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ON A COMPLETENESS THEOREM FOR NON SELF-ADJOINT OPERATOR-FUNCTIONS

Abstract

In this work the N-tiple completeness theorem for a class of non self-adjoint operator-functions is proved.

In spite of that the main operator may be not compact, but its some natural power is a compact operator.

In the present paper the N-tiple completeness of a system of eigen and adjoint elements (e.a.) of the operator function

$$A(\lambda)=\sum_{k=1}^{n-1}(\lambda-a)^{-k}B_kT^k+\sum_{k=0}^{n}(\lambda-a)^{-k}z_kT^k+B(\lambda) \tag{1}$$

in the Hilbert space is established.

Assume that

1) T is a complete operator of finite order and T^n is a compact and scalar type operator and characteristic numbers of the operator T lie on finite number of rays outgoing from the origin of coordinates
2) z_k $\left(k=\overline{0,n}\right)$ are arbitrary complex numbers, where $z_0\neq 1$, $z_n\neq 0$
3) B_k are arbitrary compact operators
4) $B(\lambda)=\sum_{k=0}^{s}A_k(\lambda-a)^k$, $A_k\,(k=0,1,...,s)$ are compact operators, where A_S is a complete operator. In the paper (1) instead of condition 1) in the present paper it is assumed that $\Re$ is a complete compact self adjoint operator of finite order. Under the conservation of the other conditions in [1] the theorem on the N-tiple completeness ($\Re$) of the operator function $A(\lambda)$ has been proved.

Note that if T isn't a scalar type operator, but its some natural power is scalar, for example, T^n is a scalar type operator, then it is impossible to assert that operator T has the system of orthogonal eigen projectors. There is corresponding example in paper [2].

This and other questions related completed us to investigate the questions of the N-tiple completeness for the operator function $A(\lambda)$ under the conditions more general than in paper [1].

Further we'll need the following very significant

Lemma 1 [2]. *Let T^n be a compact scalar type operator. If μ_j^n $(\mu_j\neq 0;\ j=1,2,...)$ is a characteristic number and P_j $(j=1,2,...)$ is a corresponding eigen projector of the operator T^n, then for any m $(m=1,2,...,n-1)$ operator T^m is representable in the form.*

$$T^m=\sum_{j=1}^{\infty}\mu_j^{-m}\left(\sum_{i=1}^{n}\alpha_i^mE_{ji}\right)P_j,\ \ \alpha_i^n=1,\ \ E_{ji}E_{jk}=\delta_{ik}E_{ji},\ \ P_j=\sum_{i=1}^{n}E_{ji}.$$

With the help of this lemma it is proved the following

Lemma 2. *Under the conditions 1)-4) $(I-A(\lambda))^{-1}$ is bounded on each ray passing through the origin of coordinates but not through the roots of the equations*

$$\sum_{k=1}^{n} z_k \mu^k h_j^k = 1$$

for the sufficiently small $|\lambda - a|$, *where* $\mu = \dfrac{1}{\lambda - a}$ *are the eigen values of the operator* T.

Proof. Let's make the substitution

$$\mu = \frac{1}{\lambda - a}$$

and consider the operator-function

$$\widetilde{A}(\mu) = \sum_{k=1}^{n-1} \mu^k B_k T^k + \sum_{k=1}^{n} \mu^k z_k T^k + \widetilde{B}(\mu), \qquad (2)$$

where $\widetilde{A}(\mu) \equiv A(\lambda)$.

It is easy to show that

$$\left(I - \sum_{k=1}^{n} \mu^k z_k T^k \right)^{-1} = \left[\prod_{k=1}^{n} (I - r_k \mu T) \right]^{-1} = \prod_{k=1}^{n} (I - r_k \mu T)^{-1},$$

where numbers r_k are one valuedly determined by the numbers z_k $(k = 1,2,...,n)$.

Consider the equation

$$y = \widetilde{A}(\mu) y + f, \quad f \in H. \qquad (3)$$

Hence we have

$$\widetilde{y} = \sum_{k=1}^{n-1} \mu^k B_k T^k \prod_{k=1}^{n} (I - r_k \mu T)^{-1} \widetilde{y} + \widetilde{B}(\mu) \prod_{k=1}^{n} (I - r_k \mu T)^{-1} \widetilde{y} + f, \qquad (4)$$

where $\widetilde{y} = \left(I - \sum_{k=1}^{n} z_k \mu^k T^k \right) y$.

From lemma 1 it follows, that

$$T = \sum_{j=1}^{\infty} h_j \left(\sum_{i=1}^{m} \alpha_i E_{ji} \right) P_j,$$

where

$$\alpha_i^n = 1, \ E_{ji} E_{ji} = \delta_{ij} E_{ji}, \ \overline{h}_j = \mu_j^{-1}, p_j = \sum_{i=1}^{n} E_{ji}.$$

Let μ be on the ray not passing through the roots of the equation

$$\sum_{k=1}^{n} z_k \mu^k h_j^k = 1, \ j = 1,2,...$$

Estimate

$$\left\| (I - r_k \mu T)^{-1} \right\|,$$

it is easy to see that

$$\left\| (I - r_k \mu T)^{-1} \right\| = \left\| \sum_{i=0}^{m-1} (r_k \mu)^i T^i \left(I - r_k^m \mu^m T^m \right)^{-1} \right\|. \qquad (5)$$

Then granting lemma 1 in the relation (5) we obtain

$$\left\| (I - r_k \mu T)^{-1} \right\| = \left\| \sum_{i=0}^{m-1} (r_k \mu)^i \left(\sum_{j=1}^{\infty} \frac{h_j^i}{1 - (r_k \mu h_j)^m} \sum_{i=1}^{m} \alpha_i^i E_{ji} \right) P_j \right\| \le M_1 \sum_{i=0}^{m-1} \sup \frac{|\mu|^i |h_j^i|}{\left| 1 - (r_k \mu h_j)^m \right|},$$

where M_1 is a constant, not depending on μ. Further incoming as in *paper [2], we obtain*

$$\left\|(I - r_k \mu T)^{-1}\right\| \le M_2 \left(\frac{1}{\sin\varepsilon}\right)^m, \qquad (6)$$

where M_2 is a constant not depending on μ and ε is the bast of angles between the rays outgoing from the origin of coordinates and passing through $r_k \mu$ and $\frac{1}{h_j}$.

Granting (6), it is easy to establish that

$$\left\|\prod_{k=1}^{n}(I - r_k \mu T)^{-1}\right\| \le M_3 \left(\frac{1}{\sin\varepsilon}\right)^{mn}, \qquad (7)$$

where M_3 is a constant not depending on μ.

By assumption T^n is a complete compact, scalar type operator. Then by the Gelfand-Makke theorem [3] it is boundedly-similar normal operator. Therefore without loosing generality we may consider it as a normal operator.

We'll denote the orthogonal projector of the invariant eigen subspace corresponding to the first l-eigen values, by P_l. It is obvious that the $P_l \to I$ (strongly) at $l \to \infty$. Then we have

$$\left\|\mu^k B_k T^k \prod_{k=1}^{n}(I - r_k \mu T)^{-1}\right\| \le \left\|B_k P_l T^k \prod_{k=1}^{n}(I - r_k \mu T)^{-1} \mu^k\right\| +$$

$$+ \left\|B_k (I - P_l)\right\| \left\|T^k \prod_{k=1}^{n}(I - r_k \mu T)^{-1} \mu^k\right\|.$$

Now using lemma 1 and the method used by obtaining the estimation (7) one can establish that

$$\left\|B_k P_l T^k \prod_{k=1}^{n}(I - r_k \mu T)^{-1}\right\| \to 0$$

for $|\mu| \to \infty$.

By condition B_k is a compact operator. Then $\left\|B_k (I - P_l)\right\|$ as much as desired little for sufficiently large l.

As above it is proved that

$$\left\|\prod_{k=1}^{n}(I - r_k \mu T)^{-1} \widetilde{B}(\mu)\right\| \to 0$$

for $|\mu| \to \infty$.

So we showed that

$$\left\|\widetilde{A}(\mu)\right\| \to 0$$

for $|\mu| \to \infty$.

Consequently, $\left\|\widetilde{A}(\mu)\right\| < 1$ for sufficiently large $|\mu|$ lying on rays, mentioned in the present lemma and then on each such ray $\left(I - \widetilde{A}(\mu)\right)^{-1}$ exists and bounded for sufficiently large $|\mu|$.

Lemma has been proved.

Theorem. *Let for the operator function* $A(\lambda)$ *the conditions 1)-4) be fulfilled.*

Then

1) (e.a.) N-tiple complete in H, where N is a sum of multiplicities of poles of the operator function $A(\lambda)$.

2) The limit point of characteristic numbers of operator-function $A(\lambda)$ may be the point a.

3) For any $\varepsilon > 0$ there exist only finite number of characteristic numbers in sufficiently small neighborhood of the point a, which satisfy no one of inequalities

$$\alpha_i - \varepsilon \le arq(\lambda - a)^{-1} \le \alpha_i + \varepsilon,$$

where α_i is a argument of rays passing through the origin of coordinates and rots of equations

$$\sum_{k=1}^{n} z_k \mu^k h_j^k = 1,$$

where h_j are eigen values of the operator T and $\mu = \dfrac{1}{\lambda - a}$.

Proof. According to M.V.Keldysh results [3], the main part of resolvent of any operator-function $K(\lambda)$ analytically depending on a spectral parameter λ in some domain D of a complex plain and being a compact operator for each value $\lambda \in D$, has the form

$$\sum_i \left[\frac{y^{(i)} z^{(i)}}{(\lambda - c)^{m_i+1}} + \frac{y^{(i)} z_1^{(i)} + y_1^{(i)} z_1}{(\lambda - c)^{m_i}} + \frac{y^{(i)} z_{m_i-1}^{(i)} + y_1^{(i)} z_{m_i-2}^{(i)} + \ldots + y_{m_i-1}^{(i)} z^{(i)}}{(\lambda - c)} \right] \qquad (8)$$

in the neighborhood of pole c the resolvent $[I - K(\lambda)]^{-1}$.

One can show that the mentioned here main part of the resolvent in the neighborhood of the pole c and in general this result is true also for the operator-function $A(\lambda)$.

Granting lemma 2 in (8) and taking the element $f(\lambda)$ in the form

$$f(\lambda) = \sum_{i=1}^{n} \frac{f_i}{(\lambda - a)^i} + \sum_{j=0}^{s} f_j \lambda^j ,$$

and also proceeding from that the operators T and A_s are complete and acting similarly as in [1] one can show that all elements f_i are zero. And this means that the system of eigen and adjoined elements of the operator function $A(\lambda)$ is N-tiple complete in the space H.

The other statements of the present paper are proved as well as in paper [1].

Note that the obtained results generalize the corresponding results [1].

In conclusion I express my gratitude to my supervisor prof. A.M.Ahmedov for the statement of the problem and helpful comments.

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Received February 10, 2000; Revised November 20, 2000.
Translated by Nazirova S.H.