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JOHN-NIRENBERG'S THEOREM FOR THE HIGH ORDER MEAN OSCILLATION AND ITS APPLICATIONS

Abstract

At the given paper the analogue of John-Nirenberg's theorem for the mean oscillation of high order is proved, and we get some estimations in terms of non-increasing rearrangement functions.

Let $Q(a,r):=\{x=(x_1,x_2,...,x_n)\in \mathbf{R}^n : |x_i-a_i|\le r, i=1,2,...,n\}$ be a closed cube with the center at the point $a=(a_1,a_2,...,a_n)\in \mathbf{R}^n$. Let's apply the orthogonalization process by means of the inner product

$$(f,g):=\frac{1}{|Q(0,1)|}\int_{Q(0,1)} f(t)g(t)dt$$

to the system of the power functions $\{x^\nu\}$, $|\nu|\le k$ situated in the partial lexicographical order ([1]), where by $|Q(a,r)|$ we denote the volume of the cube $Q(a,r)$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, $x^\nu=x_1^{\nu_1}\cdot x_2^{\nu_2}\cdot\dots\cdot x_n^{\nu_n}$, $|\nu|=\nu_1+\nu_2+\dots+\nu_n$, ν_i $(i=1,2,...,n)$ are the non-negative integers. The result of the orthogonalization process we'll denote by $\{\psi_\nu\}$, $|\nu|\le k$. The system $\{\psi_\nu\}$, $|\nu|\le k$ is orthogonal and normed.

For the locally summable function f we'll suppose [2], [3]

$$P_{k,Q(a,r)}f(x):=\sum_{|\nu|\le k}\left(\frac{1}{|Q(a,r)|}\int_{Q(a,r)} f(t)\psi_\nu\left(\frac{t-a}{r}\right)dt\right)\psi_\nu\left(\frac{x-a}{r}\right).$$

$P_{k,Q(a,r)}f$ is the polynomial of degree no higher k.

Let Φ be a class of all positive monotonically increasing on $(0,+\infty)$ functions, $\varphi\in\Phi$ and

$$\Phi_{2r}(x):=\int_x^{2r}\frac{\varphi(t)}{t}dt,\ \ x\in[0,2r].$$

The next estimation is true, which is analogue of John-Nirenberg's theorem [4] for the case of mean oscillation of high order.

Theorem 1. *Let Q_0 be some cube in $\mathbf{R}^n$ with edges parallel to coordinate axes $f\in L^1(Q_0)$, $\varphi\in\Phi$ and*

$$\|f\|_{\varphi,k}:=\sup\left\{\frac{1}{\varphi(r)}\cdot\frac{1}{|Q(a,r)|}\int_{Q(a,r)}|f(t)-P_{k,Q(a,r)}f(t)|dt : Q(a,r)\subset Q_0\right\}<+\infty, \qquad (1)$$

where the supremum is taken on all cubes $Q(a,r)$ contained in cube Q_0. Let $Q=Q(a,r)$ be any cube from Q_0. Then

1^0) if $\Phi_{2r}(0)<+\infty$, then

$$m\{x\in Q:|f(x)-P_{k,Q}f(x)|>\lambda\}\le\begin{cases}B\cdot\left(\Phi_{2r}^{-1}\left(\frac{b\lambda}{\|f\|_{\varphi,k}}\right)\right)^n & when\ 0<\lambda<\frac{1}{b}\|f\|_{\varphi,k}\cdot\Phi_{2r}(0),\\ 0 & when\ \lambda\ge\frac{1}{b}\|f\|_{\varphi,k}\cdot\Phi_{2r}(0);\end{cases}$$

2^0) if $\Phi_{2r}(0)=+\infty$, then for any number $\lambda>0$

$$m\{x\in Q:|f(x)-P_{k,Q}f(x)|>\lambda\}\le B\cdot\left(\Phi_{2r}^{-1}\left(\frac{b\lambda}{\|f\|_{\varphi,k}}\right)\right)^n,$$

where Φ_{2r}^{-1} is a function inverse to the function Φ_{2r}, mE denotes Lebesgue measure of the set $E\subset\mathbf{R}^n$. And the positive constants B and b don't depend on φ, f, Q_0, Q and λ.

Proof. The case $\|f\|_{\varphi,k}=0$ is evident. Let $\|f\|_{\varphi,k}\neq 0$ and $Q=Q(a,r)$ is an arbitrary cube such that $Q\subset Q_0$. Let's denote $g(x):=\frac{1}{\|f\|_{\varphi,k}}(f(x)-P_{k,Q}f(x))$. Then we have

$$P_{k,Q}g(x)=\frac{1}{\|f\|_{\varphi,k}}P_{k,Q}(f-P_{k,Q})(x)=\frac{1}{\|f\|_{\varphi,k}}(P_{k,Q}f(x)-P_{k,Q}f(x))\equiv 0,$$

$$\frac{1}{|Q|}\int_Q|g(t)|dt=\frac{1}{|Q|}\int_Q|g(t)-P_{k,Q}g(t)|dt=\frac{1}{\|f\|_{\varphi,k}}\cdot\frac{1}{|Q|}\int_Q|f(t)-P_{k,Q}f(t)|dt\le$$

$$\le\frac{1}{\|f\|_{\varphi,k}}\cdot\|f\|_{\varphi,k}\cdot\varphi(r)=\varphi(r)<2^n\cdot\varphi(r).$$

Therefore applying Calderon-Zygmund's theorem on decomposition [5] we find the system of non-intersecting cubes $\{Q_j^{(1)}\}$ (here "non-intersecting" means that their interior is mutually disjoint) from Q such that

1) $|g(x)|\le 2^n\cdot\varphi(r)$ almost everywhere in $Q\setminus\bigcup_j Q_j^{(1)}$;

2) $2^n\cdot\varphi(r)\le\frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}}|g(y)|dy\le 2^{2n}\cdot\varphi(r)$ for any j;

3) $$\sum_j|Q_j^{(1)}|\le\frac{1}{2^n\cdot\varphi(r)}\sum_j\int_{Q_j^{(1)}}|g(y)|dy\le\frac{1}{2^n\cdot\varphi(r)}\int_Q|g(y)|dy=$$

$$=\frac{1}{2^n\cdot\varphi(r)}\cdot|Q|\cdot\frac{1}{|Q|}\int_Q|g(y)-P_{k,Q}g(y)|dy\le\frac{1}{2^n}\cdot|Q|.$$

Let's take any from the cubes $Q_j^{(1)}$, for simplicity let's denote it by the $Q^{(1)}$ and apply Calderon-Zygmund's theorem to the function $(g(x)-P_{k,Q^{(1)}}g(x))X_{Q^{(1)}}(x)$, where X_E is the characteristic function of the set $E\subset\mathbf{R}^n$. Since

$$\frac{1}{|Q^{(1)}|}\int_{Q^{(1)}}|g(y)-P_{k,Q^{(1)}}g(y)|dy=\frac{1}{\|f\|_{\varphi,k}}\cdot\frac{1}{|Q^{(1)}|}\int_{Q^{(1)}}|f(y)-P_{k,Q^{(1)}}f(y)|dy,$$

then by virtue of theorem's conditions we get

$$\frac{1}{|Q^{(1)}|}\int_{Q^{(1)}}|g(y)-P_{k,Q^{(1)}}g(y)|dy\le\varphi(r_{Q^{(1)}})<2^n\cdot\varphi(r_{Q^{(1)}}),$$

where $r_{Q^{(1)}}$ is a half of the edge of the cube $Q^{(1)}$ and therefore there exists a system of non-intersecting cubes $Q_j^{(2)}$ from $Q^{(1)}$ such that

1') $|g(y)-P_{k,Q^{(1)}}g(y)|\le 2^n\cdot\varphi(r_{Q^{(1)}})$, almost everywhere in $Q^{(1)}\setminus\bigcup_j Q_j^{(2)}$;

2') $2^n\cdot\varphi(r_{Q^{(1)}})\le\frac{1}{|Q_j^{(2)}|}\int_{Q_j^{(2)}}|g(y)-P_{k,Q^{(1)}}g(y)|dy\le 2^{2n}\cdot\varphi(r_{Q^{(1)}})$ for any j;

3') $\sum_j|Q_j^{(2)}|\le\frac{1}{2^n\cdot\varphi(r_{Q^{(1)}})}\sum_j\int_{Q_j^{(2)}}|g(y)-P_{k,Q^{(1)}}g(y)|dy\le$

$$\le\frac{1}{2^n}\cdot\frac{1}{\varphi(r_{Q^{(1)}})}\int_{Q^{(1)}}|g(y)-P_{k,Q^{(1)}}g(y)|dy\le\frac{1}{2^n}\cdot|Q^{(1)}|.$$

From the definition of polynomial's $P_{k,Q}f$ it is easy to see that

$$|P_{k,Q}f(x)|\le c\cdot\frac{1}{|Q|}\int_Q|f(t)|dt,\quad x\in Q,$$

where the constant $c>0$ doesn't depend on f, Q, x. Allowing for this from 1') and 2) we'll get that for almost all $x\in Q^{(1)}\setminus\bigcup_j Q_j^{(2)}$

$$|g(x)|\le|g(x)-P_{k,Q^{(1)}}g(x)|+|P_{k,Q^{(1)}}g(x)|\le 2^n\cdot\varphi(r_{Q^{(1)}})+c\cdot 2^{2n}\cdot\varphi(r)\le$$
$$\le 2^{2n}\cdot\max\{1,c\}\cdot(\varphi(r_{Q^{(1)}})+\varphi(r)).\qquad(2)$$

From 3) it follows $(2r_{Q^{(1)}})^n\le\frac{1}{2^n}\cdot(2r)^n$ or $r_{Q^{(1)}}\le\frac{1}{2}r$. Therefore from the inequality (2) we get that for almost all $x\in Q^{(1)}\setminus\bigcup_j Q_j^{(2)}$ the inequality

$$|g(x)|\le c_0\cdot 2n\left(\varphi\left(\frac{r}{2}\right)+\varphi(r)\right)$$

is true, where $c_0=\max\{1,c\}$.

From the correlations 3) and 3') it follows, that

$$\sum_j|Q_j^{(2)}|\le\frac{1}{2^n}\sum_l|Q_j^{(1)}|\le\left(\frac{1}{2^n}\right)^2\cdot|Q|.$$

Here the summation is conducted by all cubes $Q_j^{(1)}$.

Continuing this process on the m-th step we'll get the system of non-intersecting cubes $Q_j^{(m)}$ such that almost everywhere in the set $Q^{(m-1)}\setminus\bigcup_j Q_j^{(m)}$ the condition

$$|g(x)| \le 2^{2n} \cdot c_0 \cdot \sum_{j=0}^{m-1} \varphi\left(\frac{r}{2^j}\right) \le \frac{2^{2n} c_0}{\ln 2} \cdot \Phi_{2r}\left(\frac{r}{2^{m-1}}\right). \tag{3}$$

is fulfilled.

Besides

$$\sum_j \left|Q_j^{(m)}\right| \le \left(\frac{1}{2^n}\right)^m \cdot |Q|, \tag{4}$$

moreover here summation is taken by all cubes $Q_j^{(m-1)}$. From the inequality (3) and (4), particularly it follows that

$$m\left\{x \in Q : |g(x)| > \frac{2^{2n} \cdot c_0}{\ln 2} \cdot \Phi_{2r}\left(\frac{r}{2^{m-1}}\right)\right\} \le \left(\frac{1}{2^n}\right)^m \cdot |Q|.$$

Hence here with help of some additional consideration we get the required statements. The theorem is proved.

Let's denote that in case when $\varphi(t) \equiv 1$, $k = 0$ the statement of the point 2^0) of theorem 1 composes contents of John-Nirenberg's theorem [4]. The statement of this point in case $\varphi \in \Phi$, $\varphi(+0) = 0$, $k = 0$ was proved in [6].

Corollary 1. *Let Q_0 be a cube in $\mathbf{R}^n$ with the edges parallel to coordinate axes, $f \in L^1(Q_0), \varphi \in \Phi$ and the condition (1) be fulfilled. Then if $Q = Q(a,r)$ is any cube from the Q_0 and $\Phi_{2r}(0) < +\infty$, then there exists such a positive constant c that for almost all $x \in Q$ the inequality*

$$\left|f(x) - P_{k,Q} f(x)\right| \le c \|f\|_{\varphi,k} \cdot \int_0^{2r} \frac{\varphi(t)}{t} dt$$

is true, where the constant c doesn't depend on f, Q_0, Q and x.

Theorem 2. *Let $f \in L^1(Q_0)$, Q_0 be a cube from the $\mathbf{R}^n$, $\varphi \in \Phi$, $\varphi(2r) = O(\varphi(r))$ $(r > 0)$ and one of the conditions*

1) $\Phi_{2r}(0) < +\infty$; *for any cube $Q = Q(a,r) \subset Q_0$ and for any number $\lambda > 0$*

$$m\left\{x \in Q : \left|f(x) - P_{k,Q} f(x)\right| > \lambda\right\} \le \begin{cases} B \cdot \left(\Phi_{2r}^{-1}(b\lambda)\right)^n & \text{when } 0 < \lambda < \frac{1}{b}\Phi_{2r}(0), \\ 0 & \text{when } \lambda \ge \frac{1}{b}\Phi_{2r}(0); \end{cases}$$

2) $\Phi_{2r}(0) = +\infty$; *for any cube $Q = Q(a,r) \subset Q_0$, and for any number $\lambda > 0$*

$$m\left\{x \in Q : \left|f(x) - P_{k,Q} f(x)\right| > \lambda\right\} \le B \cdot \left(\Phi_{2r}^{-1}(b\lambda)\right)^n$$

is fulfilled, where the positive constants B and b don't depend on Q and λ.

Then when $1 \le p < \infty$ for the any cube $Q = Q(a,r) \subset Q_0$ it holds the next inequality

$$\left(\frac{1}{|Q|} \int_Q \left|f(x) - P_{k,Q} f(x)\right|^p dx\right)^{1/p} \le \frac{c}{b} \cdot \varphi(r),$$

where the constant $c > 0$ depends only on B, p, n and the function φ.

Proof. Let the condition 1) be fulfilled and let $Q = Q(a,r) \subset Q_0$ be any cube. If

$$m(\lambda) := m\left\{x \in Q : \left|f(x) - P_{k,Q} f(x)\right| > \lambda\right\},$$

then we have

$$\int_Q |f(x)-P_{k,Q}f(x)|^p dx = p\int_0^\infty \lambda^{p-1}\cdot m(\lambda)d\lambda \le$$

$$\le p\int_0^{\frac{1}{b}\Phi_{2r}(0)} \lambda^{p-1}\cdot B\cdot\left(\Phi_{2r}^{-1}(b\lambda)\right)^n d\lambda = Bp\cdot\int_0^{\frac{1}{b}\Phi_{2r}(0)} \lambda^{p-1}\left(\Phi_{2r}^{-1}(b\lambda)\right)^n d\lambda =$$

$$= Bp\cdot\int_0^{2r}\left(\frac{1}{b}\Phi_{2r}(x)\right)^{p-1} x^n\cdot\frac{1}{b}\cdot\frac{\varphi(x)}{x}dx = \frac{B\cdot p}{b^p}\cdot\int_0^{2r}\left(\int_x^{2r}\frac{\varphi(t)}{t}dt\right)^{p-1} x^{n-1}\varphi(x)dx \le$$

$$\le \frac{Bp}{b^p}(\varphi(2r))^p\cdot\int_0^{2r}\left(\ln\frac{2r}{x}\right)^{p-1} x^{n-1}dx = \frac{Bp}{b^p}(\varphi(2r))^p\cdot\int_0^\infty\left(\frac{t}{n}\right)^{p-1}\left(2re^{-\frac{t}{n}}\right)^{n-1}\cdot\frac{1}{n}\cdot 2r\cdot e^{-\frac{t}{n}}dt =$$

$$= \frac{Bp}{(bn)^p}\cdot(\varphi(2r))^p(2r)^n\cdot\int_0^\infty t^{p-1}\cdot e^{-t}dt = \frac{Bp}{(bn)^p}(\varphi(2r))^p\cdot|Q|\cdot\Gamma(p).$$

Hence

$$\left(\frac{1}{|Q|}\int_Q |f(x)-P_{k,Q}f(x)|^p dx\right)^{1/p} \le \frac{(Bp\Gamma(p))^{1/p}}{bn}\cdot\varphi(2r) \le \frac{c}{b}\varphi(r),\ \ r>0.$$

Case 2) is analogously considered. The theorem is proved.

Corollary 2. *Let* $f\in L^1(Q_0)$, Q_0 *be a cube from* $\mathbf{R}^n$, $\varphi\in\Phi$, $\varphi(2r)=O(\varphi(r))$ $(r>0)$ *and the condition 1) be fulfilled. Then when* $1\le p<\infty$ *for any cube* $Q=Q(a,r)\subset Q_0$ *it holds the inequality*

$$\frac{1}{|Q|}\int_Q |f(t)-P_{k,Q}f(t)|dt \le \left(\frac{1}{|Q|}\int_Q |f(t)-P_{k,Q}f(t)|^p dt\right)^{1/p} \le c\cdot\|f\|_{\varphi,k}\cdot\varphi(r),$$

where the constant $c>0$ *doesn't depend on* f, Q_0 *and* Q.

Let $h(x)$ be a given measurable function on $\mathbf{R}^n$, $m_h(\lambda):=m\{x:|h(x)|>\lambda\}$ be the function of distribution of the function $|h(x)|$. Non-increasing rearrangement of the function h is determined by the equality (see [5])

$$h^*(t):=\inf\{\lambda: m_h(\lambda)\le t\}.$$

Both functions $h^*(t)$ and $m_h(\lambda)$ are non-negative, non-increasing and continuous on the right. Since

$$m_{h^*}(\lambda)=m\{t:|h^*(t)|>\lambda\}=m\{t:0\le t<m_h(\lambda)\}=m_h(\lambda),$$

then the functions h^* and h have the same distribution functions. Hence we get that when $1\le p\le\infty$

$$\|h^*\|_{L^p(0,+\infty)}=\|h\|_{L^p(\mathbf{R}^n)}.$$

Theorem 3. *Let* Q_0 *be some cube in* $\mathbf{R}^n$ *with the edges, parallel to coordinate axes,* $f\in L^1(Q_0)$, $\varphi\in\Phi$ *and the condition (1) be fulfilled. If* $Q=Q(a,r)$ *is an arbitrary cube from* Q_0, *then the inequality*

$$\left(\left(f-P_{k,Q}f\right)X_Q\right)^*(t)\le c\|f\|_{\varphi,k}\cdot\int_{\frac{1}{2}\sqrt[n]{t}}^{2r}\frac{\varphi(\tau)}{\tau}d\tau,\quad 0\le t\le 2^n|Q|,$$

is true, where X_Q *is a characteristic function of the cube* Q *and the constant* $c>0$ *doesn't depend on* φ, f, Q_0, Q *and* t.

Proof. By fulfilling the conditions of the theorem the statements of theorem 1 are true. Let's denote $2^n\left(\Phi_{2r}^{-1}\left(\frac{b\lambda}{\|f\|_{\varphi,k}}\right)\right)^n=t$. Then if $0<t\le 2^n|Q|$, then $\lambda=\frac{1}{b}\|f\|_{\varphi,k}\cdot\Phi_{2r}\left(\frac{1}{2}\sqrt[n]{t}\right)$. If $t=0$, then $\lambda\ge\frac{1}{b}\|f\|_{\varphi,k}\cdot\Phi_{2r}(0)$ (this holds when $\Phi_{2r}(0)<+\infty$). So, if $0\le t\le 2^n|Q|$ and $\lambda\ge\frac{1}{b}\|f\|_{\varphi,k}\cdot\Phi_{2r}\left(\frac{1}{2}\sqrt[n]{t}\right)$, then by virtue of theorem 1 we have

$$m_{(f-P_{k,Q}f)X_Q}(\lambda)=m\{x\in Q:|f(x)-P_{k,Q}f(x)|>\lambda\}\le t,$$

and therefore

$$\left(\left(f-P_{k,Q}f\right)X_Q\right)^*(t)=\inf\{\lambda: m_{(f-P_{k,Q}f)X_Q}(\lambda)\le t\}\le$$
$$\le\frac{1}{b}\|f\|_{\varphi,k}\cdot\Phi_{2r}\left(\frac{1}{2}\sqrt[n]{t}\right)=\frac{1}{b}\|f\|_{\varphi,k}\cdot\int_{\frac{1}{2}\sqrt[n]{t}}^{2r}\frac{\varphi(\tau)}{\tau}d\tau.$$

The theorem is proved.

Its is also true the following

Theorem 4. *Let* $f\in L^1(Q_0)$, Q_0 *be a cube in* $\mathbf{R}^n$, $\varphi\in\Phi$, $\varphi(2r)=O(\varphi(r))$ $(r>0)$ *and for any cube* $Q=Q(a,r)\subset Q_0$ *the condition*

$$\left(\left(f-P_{k,Q}f\right)X_Q\right)^*(t)\le A\cdot\int_{\frac{1}{2}\sqrt[n]{t}}^{2r}\frac{\varphi(\tau)}{\tau}d\tau,\quad 0\le t\le 2^n|Q|,$$

be fulfilled, where the constant $A>0$ *doesn't depend on* Q *and* t. *Then when* $1\le p<\infty$ *for any cube* $Q=Q(a,r)\subset Q_0$ *the inequality*

$$\left(\frac{1}{|Q|}\int_Q|f(x)-P_{k,Q}f(x)|^p dx\right)^{1/p}\le c\cdot A\cdot\varphi(r),\ \ r>0$$

is true, where the constant $c>0$ *depends only on* p,n *and* φ.

Theorem 5. *Let either* Q_0 *be a cube in* $\mathbf{R}^n$ *and* $f\in L^1(Q_0)$, *or* $Q_0=\mathbf{R}^n$ *and* $f\in L^1_{loc}(\mathbf{R}^n)$, *and let* $\varphi\in\Phi$, $\varphi(2r)=O(\varphi(r))$ $(r>0)$. *Then the next conditions are equivalent*

I. *For any cube* $Q=Q(a,r)\subset Q_0$

$$\frac{1}{|Q|}\int_Q|f(t)-P_{k,Q}f(t)|dt\le c\cdot\varphi(r),$$

where the constant $c>0$ *doesn't depend on* Q;

II. *For any cube* $Q=Q(a,r)\subset Q_0$

1) if $\Phi_{2r}(0)<+\infty$, *then*

$$m\{x \in Q : |f(x) - P_{k,Q} f(x)| > \lambda\} \le \begin{cases} B \cdot \left(\Phi_{2r}^{-1}(b\lambda)\right)^n & when\ 0 < \lambda < \frac{1}{b}\Phi_{2r}(0), \\ 0 & when\ \lambda \ge \frac{1}{b}\Phi_{2r}(0); \end{cases}$$

2) if $\Phi_{2r}(0) = +\infty$, then for any $\lambda > 0$

$$m\{x \in Q : |f(x) - P_{k,Q} f(x)| > \lambda\} \le B \cdot \left(\Phi_{2r}^{-1}(b\lambda)\right)^n ,$$

where the positive constants B and b don't depend on Q and λ;

III. At $1 \le p < \infty$ for any cube $Q = Q(a,r) \subset Q_0$

$$\left(\frac{1}{|Q|}\int_Q |f(x) - P_{k,Q} f(x)|^p dx\right)^{1/p} \le c \cdot \varphi(r),$$

where the constant $c > 0$ doesn't depend on Q;

IV. For any cube $Q = Q(a,r) \subset Q_0$

$$\left(\left(f - P_{k,Q} f\right) X_Q\right)^*(t) \le c \cdot \int_{\frac{1}{2}\sqrt[n]{t}}^{2r} \frac{\varphi(\tau)}{\tau} d\tau \ , \ 0 \le t \le 2^n |Q|,$$

where the constant $c > 0$ doesn't depend on Q and t.

Proof. We get the equivalence of the given conditions from the chain of implications, which is established by applying theorems 1-4

I. $\Rightarrow$ II. $\Rightarrow$ III. $\Rightarrow$ I. $\Rightarrow$ IV. $\Rightarrow$ III.

The theorem is proved.

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