2}\left\|P^{\frac{1}{2}}\right\|^2+(c_1+c_2)d^2\gamma+\frac{\gamma\varepsilon_1 d^2}{2}\right)\|\nabla u\|^2+\frac{\gamma}{2\varepsilon_1}\|Pg\|^2.\end{gathered} \tag{10}$$

Using the conditions on c_1, c_2 we choose $\gamma,\delta,\varepsilon$ from the inequality

$$-\gamma+\frac{\delta}{2}+\frac{\gamma\,\delta\,d^2}{2}\left\|P^{\frac{1}{2}}\right\|+(c_1+c_2)\gamma d^2+\frac{\gamma\varepsilon_1 d^2}{2}\le 0.$$

Then from (10) we obtain

$$\frac{d}{dt}\Phi_\gamma(u)+\delta\Phi_\gamma(u)\le k_1, \tag{11}$$

where $k_1=\frac{\gamma}{2\varepsilon_1}\|Pg\|^2$. From (11) we have

$$\Phi_\gamma(u)\le\Phi_\gamma(u_0)e^{-\delta t}+\frac{k_1}{\delta}\le k_2 \tag{12}$$

On the other hand from the expression (7) and the condition (6) it follows that

$$\frac{1}{2}\|\nabla u\|^2 = \Phi_\gamma(u) - (\mathcal{F}(u),1) - \frac{\gamma}{2}(Pu,u) + (Pg,u) \le \Phi_\gamma(u) + \\ + c_2 d^2\|\nabla u\|^2 + \varepsilon_2\|\nabla u\|^2 + \frac{d^2}{4\varepsilon_2}\|Pg\|^2 . \tag{13}$$

Allowing for that $c_2 d^2 < \frac{1}{2}$ and choosing ε_2 such that

$$l \equiv \frac{1}{2} - c_2 d^2 - \varepsilon_2 > 0,$$

From (13) we obtain

$$\|\nabla u\|^2 \le \frac{1}{l}\Phi_\gamma(u) + \frac{d^2}{4l\varepsilon_2}\|Pg\|^2 . \tag{14}$$

From (12) and (4) it follows that

$$\|\nabla u\|^2 \le k_3 , \tag{14'}$$

where k_3 depends only on $c_1, c_2, \|Pg\|$ and independent of R. Tending R to infinity and allowing for lower semi-continuity of norm we obtain the apriori estimation (14') for the solution of (1)-(3). With its help the existence and uniqueness of the solution $u \in C\left(\mathbf{R}^+; \overset{\circ}{W}{}_2^1(\Omega)\right)$ of the problem (1)-(3) and continuity of the semi-group V_t, $t \in \mathbf{R}^+$ of the corresponding (1)-(3) is proved as well as in [7]. Moreover, the inequalities (12)-(14) show the existence of the bounded absorbing set $B_0 = \left\{u \in \overset{\circ}{W}{}_2^1(\Omega); \Phi_\gamma(u) \le \frac{2k_1}{\delta}\right\}$, V_t, $t \in \mathbf{R}^+$ for the semi-group . The theorem is proved.

Now for proving theorem on the existence of minimal global B - attractor for the semi-group V_t, $t \in \mathbf{R}^+$, we cite some auxiliary results.

Lemma 1. *Let the conditions of theorem 1 be satisfied, in addition let the inequality $2c_3 d^2 < 1$; $p < 1$ for $n = 2$, where c_3 is a constant from the condition (5), p from (6), be valid. Then for the solution u of the problem (1)-(3) the next weight estimation holds*

$$\sup_{1 \le t < +\infty} \|u(x,t)\psi(x,t)\|^2_{L_2(\Omega)} \le c , \tag{15}$$

where c boundedly depends on $g, \|\nabla u_0\|$, $\psi(x,t)$ is a weight function mentioned below.

Proof. For this we use the weight function

$$\psi(x,t) = \left(1 + |x|^2\right)^{\frac{1}{2}} \left(1 - e^{-\frac{t}{\left(1+|x|^2\right)^{\frac{1}{2}}}}\right).$$

It's easy to check that it has the following properties

1. $\psi(x,t) \ge 0$, $\psi(x,0) = 0$;
2. all derivatives of ψ are bounded functions;
3. $\psi(x,t)$ is equivalent to the fixed t when $|x| \to \infty$;
4. $\psi(x,t)$ is equavalent to the function $y = \left(1 + |x|^2\right)^{\frac{1}{2}}$, when $t \to \infty$;
5. $\psi(x,t)$ tends to infinity when $(|x|, t) \to \infty$.

Taking into account the second property of a weight function it's obvious that for the solution u of the problem (1)-(3) the expression $\left\|u\psi^{\frac{1}{2}}\right\|_{L_2(\Omega)}$ is bounded for the fixed value of t.

For proving (15) we write the equation (1') in the following form

$$Pu_t - \Delta u + f(u) + \lambda_0 u = g_1, \tag{16}$$

where $g_1 = Pg + \lambda_0 u$, λ_0 is some positive constant. Multiply (16) scalarly in $L_2(\Omega)$ by $-\Delta u\psi(x,t)$ we get

$$(Pu_t, -\Delta u\psi) + \left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \lambda_0\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 + \lambda_0(u, \nabla u\nabla\psi) -$$

$$-(f(u), \Delta u\psi) = -(g_1, \Delta u\psi),$$

$$(u, u_t\psi) - (u, 2\nabla(Pu_t)\nabla\psi + Pu_t\Delta\psi) + \left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \lambda_0\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 =$$

$$= -(g_1, \Delta u\psi) - (\nabla u, f(u)\nabla\psi) - \left(f'(u), |\nabla u|^2\psi\right) - \lambda_0(u, \nabla u\nabla\psi),$$

$$\frac{1}{2}\frac{d}{dt}\left\|u\psi^{\frac{1}{2}}\right\|^2 + \left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \lambda_0\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 = \frac{1}{2}(u^2, \psi_t) + 2(u, \nabla(Pu_t)\nabla\psi) +$$

$$+ (u, Pu_t\Delta\psi) - (\nabla u, f(u)\nabla\psi) - \left(f'(u), |\nabla u|^2\psi\right) - (g_1, \Delta u\psi) - \lambda_0(u, \nabla u\nabla\psi),$$

$$\frac{1}{2}\frac{d}{dt}\left\|u\psi^{\frac{1}{2}}\right\|^2 + \left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \lambda_0\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 = \frac{1}{2}(u^2, \psi_t) - (Pu_t, 2\nabla u\nabla\psi + u\Delta\psi) -$$

$$-(\nabla u, f(u)\nabla\psi) - \left(f'(u), |\nabla u|^2\psi\right) - (g_1, \Delta u\psi) - \lambda_0(u, \nabla u\nabla\psi). \tag{17}$$

Using the equation (1') in right hand side of (17) we have

$$\frac{1}{2}\frac{d}{dt}\left\|u\psi^{\frac{1}{2}}\right\|^2 + \left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \lambda_0\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 = \frac{1}{2}(u^2, \psi_t) - 2(\Delta u, \nabla u\nabla\psi) -$$

$$-(\Delta u, u\Delta\psi) + (f(u), \nabla u\nabla\psi) + (f(u), u\Delta\psi) - 2(Pg, \nabla u\nabla\psi) - (Pg, u\Delta\psi) -$$

$$-\left(f'(u), |\nabla u|^2\psi\right) - (g_1, \Delta u\psi) - \lambda_0(u, \nabla u\nabla\psi). \tag{18}$$

Now in order to estimate the right hand side of (18) we use the conditions (5), (6), the properties of the weight function ψ discribed above, the inequality (4') and apply Poincare, Cauchy and Young inequalities. All calculations are led for $t \geq t_0 = 1$

$$|(\Delta u, \nabla u\nabla\psi)| \leq \left\|\Delta u\psi^{\frac{1}{2}}\right\|\left\|\nabla u\frac{\nabla\psi}{\psi^{\frac{1}{2}}}\right\| \leq \frac{\varepsilon_1}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{1}{2\varepsilon_1}\left\|\nabla u\frac{\nabla\psi}{\psi^{\frac{1}{2}}}\right\| \leq \frac{\varepsilon_1}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 +$$

$$+\frac{c_5}{2\varepsilon_1\psi(1)}\|\nabla u\|^2 \leq \frac{\varepsilon_1}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{k_3c_5}{2\varepsilon_1\psi(1)} \leq \frac{\varepsilon_1}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + c_6;$$

$$\left|(u^2, \psi_t)\right| \leq c_7\|u\|^2 \leq c_7 d^2\|\nabla u\|^2 \leq c_7 d^2 k_3 \leq c_8;$$

$$\left|(\Delta u, u\Delta\psi)\right| \le \left\|\Delta u\psi^{\frac{1}{2}}\right\|\left\|u\frac{\Delta\psi}{\psi^{\frac{1}{2}}}\right\| \le \frac{\varepsilon_2}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{1}{2\varepsilon_2}\left\|u\frac{\Delta\psi}{\psi^{\frac{1}{2}}}\right\|^2 \le$$

$$\le \frac{\varepsilon_2}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{c_5 d^2}{2\varepsilon_2\psi(1)}\|\nabla u\|^2 \le \frac{\varepsilon_2}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + c_9 .$$

For the following two estimations of members in the right hand side of (17) we'll apply the imbedding theorem of $\overset{\circ}{W}{}_2^1(\Omega)$ in $L_q(\Omega)$.

In the second estimation we'll also use the multiplicative inequality from [7]

$$\left\|D^i v\right\|_{Lp_1} \le \tilde{c}_i \left\|D^m v\right\|_{Lp_2}^a \|v\|_{Lp_3}^{1-a}, \tag{19}$$

where $\frac{j}{m} \le a \le 1$, $\frac{1}{p_1} = \frac{j}{n} + a\left(\frac{1}{p_2} - \frac{m}{n}\right) + (1-a)\frac{1}{p_3}$, D^i are the i-th order derivatives, $\tilde{c} > 0$. In addition by virtue of that $p < 1$ for $n = 3$ and p is arbitrary for $n = 2$, then we can choose the index a in the right hand side of (19) such that $2(p+2)a < 1$. Thus we have

$$\left|(f(u), u\Delta\psi)\right| \le c_{10}\left(\|u\|_{L_1} + \|u\|_{L_{p+3}}^{p+3}\right) \le c_{11}\|\nabla u\|^{p+3} \le c_{12};$$

$$\left|(f(u), \nabla u\nabla\psi)\right| \le c_{13}\left(|u|^{p+2} + 1, |\nabla u|\right) \le c_{13}\left(\|u\|_{L_2(p+2)}^{2(p+2)} + 1\right)\|\nabla u\| \le$$

$$\le c_{14}\|\Delta u\|^{2(p+2)a}\|\nabla u\|^{2(p+2)(1-a)+1} + c_{13}\|\nabla u\| < c_{15}\left(\|\Delta u\|^{2(p+2)a} + 1\right) \le$$

$$\le \frac{c_{15}}{(\psi(1))^{(p+1)a}}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^{2(p+2)a} + c_{15} \le \frac{\varepsilon_3}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + c_{16};$$

$$-\left(f'(u), |\nabla u|^2\psi\right) \le c_3\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2;$$

$$\left|(Pg, \nabla u\nabla\psi)\right| \le c_5\|Pg\|\|\nabla u\| \le c_{17};$$

$$\left|(Pg, u\Delta\psi)\right| \le c_{18}\|Pg\|\|u\| \le c_{18}d\|Pg\|\|\nabla u\| \le c_{19};$$

$$\left|(g_1, \Delta u\psi)\right| \le \left|(Pg, \Delta u\psi)\right| + \lambda_0\left|(u, \Delta u\psi)\right| \le \frac{\varepsilon_4}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{1}{2\varepsilon_4}\|Pg\|^2 +$$

$$+ \lambda_0\left\|\Delta u\psi^{\frac{1}{2}}\right\|\left\|u\psi^{\frac{1}{2}}\right\| \le \frac{\varepsilon_4}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{1}{2\varepsilon_4}\|Pg\|^2 + \lambda_0 d\left\|\Delta u\psi^{\frac{1}{2}}\right\|\left\|\nabla\left(u\psi^{\frac{1}{2}}\right)\right\| \le$$

$$\le \frac{\varepsilon_4}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{\varepsilon_5}{2}\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{(\lambda_0 d)^2}{\varepsilon_5}\left(\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 + \left\|u\psi^{-\frac{1}{2}}\nabla\psi\right\|^2\right) + \frac{1}{2\varepsilon_4}\|Pg\|^2 \le$$

$$\le \frac{1}{2}(\varepsilon_4 + \varepsilon_5)\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2 + \frac{(\lambda_0 d)^2}{\varepsilon_5}\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2 + \frac{(\lambda_0 d)^2}{\psi(1)}c_5 d^2\|\nabla u\|^2 + \frac{1}{2\varepsilon_4}\|Pg\|^2 \le$$

$$\le \frac{1}{2}(\varepsilon_4+\varepsilon_5)\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2+\frac{(\lambda_0 d)^2}{\varepsilon_5}\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2+c_{20};$$

$$|(u,\nabla u\nabla\psi)|\le c_5\|u\|\|\nabla u\|\le c_5 dk_3\le c_{21}.$$

Taking into account obtained estimation in (18) we have

$$\frac{1}{2}\frac{d}{dt}\left\|u\psi^{\frac{1}{2}}\right\|^2+\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2+\lambda_0\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2\le\left(\varepsilon_1+\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2}+\frac{\varepsilon_4}{2}+\frac{\varepsilon_5}{2}\right)\times$$

$$\times\left\|\Delta u\psi^{\frac{1}{2}}\right\|^2+\left(c_3+\frac{(\lambda_0 d)^2}{\varepsilon_5}\right)\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2+2c_6+\frac{1}{2}c_8+c_9+c_{12}+c_{16}+2c_{17}+$$

$$+c_{19}+c_{20}+\lambda_0 c_{21}. \qquad (20)$$

We choose the positive parameters ε_i,λ_0 such that the next inequalities are satisfied

$$\varepsilon_1+\frac{1}{2}\sum_{i=2}^{5}\varepsilon_i\le 1,$$
$$c_3+\frac{(\lambda_0 d)^2}{\varepsilon_5}-\lambda_0<0. \qquad (21)$$

Such choic is possible by virtue of the conditions of theorem $2c_3d^2<1$. Then denoting $\lambda_1\equiv 2\left(\lambda_0-c_3-\frac{(\lambda_0 d)^2}{\varepsilon_5}\right)>0$ from (20) and (21) we obtain

$$\frac{d}{dt}\left\|u\psi^{\frac{1}{2}}\right\|^2+\lambda_1\left\|\nabla u\psi^{\frac{1}{2}}\right\|^2\le c_{22}. \qquad (22)$$

Now using lower boundedness of the function $\psi(x,t)$ in $\Omega\times[1;+\infty]$ and the inequality (14') we have

$$\left\|\nabla u|\psi^{\frac{1}{2}}\right\|^2=\left\|\nabla\left(u\psi^{\frac{1}{2}}\right)\right\|^2-\frac{1}{4}\left\|u|\nabla\psi|\psi^{-\frac{1}{2}}\right\|^2-\int_\Omega u(\nabla u\cdot\nabla\psi)dx\ge d^{-2}\left\|u\psi^{\frac{1}{2}}\right\|^2-$$

$$-\frac{1}{4\psi(1)}\|u|\nabla\psi|\|^2-\int_\Omega|u||(\nabla u\cdot\nabla\psi)|dx\ge d^{-2}\left\|u\psi^{\frac{1}{2}}\right\|^2-\frac{c_5d^2}{4\psi(1)}\|\nabla u\|^2-c_5d^2\|\nabla u\|^2\ge$$

$$\ge d^{-2}\left\|u\psi^{\frac{1}{2}}\right\|^2-\left(1+\frac{1}{4\psi(1)}\right)c_5d^2k_3\ge d^{-2}\left\|u\psi^{\frac{1}{2}}\right\|^2-c_{23}.$$

Taking into account the last inequality in (22) we obtain

$$\frac{d}{dt}\left\|u\psi^{\frac{1}{2}}\right\|^2+\lambda_1 d^{-2}\left\|u\psi^{\frac{1}{2}}\right\|^2\le c_{24}.$$

Integrating this inequality from $t_0=1$ to some t we obtain

$$\left\|u\psi^{\frac{1}{2}}\right\|^2\le\frac{c_{24}d^2}{\lambda_1}\left(1-e^{-\lambda_1 d^{-2}t}\right)+e^{\lambda_1 d^{-2}(1-t)}\left\|u(1)\psi^{\frac{1}{2}}(1)\right\|^2, \qquad (23)$$

where c_{24} independent on time. From (23) the estimation (15) is obtained. In the completion of proving the lemma we note that as the initial time t_0 we could take any $t_0 > 0$.

From theorem 1 and lemma 1 we have in particular the following

Remark 1. Let the conditions of Lemma 1 be satisfied. Then the following inequality is valid

$$\overline{\lim_{t\to+\infty}} \left\| u(x,t)\psi^{\frac{1}{2}}(x,t) \right\| \le c, \tag{24}$$

where c is some independent on the initial value u_0, positive constant, $u(x,t)$ is a solution of the problem (1)-(3).

Lemma 2. *Let the conditions of lemma 1 be fulfilled. Then any sequence of the form $\{u_n\}_{n=1}^{\infty}$, where $u_n \equiv u(x,t_n)$, t_n a sequence of positive numbers tending to infinity, is compact in the space $L_2(\Omega)$.*

Proof. By virtue of theorem 1 any sequence of $\{u_n\}^{\infty}$ is bounded in $\overset{\circ}{W}{}_2^1(\Omega)$. Consequently, from it we can choose a sequence that we also denote by u_n, weakly convergent in $\overset{\circ}{W}{}_2^1(\Omega)$ to some $\sigma \in \overset{\circ}{W}{}_2^1(\Omega)$. We choose now the sequence $\{\varepsilon_i\}$ monotonically tending to zero of positive numbers such that

$$\int_{\Omega\setminus\Omega^{R_i}} |\sigma|^2 dx < \frac{\varepsilon_i}{3}, \tag{25}$$

where $\Omega^{R_i} = \Omega_{R_i} \cap \Omega$, $\Omega_{R_i} = \left\{x \in \mathbf{R}^{n+1}, |x| < R_i\right\}$, R_i is a sequence of positive numbers corresponding to $\{\varepsilon_i\}$ which tends to infinity. By virtue of (24) and property 5 of the weight function $\psi(x,t)$ from the expression

$$\int_{\Omega\setminus\Omega^{R_i}} |u_n|^2 dx = \int_{\Omega\setminus\Omega^{R_i}} |u_n|^2 \psi(x,t_n)\frac{1}{\psi(x,t_n)} dx$$

it follows that we can find such sequence $\{u_n\}_{n=1}^{\infty}$ that for $t_n \ge R_i$

$$\int_{\Omega\setminus\Omega^{R_i}} |u_n(x,t_n)|^2 dx < \frac{\varepsilon_i}{3} \tag{26}$$

is satisfied.

On the other side by virtue of compact imbedding of the space $\overset{\circ}{W}{}_2^1\left(\overline{\Omega}^{R_i}\right)$ in $L_2\left(\overline{\Omega}^{R_i}\right)$ from $\{u_n\}$ we can select such sequence $\left\{u_n^i\right\}$ strongly convergent to $\sigma \in L_2\left(\overline{\Omega}^{R_i}\right)$. Then by virtue of (25), (26) leading the diagonalization process we can choose such sequance $\left\{u_i^i\right\}_{i=1}^{\infty}$ which tends to σ in $L_2(\Omega)$. The lemma has been proved.

Theorem 2. *Let the conditions (5), (6) where $c_3 d^2 < 1$ and $u_0 \in L_2(\Omega)$ be fulfilled. Then for any positive to t_0, T, $(t_0 < T)$ there exists the unique solution u of the problem (1)-(3) such that*

$$u \in L_\infty(0,T;L_2(\Omega)) \cap L_\infty\left(t_0,T;\overset{\circ}{W}{}_2^1(\Omega)\right).$$

Proof of the theorem is based on the following apriori estimations. By multiplying the equation (1) scalarly in $L_2(\Omega)$ by u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\|\nabla u\|^2+(f'(u)\nabla u,\nabla u)=(g,u),$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\|\Delta u\|^2\le c_3\|\nabla u\|^2+\|g\|\|u\|. \tag{27}$$

By virtue of the inequalities $\|\nabla u\|\le d\|\Delta u\|, \|u\|\le d^2\|\Delta u\|$ that it is easily provable with the help of the Poincare inequality and the conditions $c_3 d^2<1$ from (27) we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\left(1-c_3 d^2-\frac{\varepsilon}{2}\right)\|\Delta u\|^2\le\frac{d^4}{2\varepsilon}\|g\|^2.$$

Introducing designation $\eta\equiv 2\left(1-c_3 d^2-\frac{\varepsilon}{2}\right)$ and integrating with respect to t the last inequality we obtain

$$\|u\|^2+\eta\int_0^T\|\Delta u\|^2 dt\le\frac{d^4}{\varepsilon}\int_0^T\|g\|^2 dt+\|u_0\|^2,$$

$$\|u\|^2+\int_0^T\|\Delta u\|^2 dt\le c_{25}\,,\ \ t\in[0,T], \tag{28}$$

where c_{25} depends on $T, \|g\|, \|u_0\|$.

Now in order to prove the second statement of the theorem we multiply (1') scalarly in $L_2(\Omega)$ on u_t we obtain

$$\left\|P^{\frac{1}{2}}u_t\right\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+(f(u),u_t)=(Pg,u_t),$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2+2(1,\mathcal{F}(u))\right)+\frac{1}{2}\left\|P^{\frac{1}{2}}u_t\right\|^2\le\frac{1}{2}\left\|P^{\frac{1}{2}}g\right\|^2.$$

Multiply the last inequality by t and taking into account the theorem integrate its

$$\frac{d}{dt}\left(t\|\nabla u\|^2+2t(1,\mathcal{F}(u))\right)+t\left\|P^{\frac{1}{2}}u_t\right\|^2\le t\left\|P^{\frac{1}{2}}g\right\|^2+\|\nabla u\|^2+2(1,\mathcal{F}(u)),$$

$$t\|\nabla u\|^2+\int_0^t s\left\|P^{\frac{1}{2}}u_t\right\|^2 ds\le c_2 t\|u\|^2+\frac{t^2}{2}\left\|P^{\frac{1}{2}}g\right\|^2+\int_0^t\|\nabla u\|^2 ds+2\int_0^t(1,\mathcal{F}(u))ds,$$

$$t\|\nabla u\|^2+\int_0^t s\left\|P^{\frac{1}{2}}u_t\right\|^2 ds\le c_2 t\|u\|^2+\frac{t^2}{2}\left\|P^{\frac{1}{2}}g\right\|^2+d^2\int_0^t\|\Delta u\|^2 ds+\widetilde{c}_4\int_0^t\|u(s)\|_{L_{p+3}}^{p+3}ds. \tag{29}$$

By virtue of (41) from (42) we have

$$t\|\nabla u\|^2+\int_0^t s\left\|P^{\frac{1}{2}}u_t\right\|^2 ds\le c_2 c_{25}t+c_{25}d^2+c_{26}t^2+\widetilde{c}_4\int_0^t\|u(s)\|_{L_{p+3}}^{p+3}ds. \tag{30}$$

We use the multiplicative inequality (19) for the estimation of the last member in the right hand side of (30) for $n=3$ (analogously for $n=2$) and obtain

$$\|u\|_{L_{p+3}}\le\widetilde{c}\|\Delta u\|^{\frac{3}{4}-\frac{3}{2(p+3)}}\|u\|^{\frac{1}{4}+\frac{3}{2(p+3)}}.$$

Then by virtue of that $p \le 1$ and (28) $(p+3)\left(\frac{3}{4}-\frac{3}{2(p+3)}\right) \le \frac{3(p+3)}{4}-\frac{3}{2} \le 1{,}5$

$$\int_0^t \|u(s)\|_{L_{p+3}}^{p+3} ds \le c\int_0^t \|\Delta u\|^{\frac{3(p+3)}{4}-\frac{3}{2}} \|u\|^{\frac{p+3}{4}+\frac{3}{2}} ds \le c_{27}\int_0^t \left(1+\|\Delta u\|^2\right) ds \le c_{28}(1+t). \tag{31}$$

Using (30) and (31) we get

$$\|\nabla u\|^2 \le \frac{1}{t}\left(c_2 c_{25} t + c_{25} d^2 + c_{26} t^2 + \tilde{c}_4 c_{28}(1+t)\right),$$

$$\|\nabla u(t)\|^2 \le \frac{c_{29}}{t}\left(1+t+t^2\right), \quad t>0, \tag{32}$$

where c_{29} depends on $\|u_0\|, \|g\|$.

The statement of the theorem is proved as well as in theorem 1, using the apriori estimations (28), (32).

According to the last theorem the operations of the semi-group $V_t : L_2(\Omega) \to L_2(\Omega)$, $t \in \mathbf{R}^+$ of the corresponding problem (1)-(3) are bounded from $L_2(\Omega)$ in $\overset{\circ}{W}{}_2^1(\Omega)$.

Now we formulate and prove the main result on the existence of an attractor of the semi-group $V_t, t \in \mathbf{R}^+$.

Theorem 3. *Let the conditions (4)-(6) where* $2c_i d^2 < 1$, $\left(i=\overline{1,3}\right)$, $p<1$ *for* $n=2$ *be valid. Then for the semi-group* $V_t, t \in \mathbf{R}^+$ *corresponding to the problem (1)-(3) there exists a minimal global* B *- attractor* $\mathcal{M}$ *in the space* $L_2(\Omega)$, *which is compact, invariant and connected set.*

Proof. By virtue of the conditions of the theorem the statements of all previous results are valid. The attractor $\mathcal{M}$ of the semi-group V_t, $t \in \mathbf{R}^+$ is constructed as w - limit set of bounded absorbing set B_0 from theorem 1:

$$\mathcal{M} = w(B_0) = \bigcap_{t \ge 0} \left[V_t(B_0)\right]_{L_2(\Omega)}, \tag{33}$$

where $[\cdot]_{L_2(\Omega)}$ means a closure by norm in $L_2(\Omega)$. Following the scheme of proof of the theorem from chapter of paper [8] that to complete the proof of the present theorem it remains to establish compactness of the set $w(B_0)$ in the space $L_2(\Omega)$. Let by $\{w_i\}_{i=1}^{\infty}$ denote some equence from the set $w(B_0)$. By definition of this set for every w_i there exists a sequence $\left\{\left(y_i^n, t_i^n\right)\right\}_{n=1}^{\infty}$ where $y_i^n \in B_0$, $t_i^n \to \infty$ such that in $L_2(\Omega)$

$$\lim_{n\to\infty} V_{t_i^n}\left(y_i^n\right) = w_i \tag{34}$$

holds.

We prove that from $\{w_i\}_{i=1}^{\infty}$ we can choose such a subsequence which we also denote by $\{w_i\}$ tending to some $w \in L_2(\Omega)$. As well as for proving lemma 2 we choose the sequence $\{w_i\}_{i=1}^{n}$ weakly convergent to w in $L_2(\Omega)$.

It's sufficient to prove that for any $\varepsilon > 0$ and some fixed $N > 0$ the uniform with respect to $i \in N$ estimation

$$\int_{\Omega\setminus\Omega^N} |w_i(x)|^2 dx < \varepsilon, \tag{35}$$

is valid, where $\Omega^N = \Omega \cap \{x \in \mathbf{R}^{n+1}, |x| < N\}$.

By virtue of (34) and the weight estimation (24) analogously to the estimation (35) for $\forall \varepsilon > 0$, $i, n \in \mathbf{N}$ and some $N > 0$ for $t_i^n > N$ the estimation

$$\int_{\Omega \setminus \Omega^N} \left| V_{t_i^n}\left(y_i^n\right) \right|^2 dx \le \varepsilon$$

is deduced.

Tending $n \to \infty$ from the last estimation we have:

$$\overline{\lim_{n \to \infty}} \int_{\Omega \setminus \Omega^N} \left| V_{t_i^n}\left(y_i^n\right) \right|^2 dx \le \varepsilon . \qquad (36)$$

From (36) and (34) by virtue of upper semi-continuity of norm in $L_2(\Omega)$ the estimation (35) follows.

Using the estimation (35) and analogously to proving lemma 2, the proof of the theorem is completed.

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Received September 18, 2000; Revised February 21, 2001.
Translated by Mamedova V.A.