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ON EXTREMAL PROBLEM FOR GOURSA-DARBOUX TYPE DIFFERENTIAL INCLUSIONS, I

Abstract

The necessary and sufficient conditions of extremum of the convex variation problems given on the set of the variable absolutely continuous functions have been obtained.

The paper consists of two parts. In the first part of the paper the necessary and sufficient conditions of extremum of the convex variation problems given on the set of two variable absolutely continuous functions are obtained. Mainly the case when the optimal solution is not internal, has been studied.

Note that the extremal problems for the convex differential inclusions are easy reduced to the convex variation problems. In the second part of the paper the nonconvex extremal problem for the differential inclusions with phase constraints is considered.

The extremal problems for multidimensional differential inclusions also have been considered in the author's papers [1-6].

1. About minimization of two-dimensional variation problems.

Two variable function $u(\cdot):[0,T]\times[0,S]\to R^n$ is called absolutely continuous in $[0,T]\times[0,S]$ if it is represented in the form

$$u(t,s)=u(0,0)+\int_0^t x(r)dr+\int_0^s y(v)dv+\int_0^t\int_0^s z(\tau,v)d\tau dv,$$

where $x(\cdot)\in L_1^n[0,T]$, $y(\cdot)\in L_1^n[0,S]$, $z(\cdot)\in L_1^n([0,T]\times[0,S])$. We'll denote the set of all absolutely continuous functions determined in $[0,T]\times[0,S]$ with the norm:

$$\|u(\cdot)\|=|u(0,0)|+\int_0^T|x(\tau)|d\tau+\int_0^S|y(v)|dv+\int_0^T\int_0^S|z(\tau,v)|d\tau dv$$

by $A^n([0,T]\times[0,S])$. It is easy to verify that one can represent any linear continuous functional v^* on $A^n([0,T]\times[0,S])$ in the form:

$$v^*(u)=\left(u(0,0)|c\right)+\int_0^T\left(u_t(t,0)|a(t)\right)dt+\int_0^S\left(u_s(0,s)|b(s)\right)ds+\int_0^T\int_0^S\left(u_{ts}(t,s)|v(t,s)\right)dtds,$$

where $c\in R^n$, $a(\cdot)\in L_\infty^n[0,T]$, $b(\cdot)\in L_\infty^n[0,S]$, $v(\cdot)\in L_\infty^n([0,T]\times[0,S])$. Further we'll denote the functional v^* by the symbol $(c,a(\cdot),b(\cdot),v(\cdot))$. We'll denote the set $A^n([0,T]\times[0,S])$ with the norm $\|u\|=\max\limits_{t,s}\|u(t,s)\|$ by $A_c^n([0,T]\times[0,S])$. It is clear that $A_c^n([0,T]\times[0,S])^*\subset A^n([0,T]\times[0,S])^*$.

Lemma 1. *The inequality*

$$\sup\left\{\int_0^T (u_t(t,0)|\gamma_1(t))dt + \int_0^S (u_s(0,s)|\eta_1(s))ds + \int_0^T\int_0^S (u_t(t,s)|\overline{\psi}_1(t,s))dtds + \int_0^T\int_0^S (u_s(t,s)|\overline{\psi}_2(t,s))dtds + \right.$$

$$\left. + \int_0^T\int_0^S (u_{ts}(t,s)|\overline{\psi}_3(t,s))dtds + (u(0,0)|a) + (u(T,S)|b) : u \in A^n([0,T]\times[0,S])\right\} < +\infty$$

is correct if and only if

$$a = -b,\ \gamma_1(t) + b + \int_0^S \overline{\psi}_1(t,v)dv = 0\ ;\ \ \eta_1(s) + b + \int_0^T \overline{\psi}_2(\tau,s)d\tau = 0,$$

$$\overline{\psi}_3(t,s) + b - \int_0^S \overline{\psi}_1(t,v)dv + \int_0^S \overline{\psi}_1(t,v)dv - \int_0^t \overline{\psi}_2(\tau,s)d\tau + \int_0^T \overline{\psi}_2(\tau,s)d\tau = 0.$$

Proof. It is clear that

$$\int_0^T\int_0^S (u_t(t,s)|\overline{\psi}_1(t,s))dtds = \int_0^T\int_0^S \left(u_t(t,s) \middle| d\left(\int_0^s \overline{\psi}_1(t,v)dv - \int_0^S \overline{\psi}_1(t,v)dv \right)\right)dt =$$

$$= \int_0^T \left(u_t(t,0) \middle| \int_0^S \overline{\psi}_1(t,v)dv \right)dt - \int_0^T\int_0^S \left(u_{ts}(t,s) \middle| \int_0^s \overline{\psi}_1(t,v)dv - \int_0^S \overline{\psi}_1(t,v)dv \right)dtds,$$

$$\int_0^T\int_0^S (u_s(t,s)|\overline{\psi}_2(t,s))dtds = \int_0^S\int_0^T \left(u_s(t,s) \middle| d\left(\int_0^t \overline{\psi}_2(\tau,s)d\tau - \int_0^T \overline{\psi}_2(\tau,s)d\tau \right)\right)ds =$$

$$= \int_0^S \left(u_s(0,s) \middle| \int_0^T \overline{\psi}_2(\tau,s)d\tau \right)ds - \int_0^T\int_0^S \left(u_{ts}(t,s) \middle| \int_0^t \overline{\psi}_2(\tau,s)d\tau - \int_0^T \overline{\psi}_2(\tau,s)d\tau \right)dtds;$$

$$(u(T,S)|b) = (u(0,0)|b) + \int_0^T (u_t(t,0)|b)dt + \int_0^S (u_s(0,s)|b)ds + \int_0^T\int_0^S (u_{ts}(t,s)|b)dtds.$$

Therefore for any $u \in A^n([0,T]\times[0,S])$

$$(u(0,0)|a+b) + \int_0^T \left(u_t(t,0) \middle| \gamma_1(t) + b + \int_0^S \overline{\psi}_1(t,v)dv \right)dt + \int_0^S \left(u_s(0,s) \middle| \eta_1(s) + b + \int_0^T \overline{\psi}_2(\tau,s)d\tau \right)ds +$$

$$+ \int_0^T\int_0^S \left(u_{ts}(t,s) \middle| \overline{\psi}_3(t,s) + b - \int_0^s \overline{\psi}_1(t,v)dv + \int_0^S \overline{\psi}_1(t,v)dv - \int_0^t \overline{\psi}_2(\tau,s)d\tau + \int_0^T \overline{\psi}_2(\tau,s)d\tau \right)dtds = 0.$$

Hence follows that

$$a = -b,\ \gamma_1(t) + b + \int_0^S \overline{\psi}_1(t,v)dv = 0,\ \ \eta_1(s) + b + \int_0^T \overline{\psi}_2(\tau,s)d\tau = 0,$$

$$\overline{\psi}_3(t,s) + b - \int_0^s \overline{\psi}_1(t,v)dv + \int_0^S \overline{\psi}_1(t,v)dv - \int_0^t \overline{\psi}_2(\tau,s)d\tau + \int_0^T \overline{\psi}_2(\tau,s)d\tau = 0.$$

Lemma has been proved.

Assume

$$A_p^n([0,T]\times[0,S]) = \{u \in A^n([0,T]\times[0,S]) : u(0,0) = a_1,$$
$$u(T,0) = a_2,\ u(0,S) = a_3,\ u(T,S) = a_4\}.$$

Lemma 2. *Let* $\varphi_1(\cdot) \in L_\infty[0,T]$, $\varphi_2(\cdot) \in L_\infty[0,S]$, $\varphi(\cdot) \in L_\infty([0,T]\times[0,S])$ *and*

$$\sup\left\{\int_0^T (u_t(t,0)|\varphi_1(t))dt + \int_0^S (u_s(0,s)|\varphi_2(s))ds + \int_0^T\int_0^S (u_{ts}(t,s)|\varphi(t,s))dtds : u \in A_p^1([0,T]\times[0,S])\right\}.$$

Then $\varphi_1(t) = const$, $\varphi_2(s) = const$, $\varphi(t,s) = const$.

Proof. Assume

$$B_1 = \left\{ x(\cdot) \in L_1[0,T] : \int_0^T x(\tau)d\tau = a_2 - a_1 \right\},\ B_2 = \left\{ y(\cdot) \in L_1[0,S] : \int_0^S y(v)dv = a_3 - a_1 \right\},$$

$$B_3 = \left\{ z(\cdot) \in L_1([0,T]\times[0,S]) : \int_0^T\int_0^S z(\tau,v)d\tau dv = a_4 + a_1 - a_2 - a_3 \right\}.$$ Since

$$u(t,s) = u(0,0) + \int_0^t x(\tau)d\tau + \int_0^s y(v)dv + \int_0^t\int_0^s z(\tau,v)d\tau dv,$$

then we'll obtain that

$$\sup_{\substack{x(\cdot)\in B_1, y(\cdot)\in B_2 \\ z(\cdot)\in B_3}} \left\{ \int_0^T (x(\tau)|\varphi_1(\tau))d\tau + \int_0^S (y(v)|\varphi_2(v))dv + \int_0^T\int_0^S (z(t,s)|\psi(t,s))dtds \right\} < +\infty.$$

Therefore

$$\sup_{x(\cdot)\in B_1} \int_0^T (x(\tau)|\varphi_1(\tau))d\tau < +\infty,\ \sup_{y(\cdot)\in B_2} \int_0^S (y(v)|\varphi_2(v))dv < +\infty,\ \sup_{z(\cdot)\in B_3} \int_0^T\int_0^S (z(t,s)|\psi(t,s))dtds < +\infty.$$

Thus by lemma 4 [2] we'll obtain that $\varphi_1 = const$, $\varphi_2 = const$. We'll show that $\psi = const$. Assume the contrary. Let $\psi(t,s) \neq const$. Then there must be found two sets Δ_1 and Δ_2, that $\mu(\Delta_1) = \mu(\Delta_2) > 0$, which nonintersecting to each-other would lie inside of $[0,T]\times[0,S]$ and there will the inequality $\beta_1 = \inf_{(t,s)\in\Delta_1} \psi(t,s) > \sup_{(t,s)\in\Delta_2} \psi(t,s) = \beta_2$ would be fulfilled.

Let's consider the function

$$z_n(t,s) = \frac{1}{TS}a_1 - \frac{1}{TS}a_2 - \frac{1}{TS}a_3 + \frac{1}{TS}a_4 + nv(t,s),$$

where

$$v(t,s) = \begin{cases} c,\ (t,s)\in\Delta_1 \\ -c, (t,s)\in\Delta_2 \\ 0,\ (t,s)\in[0,T]\times[0,S]\setminus(\Delta_1\cup\Delta_2),\ c>0. \end{cases}$$

By mean value theorem we'll obtain

$$\int_0^T\int_0^S z_n(t,s)\psi(t,s)dtds = \frac{a_1 - a_2 - a_3 + a_4}{TS}\int_0^T\int_0^S \psi(t,s)dtds + n\int_0^T\int_0^S v(t,s)\psi(t,s)dtds =$$

$$= \frac{a_1 - a_2 - a_3 + a_4}{TS}\int_0^T\int_0^S \psi(t,s)dtds + nc\int_{\Delta_1} \psi(t,s)dtds - nc\int_{\Delta_2} \psi(t,s)dtds \geq$$

$$\geq \frac{a_1 - a_2 - a_3 + a_4}{TS}\int_0^T\int_0^S \psi(t,s)dtds + nc(\beta_1 - \beta_2)\mu(\Delta_1).$$

Hence we'll obtain that $\lim_{n\to\infty}\int_0^T\int_0^S z_n(t,s)\psi(t,s)dtds = +\infty$. We came to the contrary.

Lemma has been proved.

Let $g:[0,T]\times[0,S]\times R^n \to \overline{R} = R\cup\{+\infty\}, \psi_1 : [0,T]\times R^n \to \overline{R}, \psi_2 : [0,S]\times R^n \to \overline{R}$ be measurable integrands $q : R^{4n} \to \overline{R}$. Let's consider subdifferentially of the functional

$$J_1(u)=\int_0^T\int_0^S g(t,s,u(t,s))dtds+\int_0^T \psi_1(t,u(t,0))dt+$$

$$+\int_0^S \psi_2(s,u(0,s))ds+q(u(0,0),u(T,0),u(0,S),u(T,S))$$

in the space $A^n([0,T]\times[0,S])$. Assume

$$F_1(u)=\int_0^T\int_0^S g(t,s,u(t,s))dtds,\ F_2(u)=\int_0^T \psi_1(t,u(t,0))dt,\ F_3(u)=\int_0^S \psi_2(s,u(0,s))ds,$$

$$F_4(u)=q(u(0,0),u(T,0),u(0,s),u(T,S)),$$

$$Q=\{u(\cdot)\in A^n([0,T]\times[0,S]):F_1(u)<+\infty\},\ Q_1=\{x(\cdot)\in W_{1,1}^n[0,T]:F_2(x)<+\infty\},$$

$$Q_2=\{y(\cdot)\in W_{1,1}^n[0,S]:F_3(y)<+\infty\}.$$

Lemma 3. *If* $F_4^*(v^*)<+\infty$, *where* $v^*=(c,a(\cdot),b(\cdot),v(\cdot))$, *then* $a(t)=d_1\in R^n$, $b(s)=d_2\in R^n$, $v(t,s)=d\in R^n$ *and*

$$\mathbf{F}_4^*(v^*)=q^*(c-d_1-d_2+d,d_1-d,d_2-d,d).$$

Proof. Let $(a_1,a_2,a_3,a_4)\in R^{4n}$ such that $q(a_1,a_2,a_3,a_4)<+\infty$. By the definition

$$F_4^*(v^*)=\sup_{u\in A^n([0,T]\times[0,S])}\Bigg\{(u(0,0)|c)+\int_0^T (u_t(t,0)|a(t))dt+\int_0^S (u_s(0,s)|b(s))ds+$$

$$+\int_0^T\int_0^S (u_{ts}(t,s)|v(t,s))dtds-q(u(0,0),u(T,0),u(0,S),u(T,S))\Bigg\}\geq(a_1|c)-q(a_1,a_2,a_3,a_4)+$$

$$+\sup\left\{\int_0^T (u_t(t,0)|a(t))dt+\int_0^S (u_s(0,s)|b(s))ds+\int_0^T\int_0^S (u_{ts}(t,s)|v(t,s))dtds:u\in A_p^n([0,T]\times[0,S])\right\}.$$

Therefore from the lemma 2 we'll obtain, that $a(t)=d_1\in R^n$, $b(t)=d_2\in R^n$, $v(t,s)=d\in R^n$. It is obvious that

$$F_4^*(v^*)=\sup_{u\in A^n([0,T]\times[0,S])}\Bigg\{(u(0,0)|c)+\int_0^T (u_t(t,0)|d_1)dt+\int_0^S (u_s(0,s)|d_2)ds+\int_0^T\int_0^S (u_{ts}(t,s)|d)dtds-$$

$$-q(u(0,0),u(T,0),u(0,S),u(T,S))\Bigg\}=\sup\{(u(0,0)|c-d_1-d_2+d)+(u(T,0)|d_1-d)+$$

$$+(u(0,S)|d_2-d)+(u(T,S)|d)-q(u(0,0),u(T,0),u(0,S),u(T,S))\}=$$

$$=q^*(c-d_1-d_2+d,d_1-d,d_2-d,d).$$

Lemma has been proved.

Corollary 1. *If* q *a convex function and* $v^*=(c,a(\cdot),b(\cdot),v(\cdot))\in\partial F_4(\overline{u})$, *then* $a(t)=d_1\in R^n$, $b(s)=d_2\in R^n$, $v(t,s)=d\in R^n$ *and*

$$(c-d_1-d_2+d,d_1-d,d_2-d,d)\in\partial q(\overline{u}(0,0),\overline{u}(T,0),\overline{u}(0,S),\overline{u}(T,S)).$$

Lemma 4. *Let* $v_1^*=(c,a_1(\cdot),b_1(\cdot),v_1(\cdot)),v_2^*=(c_2,a_2(\cdot),b_2(\cdot),v_2(\cdot))$ *such that* $F_2^*(v_1^*)<+\infty$, $F_3^*(v_2^*)<+\infty$. *Then* $b_1(s)=0$, $v_1(t,s)=0$, $a_2(t)=0$, $v_2(t,s)=0$.

Proof. By the definition we have that

$$F_2^*(v_1^*)=\sup_u\Bigg\{(u(0,0)|c_1)+\int_0^T (u_t(t,0)|a_1(t))dt+\int_0^S (u_s(0,s)|b_1(s))ds+\int_0^T\int_0^S (u_{ts}(t,s)|v_1(t,s))dt-$$

$$-\int_0^T \psi_1(t,u(t,0))dt\Bigg\} = \sup_{\substack{d\in R^n, x(\cdot)\in L_1^n[0,T]\\ y(\cdot)\in L_1^n[0,S], z(\cdot)\in L_1^n([0,T]\times[0,S])}} \Bigg\{(d|c_1) + \int_0^T (x(t)|a_1(t))dt +$$

$$+\int_0^S (y(s)|b_1(s))ds + \int_0^T\int_0^S (z(t,s)|v_1(t,s))dtds - \int_0^T \psi_1\left(t, d + \int_0^t x(\tau)d\tau\right)dt\Bigg\} = \sup_{d,x(\cdot)}\{(d|c_1) +$$

$$+\int_0^T (x(t)|a_1(t))dt - \int_0^T \psi_1\left(t, d+\int_0^t x(\tau)d\tau\right)dt\Bigg\} + \sup_{y(\cdot),z(\cdot)}\left\{\int_0^S (y(s)|b_1(s))ds + \int_0^T\int_0^S (z(t,s)|v_1(t,s))dtds\right\}.$$

Therefore hence obtain, that if $F_2^*(v_1^*) < +\infty$, then $b_1(s) = 0, v_1(t,s) = 0$. Analogously it is verified that if $F_3^*(v_2^*) < +\infty$, then $a_2(t) = 0,\ v_2(t,s) = 0$. Lemma has been proved.

Lemma 5. *Let g,ψ_1 and ψ_2 be convex normal integrants, q be convex, there exist such $\alpha_1(\cdot)\in L_1([0,T]\times[0,S]), \alpha_2(\cdot)\in L_1[0,T], \alpha_3(\cdot)\in L_1(0,S)$ and $\alpha>0$, that $\alpha_1(t,s)-\alpha|z|^k \le g(t,s,z), \alpha_2(t)-\alpha|x|^k \le \psi_1(t,x), \alpha_3(s)-\alpha|y|^k \le \psi_2(s,y)$ for some $k\ge 0$; there exist the functions $u_0(\cdot)\in A^n([0,T]\times[0,S])$ and the number $r>0$ such that for $y\in R^n, |y|\le r$ the functions $g(t,s,u_0(t,s)+y)$, $\psi_1(t,u_0(t,0)+y), \psi_2(s,u_0(0,s)+y)$ are summable, $q(u_0(0,0),u_0(T,0),u_0(0,S),u_0(T,S))$ is finite. Then $v^*=(c,a(\cdot),b(\cdot),v(\cdot))\in\partial J_1(\overline{u})$ if and only if there exist measures $\lambda\in frm([0,T]\times[0,S])^n$, $\mu\in frm[0,T]^n, \gamma\in frm[0,S]^n$ and $\overline{c},d_1,d_2,d\in R^n$ such that*

1) $v^*(u) = \int_0^T\int_0^S (u(t,s)|d\lambda) + \int_0^T (u(t,0)|d\mu) + \int_0^S (u(0,s)|d\gamma) + (u(0,0)|\overline{c}-d_1-d_2+d) +$
$+(u(T,0)|d_1-d) + (u(0,S)|d_2-d) + (u(T,S)|d)$,

2) $\omega(t,s)\in\partial g(t,s,\overline{u}(t,s))$; *3)* $\omega_1(t)\in\partial\psi_1(t,\overline{u}(t,0))$; *4)* $\omega_2(s)\in\partial\psi_2(s,\overline{u}(0,s))$,

5) $(\overline{c}-d_1-d_2+d, d_1-d, d_2-d, d)\in\partial q(\overline{u}(0,0),\overline{u}(T,0),\overline{u}(0,S),\overline{u}(T,S))$,

6) $\max\left\{\int_0^T\int_0^S (u(t,s)|d\lambda_s) : u\in Q\right\} = \int_0^T\int_0^S (\overline{u}(t,s)|d\lambda_s)$,

7) $\max\left\{\int_0^T (x(t)|d\mu_s) : x\in Q_1\right\} = \int_0^T (\overline{u}(t,0)|d\mu_s)$,

8) $\max\left\{\int_0^S (y(s)|d\gamma_s) : y\in Q_2\right\} = \int_0^S (\overline{u}(0,s)|d\gamma_s)$,

where $\lambda(E) = \int_E \omega(t,s)dtds + \lambda_s(E), \mu(E_1) = \int_{E_1}\omega_1(t)dt + \mu_s(E_1), \gamma(E_2) = \int_{E_2}\omega_2(s)ds + \gamma_s(E_2)$

are Lebesque expansions λ,μ and γ, respectively.

Proof. By Rockfeller theorem we'll obtain that

$$\partial J_1(\overline{u}) = \partial F_1(\overline{u}) + \partial F_2(\overline{u}) + \partial F_3(\overline{u}) + \partial F_4(\overline{u}).$$

Therefore if $v^* = (c,a(\cdot),b(\cdot),v(\cdot))\in\partial J_1(\overline{u})$, then $v^* = v_1^*+v_2^*+v_3^*+v_4^*$, where $v_i^*\in\partial F_i(\overline{u})$. Analogously to the statement 2.2.1 [5] it is proved that $v_1^*\in\partial F_1(\overline{u})$ if and only if there exists $z_1^*\in A_c^n([0,T]\times[0,S])^*$ that $z_1^*\in\partial F_1(\overline{u})$ and $v_1^*(u) = z_1^*(u)$

for any $u \in A^n([0,T]\times[0,S])$. By the corollary 2.2.1 [5] $z_1^*(u)=\int_0^T\int_0^S (u(t,s)|d\lambda)=$ $=\int_0^T\int_0^S (u(t,s)|\omega(t,s))dtds+\int_0^T\int_0^S (u(t,s)|d\lambda_s)$ belongs to $\partial F_1(\overline{u})$ only in that case when $\omega(t,s)\in\partial g(t,\overline{u}(t,s))$ and $\max\limits_{u\in Q}\int_0^T\int_0^S (u(t,s)|d\lambda_s)=\int_0^T\int_0^S (\overline{u}(t,s)|d\lambda_s)$.

If $v_2^*=(c_2,a_2(\cdot),b_2(\cdot),v_2(\cdot))$, then by the statement 1.5.1 [7] $F_2(\overline{u})+F^*(v_2^*)=v_2^*(\overline{u})$. Since $F^*(v_2^*)<+\infty$, then from lemma 4 it follows that $b_2(s)=0$, $v_2(t,s)=0$. Therefore by lemma 1 and lemma 2 [2] $v_2^*\in\partial F_2(\overline{u})$ if and only if there exists measure $\mu\in frm[0,T]^n$ such that $v_2^*(u)=\int_0^T (u(t,0)|d\mu)=\int_0^T (u(t,0)|\omega_1(t))dt+\int_0^T (u(t,0)|d\mu_s)$, where $\omega_1(t)\in\partial\psi_1(t,\overline{u}(t,0))$ and $\max\left\{\int_0^T (x(t)|d\mu_s): x\in Q_1\right\}=\int_0^T (\overline{u}(t,0)|d\mu_s)$.

Analogously we'll obtain that $v_3^*=(c_3,a_3(\cdot),b_3(\cdot),v_3(\cdot))\in\partial F_3(\overline{u})$ if and only if $a_3(t)=0$, $v_3(t,s)=0$ and there exists measure $\gamma\in frm[0,S]^n$, where $\gamma(E_2)=$ $=\int_{E_2}\omega_2(s)ds+\gamma_s(E_2)$, $E_2\subset[0,S]$ such that $v_3^*(u)=\int_0^S (u(0,s)|d\gamma)$, $\omega_2(s)\in\partial\psi_2(s,\overline{u}(0,s))$ and $\max\left\{\int_0^S (y(s)|d\gamma_s): y\in Q_2\right\}=\int_0^S (\overline{u}(0,s)|d\gamma_s)$.

From lemma 3 it follows that if $v^*=(c_4,a_4(\cdot),b_4(\cdot),v_4(\cdot))\in\partial F_4(\overline{u})$, then $a_4(t)=d_1\in R^n$, $b_4(s)=d_2\in R^n$, $v_4(t,s)=d\in R^n$ and $(c_4-d_1-d_2+d,d_1-d,d_2-d,d)\in$ $\in\partial q(\overline{u}(0,0),\overline{u}(T,0),\overline{u}(0,S),\overline{u}(T,S))$. Lemma has been proved.

Lemma 6. *If g,ψ_1 and ψ_2 are convex normal integrants and there exists a number $r>0$ such that for $y\in R^n, |y|\le r$ the functions $g(t,s,\overline{u}(t,s)+y)$, $\psi_1(t,\overline{u}(t,0)+y)$, $\psi_2(s,\overline{u}(0,s)+y)$ are summable and $q(\overline{u}(0,0),\overline{u}(T,0),\overline{u}(0,S),\overline{u}(T,S))$ is finite, then $v^*=(c,a(\cdot),b(\cdot),v(\cdot))\in\partial J_1(\overline{u})$ if and only if $v(\cdot)\in A^n([0,T]\times[0,S])$, $a(\cdot)\in W_{1,1}^n[0,T]$, $b(\cdot)\in W_{1,1}^n[0,S]$, $v(T,s)=v(t,S)=const$, $v_{ts}(t,s)\in\partial g(t,s,\overline{u}(t,s))$, $v_t(t,0)-\dot a(t)\in\partial\psi_1(t,\overline{u}(t,0))$, $v_s(0,s)-\dot b(s)\in\partial\psi_2(s,\overline{u}(0,s))$ and*

$$(c-a(0)-b(0)+v(0,0),a(T)-v(T,0),b(S)-v(0,S),v(T,S))\in$$
$$\in\partial q(\overline{u}(0,0),\overline{u}(T,0),\overline{u}(0,S),\overline{u}(T,S)).$$

Proof. If $v^*=(c,a(\cdot),b(\cdot),v(\cdot))\in\partial J_1(\overline{u})$ then by theorem we'll obtain that $v^*=v_1^*+v_2^*+v_3^*+v_4^*$, where $v_i^*\in\partial F_i(\overline{u})$. Analogously to the statement 2.2.1 [5] it is proved that $v_1^*\in\partial F_1(\overline{u})$ if and only if there exists $z_1^*\in A_c^n([0,T]\times[0,S])^*$ that $z_1^*\in\partial F_1(\overline{u})$ and $v_1^*(u)=z_1^*(u)$. By the corollary 2.2.1 [5] $z_1^*(u)=\int_0^T\int_0^S (u(t,s)|\omega(t,s))dtds$, where $\omega(t,s)\in\partial g(t,s,u(t,s))$. It is clear that

$$\int_0^T\int_0^S (u(t,s)|\omega(t,s))dtds = \int_0^T\int_0^S\left(u(t,s)\middle| d\left(\int_0^s \omega(t,v)dv - \int_0^S \omega(t,v)dv\right)\right)dt = \int_0^T\left(u(t,0)\middle|\int_0^S \omega(t,v)dv\right)dt -$$

$$-\int_0^T\int_0^S\left(u_s(t,s)\middle|\int_0^s \omega(t,v)dv - \int_0^S \omega(t,v)dv\right)dtds = -\int_0^S\int_0^T\left(u_s(t,s)\middle|\left(d\int_0^t\int_0^s \omega(\tau,v)d\tau dv -\right.\right.$$

$$\left.\left.-\int_0^t\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)dvd\tau + \int_0^T\int_0^S \omega(\tau,v)d\tau dv\right)\right)ds +$$

$$+\int_0^T\left(u(t,0)\middle| d\left(\int_0^t\int_0^S \omega(\tau,v)dvd\tau - \int_0^T\int_0^S \omega(\tau,v)d\tau dv\right)\right) = \left(u(0,0)\middle|\int_0^T\int_0^S \omega(\tau,v)d\tau dv\right) +$$

$$+\int_0^T\left(u_t(t,0)\middle|\int_0^T\int_0^S \omega(\tau,v)d\tau dv - \int_0^t\int_0^S \omega(\tau,v)d\tau dv\right)dt +$$

$$+\int_0^S\left(u_s(0,s)\middle|\int_0^T\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)d\tau dv\right)ds + \int_0^T\int_0^S\left(u_{ts}(t,s)\middle|\int_0^t\int_0^s \omega(\tau,v)d\tau dv -\right.$$

$$\left.-\int_0^t\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)d\tau dv + \int_0^T\int_0^S \omega(\tau,v)d\tau dv\right)dtds.$$

From the inequality $v_1^*(u) = z_1^*(u)$, hence we'll obtain that $c_1 = \int_0^T\int_0^S \omega(\tau,v)d\tau dv$,

$$a_1(t) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv - \int_0^t\int_0^S \omega(\tau,v)d\tau dv,\ b_1(s) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)d\tau dv,$$

$$v_1(t,s) = \int_0^t\int_0^s \omega(\tau,v)d\tau dv - \int_0^t\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)d\tau dv + \int_0^T\int_0^S \omega(\tau,v)d\tau dv.$$

Analogously to lemma 5 we'll obtain that $v_2^* = (c_2, a_2(\cdot), 0, 0)$, $v_3^* = (c_3, 0, b_3(\cdot), 0)$, $v_4^* = (\bar{c}, d_1, d_2, d)$. Besides there exist $\omega_1(t) \in \partial\psi_1(t, \bar{u}(t,0)), \omega_2(s) \in \partial\psi_2(s, \bar{u}(0,s))$ such that

$$(u(0,0)|c_2) + \int_0^T (u_t(t,0)|a_2(t))dt = \int_0^T (u(t,0)|\omega_1(t))dt,$$

$$(u(0,0)|c_3) + \int_0^S (u_s(0,s)|b_3(s))ds = \int_0^S (u(0,s)|\omega_2(s))ds.$$

Since

$$\int_0^T (u(t,0)|\omega_1(t))dt = \int_0^T\left(u(t,0)\middle| d\left(\int_0^t \omega_1(\tau)d\tau - \int_0^T \omega_1(\tau)d\tau\right)\right) = \left(u(0,0)\middle|\int_0^T \omega_1(\tau)d\tau\right) +$$

$$+\int_0^T\left(u_t(t,0)\middle|\int_0^T \omega_1(\tau)d\tau - \int_0^t \omega_1(\tau)d\tau\right),$$

$$\int_0^S (u(0,s)|\omega_2(s))ds = \int_0^S\left(u(0,s)\middle| d\left(\int_0^s \omega_2(v)dv - \int_0^S \omega_2(v)dv\right)\right) = \left(u(0,0)\middle|\int_0^S \omega_2(v)dv\right) +$$

$$+\int_0^S \left(u_s(0,s) \Bigg|_0^S \int \omega_2(v)dv - \int_0^s \omega_2(v)dv \right) ds,$$

then we'll obtain that $c_2 = \int_0^T \omega_1(\tau)d\tau$, $a_2(t) = \int_0^T \omega_1(\tau)d\tau - \int_0^t \omega_1(\tau)d\tau$, $c_3 = \int_0^S \omega_2(v)dv$, $b_3(s) = \int_0^S \omega_2(v)dv - \int_0^s \omega_2(v)dv$. Thus we'll get that

$$c = \int_0^T\int_0^S \omega(\tau,v)d\tau dv + \int_0^T \omega_1(\tau)d\tau + \int_0^S \omega_2(v)dv + \bar{c},$$

$$a(t) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv - \int_0^t\int_0^S \omega(\tau,v)d\tau dv + \int_0^T \omega_1(\tau)d\tau - \int_0^t \omega_1(\tau)d\tau + d_1,$$

$$b(s) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)d\tau dv + \int_0^S \omega_2(v)dv - \int_0^s \omega_2(v)dv + d_2,$$

$$v(t,s) = \int_0^t\int_0^s \omega(\tau,v)d\tau dv - \int_0^t\int_0^S \omega(\tau,v)d\tau dv - \int_0^T\int_0^s \omega(\tau,v)d\tau dv + \int_0^T\int_0^S \omega(\tau,v)d\tau dv + d.$$

Therefore $v_{ts}(t,s) = \omega(t,s), \dot{b}(s) = -\int_0^T \omega(\tau,s)d\tau - \omega_2(s)$, $\dot{a}(t) = -\int_0^S \omega(t,v)dv - \omega_1(t)$, $a(T) = d_1$, $b(S) = d_2, a(0) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv + \int_0^T \omega_1(\tau)d\tau + d_1$, $b(0) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv +$

$$+\int_0^S \omega_2(v)dv + d_2,\ v(T,s) = v(t,S) = d,\ v(0,0) = \int_0^T\int_0^S \omega(\tau,v)d\tau dv + d,$$

$\bar{c} = c - a(0) - b(0) + v(0,0) + d_1 + d_2 - d, \omega_1(t) = -\dot{a}(t) - \int_0^S \omega(\tau,v)dv = -\dot{a}(t) + v_t(t,0)$,

$$\omega_2(s) = -\dot{b}(s) - \int_0^T \omega(\tau,s)d\tau = -\dot{b}(s) + v_s(0,s).$$

Thus we'll obtain that
$v(\cdot) \in A^n([0,T]\times[0,S]), a(\cdot) \in W_{1,1}^n[0,T], b(\cdot) \in W_{1,1}^n[0,S], v(T,s) = v(t,S)$ for $t \in [0,T], s \in [0,S]$,

$$v_{ts}(t,s) \in \partial g(t,s,\bar{u}(t,s)),\ v_t(t,0) - \dot{a}(t) \in \partial\psi_1(t,\bar{u}(t,0)), v_s(0,s) - \dot{b}(s) \in \partial\psi_2(s,\bar{u}(0,s))$$

and $(c - a(0) - b(0) + v(0,0), a(T) - v(T,0), b(s) - v(0,S), v(T,S)) \in$
$\in \partial q(\bar{u}(0,0), \bar{u}(T,0), \bar{u}(0,S), \bar{u}(T,S))$. The sufficiency of the theorem is verified immediately. Lemma has been proved.

Let $f:[0,T]\times[0,S]\times R^{4n} \to \bar{R}$, $\varphi_1:[0,T]\times R^{2n} \to \bar{R}$, $\varphi_2:[0,S]\times R^{2n} \to \bar{R}$ be measurable integrants. Let's consider minimization of the functional

$$J(u) = \int_0^T\int_0^S f(t,s,u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))dtds + \int_0^T \varphi_1(t,u(t,0),u_t(t,0))dt +$$

$$+\int_0^S \varphi_2(s,u(0,s),u_s(0,s))ds + q(u(0,0),u(T,0),u(0,S),u(T,S))$$

in the space $A^n([0,T]\times[0,S])$.

Assume $f^0(t,s,y;v)=\inf\{(\omega|v)+f(t,s,y,\omega):\omega\in R^{3n}\}$, $\varphi_1^0(t,y,v_1)=\inf\{(\omega_1|v_1)+$ $+\varphi_1(t,y,\omega_1):\omega_1\in R^n\}$, $\varphi_2^0(s,y,v_2)=\inf\{(\omega_2|v_2)+\varphi_2(s,y,\omega_2):\omega_2\in R^n\}$, where $v\in R^{3n}$, $v_1\in R^n$, $v_2\in R^n$.

Theorem 1. *Let f,φ_1 and φ_2 be convex normal integrants, q be convex. In order that $\bar u$ will be minimal paint of the functional $J(u)$ in the space $A^n([0,T]\times[0,S])$ is sufficient, but if there exist the functions $\tilde u\in A^n([0,T]\times[0,S])$, $\alpha(\cdot)\in L_1([0,T]\times[0,S])$, $\beta(\cdot)\in L_1[0,S]$, $v(\cdot)\in L_1[0,T]$, numbers $c\ge 0, r>0$ such that for $y\in R^n, |y|\le r$*

$$f(t,s,\tilde u(t,s)+y,z_1,z_2,\tilde u_{ts}(t,s))\le\alpha(t,s)+c|(z_1,z_2)|,$$

the functions $\varphi_1(t,\tilde u(t,0)+y,\tilde u_t(t,0))$, $\varphi_2(s,\tilde u(0,s)+y,\tilde u_s(0,s))$ are summable, $-\alpha(t,s)-c|z|\le f(t,s,z)$, $v(t)-c|x|\le\varphi_1(t,x)$, $\beta(s)-c|x|\le\varphi_2(s,x)$ and the function $q(\tilde u(0,0),\cdot)$ is continuous at the point $(\tilde u(T,0),\tilde u(0,S),\tilde u(T,S))$, then it is also necessary that 'll be find the functions $\bar p_1\in L_\infty^n([0,T]\times[0,S])$, $\bar p_2\in L_\infty^n([0,T]\times[0,S])$, measures $\lambda\in frm([0,T]\times[0,S])^n$, $\mu\in frm[0,T]^n$, $\gamma\in frm[0,S]^n$, the functional $\bar v^*=$ $=(0,v_1(\cdot),v_2(\cdot),v(\cdot))\in A^n([0,T]\times[0,S])^*$ and the vectors $\bar c,d_1,d_2,d\in R^n$ such that

1) $v^*(u)=\int_0^T\int_0^S(u(t,s)|d\lambda)+\int_0^T(u(t,0)|d\mu)+\int_0^S(u(0,s)|d\gamma)+(u(0,0)|\bar c-d_1-d_2+d)+$ $+(u(T,0)|d_1-d)+(u(0,S)|d_2-d)+(u(T,S)|d)$,

2) $$\omega(t,s)\in\partial f^0\left(t,s,\bar u(t,s),\bar p_1(t,s),\bar p_2(t,s),v(t,s)+\int_0^s\bar p_1(t,v)dv-\int_0^S\bar p_1(t,v)dv+\right.$$
$$\left.+\int_0^t\bar p_2(\tau,s)d\tau-\int_0^T\bar p_2(\tau,s)d\tau\right.,$$

3) $$\omega_1(t)\in\partial\varphi_1^0\left(t,\bar u(t,0);v_1(t)-\int_0^S\bar p_1(t,v)dv\right),$$

4) $$\omega_2(s)\in\partial\varphi_2^0\left(s,\bar u(0,s);v_2(s)-\int_0^T\bar p_2(\tau,s)d\tau\right),$$

5) $(\bar c-d_1-d_2+d,d_1-d,d_2-d,d)\in\partial q(\bar u(0,0),\bar u(T,0),\bar u(0,S),\bar u(T,S))$,

6) $$f^0\left(t,s,\bar u(t,s);\bar p_1(t,s),\bar p_2(t,s),v(t,s)+\int_0^s\bar p_1(t,v)dv-\int_0^S\bar p_1(t,v)dv+\int_0^t\bar p_2(\tau,s)d\tau-\right.$$
$$\left.-\int_0^T\bar p_2(\tau,s)d\tau\right)=(\bar u_t(t,s)|\bar p_1(t,s))+(\bar u_s(t,s)|\bar p_2(t,s))+\left(\bar u_{ts}(t,s)\middle|v(t,s)+\int_0^s\bar p_1(t,v)dv-\right.$$
$$\left.-\int_0^S\bar p_1(t,v)dv+\int_0^t\bar p_2(\tau,s)d\tau-\int_0^T\bar p_2(\tau,s)d\tau\right)+f(t,s,\bar u(t,s),\bar u_t(t,s),\bar u_s(t,s),\bar u_{ts}(t,s)),$$

7) $$\varphi_1^0\left(t,\bar u(t,0);v_1(t)-\int_0^S\bar p_1(t,v)dv\right)=\left(\bar u_t(t,0)\middle|v_1(t)-\int_0^S\bar p_1(t,v)dv\right)+\varphi_1(t,\bar u(t,0),\bar u_t(t,0)),$$

8) $\varphi_2^0\left(s,\overline{u}(0,s);v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)=\left(\overline{u}_s(0,s)\middle|v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)+$
$+\varphi_2(s,\overline{u}(0,s),\overline{u}_s(0,s)),$

9) $\max_{u\in Q}\int_0^T\int_0^S(u(t,s)|d\lambda_s)=\int_0^T\int_0^S(\overline{u}(t,s)|d\lambda_s),$

10) $\max_{u\in Q_1}\int_0^T(x(t)|d\mu_s)=\int_0^T(\overline{u}(t,0)|d\mu_s),$

11) $\max_{u\in Q_2}\int_0^S(y(s)|d\gamma_s)=\int_0^S(\overline{u}(0,s)|d\gamma_s),$

where $\lambda(E)=\int_E\omega(t,s)dtds+\lambda_s(E),\mu(E_1)=\int_{E_1}\omega_1(t)dt+\mu_s(E_1),\gamma(E_2)=\int_{E_2}\omega_2(s)ds+\gamma_s(E_2)$

are Lebesque expansions λ,μ,γ *respectively,*

$$Q=\left\{u\in A^n([0,T]\times[0,S]):\int_0^T\int_0^S f^0\left(t,s,u(t,s);\overline{p}_1(t,s),\overline{p}_2(t,s),v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\right.\right.$$
$$\left.\left.-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\int_0^T\overline{p}_2(\tau,s)d\tau\right)dtds<+\infty\right\},$$

$$Q_1=\left\{x(\cdot)\in W_{1,1}^n[0,T]:\int_0^T\varphi_1^0\left(t,x(t):v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)dt<+\infty\right\},$$

$$Q_2=\left\{y(\cdot)\in W_{1,1}^n[0,S]:\int_0^S\varphi_2^0\left(s,y(s);v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)ds<+\infty\right\}.$$

Proof. The sufficiency of the theorem is verified immediately. Let's consider the functional

$$\Phi(u,z)=\int_0^T\int_0^S f(t,s,u(t,s),u_t(t,s)+z_t(t,s),u_s(t,s)+z_s(t,s),u_{ts}(t,s)+z_{ts}(t,s))dtds+$$
$$+\int_0^T\varphi_1(t,u(t,0),u_t(t,0)+z_t(t,0))dt+\int_0^S\varphi_2(s,u(0,s),u_s(0,s)+z_s(0,s))ds+$$
$$+q(u(0,0),u(T,0),u(0,S),u(T,S)),$$

where $z\in A_0^n=\{u\in A^n([0,T]\times[0,S]):u(0,0)=0\}$. Assume $h(z)=\inf\{\phi(u,z):u\in A^n\}$. Let's show that h is subdifferentiable at zero, i.e. the problem $\inf\{J(u):u\in A^n([0,T]\times[0,S])\}$ is stable. By the corollary $2A$ [8] the functional

$$J_0(z,z_1,z_2)=\int_0^T\int_0^S f(t,s,\widetilde{u}(t,s)+z(t,s),\widetilde{u}_t(t,s),\widetilde{u}_s(t,s),\widetilde{u}_{ts}(t,s))dtds+$$
$$+\int_0^T\varphi_1(t,\widetilde{u}(t,0)+z_1(t),\widetilde{u}_t(t,0))dt+\int_0^S\varphi_2(s,\widetilde{u}(0,s)+z_2(s),\widetilde{u}_s(0,s))ds+$$
$$+q(\widetilde{u}(0,0),\widetilde{u}(T,0)+c_1,\widetilde{u}(0,S)+c_2,\widetilde{u}(T,S)+c_3)$$

is continuous on $C^n([0,T]\times[0,S])\times C^n[0,T]\times C^n[0,S]\times R^{3n}$ at the point "0". Since for $z\in A_0^n$

$$\|z(t,0)\|_{C^n} + \|z(0,s)\|_{C^n} + \|z(t,s)\|_{C^n} \le 3\|z\|_{A^n}$$

then functional

$$J_1(z) = \int_0^T\int_0^S f(t,s,\widetilde{u}(t,s)+z(t,s),\widetilde{u}_t(t,s),\widetilde{u}_s(t,s),\widetilde{u}_{ts}(t,s))dtds +$$
$$+\int_0^T \varphi_1(t,\widetilde{u}(t,0)+z(t,0),\widetilde{u}_t(t,0))dt + \int_0^S \varphi_2(s,\widetilde{u}(0,s)+z(0,s),\widetilde{u}_s(0,s))ds +$$
$$+q(\widetilde{u}(0,0),\widetilde{u}(T,0)+z(T,0),\widetilde{u}(0,S)+z(0,S),\widetilde{u}(T,S)+z(T,S))$$

is continuous at zero point with respect to the topology A_0^n. From convexity and continuity of $J_1(u)$ at zero if follows that there exist such $\alpha>0$ and M, that $J_1(u)\le M$ for any $u_2 \in \{z\in A_0^n : \|z\|\le\alpha\}$. If $\|z(\cdot)\|_{A_0^n}\le\alpha$, $z(\cdot)\in A_0^n$, then assuming $u_2(t,s)=\widetilde{u}(t,s)-z(t,s)$ we obtain that

$$h(z)=\inf_u \Phi(u,z)\le\Phi(u_z,z)=J_1(z)\le M.$$

Using the statement 1.5.2 [7] hence follows that h is subdifferentiable at zero point. Therefore from note 3.2.3 and from statement 3.2.4 [7] it implies that all solutions $\overline{u}(\cdot)$ of the problem $\inf\{J(u): u\in A^n\}$ and all solutions $\overline{v}^* = (v_1(\cdot),v_2(\cdot),v(\cdot))$ of the problem $\sup\{-\Phi^*(0,-v^*): v^*\in (A_0^n)^*\}$ are related by the extremal correlation

$$\Phi(\overline{u},0)+\Phi^*(0;-\overline{v}^*)=0. \qquad (1)$$

By definition

$$\Phi^*(0,-\overline{v}^*) = \sup_{\substack{u\in A^n\\ z\in A_0^n}}\Big\{-\int_0^T (z_t(t,0)|v_1(t))dt - \int_0^S (z_s(0,s)|v_2(s))ds - \int_0^T\int_0^S (z_{ts}(t,s)|v(t,s))dtds -$$
$$-\int_0^T\int_0^S f(t,s,u(t,s),u_t(t,s)+z_t(t,s),u_s(t,s)+z_s(t,s),u_{ts}(t,s)+z_{ts}(t,s))dtds -$$
$$-\int_0^T \varphi_1(t,u(t,0),u_t(t,0)+z_t(t,0))dt - \int_0^S \varphi_2(s,u(0,s),u_s(0,s)+z_s(0,s))ds -$$
$$-q(u(0,0),u(T,0),u(0,S),u(T,S))\Big\} = \sup_{u\in A^n}\Big\{\int_0^T (u_t(t,0)|v_1(t))dt + \int_0^S (u_s(0,s)|v_2(s))ds +$$
$$+\int_0^T\int_0^S (u_{ts}(t,s)|v(t,s))dtds + \sup_{z\in A^n}\Big\{-\int_0^T (z_t(t,0)|v_1(t))dt - \int_0^S (z_s(0,s)|v_2(s))ds -$$
$$-\int_0^T\int_0^S (z_{ts}(t,s)|v(t,s))dtds - \int_0^T\int_0^S f(t,s,u(t,s),z_t(t,s),z_s(t,s),z_{ts}(t,s))dtds -$$
$$-\int_0^T \varphi_1(t,u(t,0),z_t(t,0))dt - \int_0^S \varphi_2(s,u(0,s),z_s(0,s))ds - q(u(0,0),u(T,0),u(0,S),u(T,S))\Big\} =$$
$$= \sup_{u\in A^n}\Big\{\int_0^T (u_t(t,0)|v_1(t))dt + \int_0^S (u_s(0,s)|v_2(s))ds + \int_0^T\int_0^S (u_{ts}(t,s)|v(t,s))dtds +$$
$$+\sup\Big\{-\int_0^T (y_1(t)|v_1(t))dt - \int_0^S (y_2(s)|v_2(s))ds - \int_0^T\int_0^S (y(t,s)|v(t,s))dtds -$$

$$-\int_0^T\int_0^S f(t,s,u(t,s),\omega_1(t,s),\omega_2(t,s),y(t,s))dtds-\int_0^T\varphi_1(t,u(t,0),y_1(t))dt-$$

$$-\int_0^S\varphi_2(s,u(0,s),y_2(s))ds-q(u(0,0),u(T,0),u(0,S),u(T,S))-$$

$$-\delta_M(y_1(\cdot),y_2(\cdot),\omega_1(\cdot),\omega_2(\cdot),y(\cdot))\}\}\},$$

where

$$y_1(\cdot)\in L_1^n[0,T],\ y_2(\cdot)\in L_1^n[0,S],\omega_1(\cdot),\omega_2(\cdot),y(\cdot)\in L_1^n([0,T]\times[0,S]),$$
$$M=\{(z_t(t,0),z_s(0,s),z_t(t,s),z_s(t,s),z_{ts}(t,s)):z(\cdot)\in A^n\}.$$

By theorem 4.4.11 [9] and lemma 1 there exist the functions $\overline{p}_1\in L_\infty^n([0,T]\times[0,S]),\overline{p}_2\in L_\infty^n([0,T]\times[0,S])$ such that

$$\Phi^*(0;-\overline{v}^*)=\sup_{u\in A^n}\left\{\int_0^T(u_t(t,0)|v_1(t))dt+\int_0^S(u_s(0,s)|v_2(s))ds+\int_0^T\int_0^S(u_{ts}(t,s)|v(t,s))dtds+\right.$$

$$+\sup\left\{-\int_0^T\left(y_1(t)|v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)dt-\int_0^S\left(y_2(s)|v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)ds-\right.$$

$$-\int_0^T\int_0^S(\omega_1(t,s)|\overline{p}_1(t,s))dtds-\int_0^T\int_0^S(\omega_2(t,s)|\overline{p}_2(t,s))dtds-$$

$$-\int_0^T\int_0^S\left(y(t,s)|v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\int_0^T\overline{p}_2(\tau,s)d\tau\right)dtds-$$

$$-\int_0^T\int_0^S f(t,s,u(t,s),\omega_1(t,s),\omega_2(t,s),y(t,s))dtds-\int_0^T\varphi_1(t,u(t,0),y_1(t))dt-$$

$$\left.\left.-\int_0^S\varphi_2(s,u(0,s),y_2(s))ds-q(u(0,0),u(T,0),u(0,S),u(T,S))\right\}\right\}=$$

$$=\sup_{u\in A^n}\left\{\int_0^T(u_t(t,0)|v_1(t))dt+\int_0^S(u_s(0,s)|v_2(s))ds+\int_0^T\int_0^S(u_{ts}(t,s)|v(t,s))dtds-\right.$$

$$-\int_0^T\int_0^S f^0(t,s,u(t,s),\overline{p}_1(t,s),\overline{p}_2(t,s),v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}(\tau,s)d\tau+\int_0^t\overline{p}_2(\tau,s)d\tau-$$

$$\left.-\int_0^T\overline{p}_2(\tau,s)d\tau\right)dtds-\int_0^T\varphi_1^0\left(t,u(t,0);v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)dt-$$

$$\left.-\int_0^S\varphi_2^0\left(s,u(0,s);v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)-q(u(0,0),u(T,0),u(0,S),u(T,S))\right\}.$$

Denoting

$$J_2(u)=\int_0^T\int_0^S f^0\left(t,s,u(t,s),\overline{p}_1(t,s),\overline{p}_2(t,s),v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}_1(t,v)+\int_0^t\overline{p}_2(\tau,s)d\tau-\right.$$

$$\left.-\int_0^T\overline{p}_2(\tau,s)d\tau\right)dtds+\int_0^T\varphi_1^0\left(t,u(t,0);v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)dt+$$

$$+\int_0^S \varphi_2^0\left(s,u(0,s);v_2(s)-\int_0^T \overline{p}_2(\tau,s)d\tau\right)ds+q(u(0,0),u(T,0),u(0,S),u(T,S))$$

we'll obtain that $\Phi^*(0;-\overline{v}^*)=J_2^*(\overline{v}^*)$. From the correlation (1) it implies that

$$J(\overline{u})+J_2^*(\overline{v}^*)=0.$$

Therefore

$$J_2^*(\overline{v}^*)=\int_0^T(\overline{u}_t(t,0)|v_1(t))dt+\int_0^S(\overline{u}_s(0,s)|v_2(s))ds+\int_0^T\int_0^S(\overline{u}_{ts}(t,s)|v(t,s))dtds-J_2(\overline{u}),$$

$$J_2(\overline{u}^*)=\int_0^T\left(\overline{u}_t(t,0)\left|v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right.\right)dt+\int_0^S\left(\overline{u}_s(0,s)\left|v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right.\right)ds+$$

$$+\int_0^T\int_0^S(\overline{u}_t(t,s)|\overline{p}_1(t,s))dtds+\int_0^T\int_0^S(\overline{u}_s(t,s)|\overline{p}_2(t,s))dtds+\int_0^T\int_0^S\left(\overline{u}_{ts}(t,s)|v(t,s)\right)+$$

$$\left.+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\int_0^T\overline{p}_2(\tau,s)d\tau\right)dtds+J(\overline{u}).$$

From the first equality it follows that $\overline{v}^*\in\partial J_2(\overline{u})$. From the second equality we have

$$f^0\left(t,s,\overline{u}(t,s),\overline{p}_1(t,s),\overline{p}_2(t,s),v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\right.$$

$$\left.-\int_0^T\overline{p}_2(\tau,s)d\tau\right)=(\overline{u}_t(t,s)|\overline{p}_1(t,s))+(\overline{u}_s(t,s)|\overline{p}_2(t,s))+\left(\overline{u}_{ts}(t,s)\left|v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\right.\right.$$

$$\left.-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\int_0^T\overline{p}_2(\tau,s)d\tau\right)+f(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s)),$$

$$\varphi_1^0\left(t,\overline{u}(t,0);v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)=\left(\overline{u}_t(t,0)\left|v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right.\right)+\varphi_1(t,\overline{u}(t,0),\overline{u}_t(t,0)),$$

$$\varphi_2^0\left(s,\overline{u}(0,s);v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)=\left(\overline{u}_s(0,s)\left|v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right.\right)+\varphi_2(s,\overline{u}(0,s),\overline{u}_s(0,s)).$$

Since for any $p(\cdot)\in L_\infty^{3n}([0,T]\times[0,S])$, $z_1(\cdot)\in L_\infty^n[0,T]$ and $z_2(\cdot)\in L_\infty^n[0,S]$ the correlation

$$f^0(t,s,\widetilde{u}(t,s)+y,p(t,s))\le((\widetilde{u}_t(t,s),\widetilde{u}_s(t,s),\widetilde{u}_{ts}(t,s))|p(t,s))+$$
$$+f(t,s,\widetilde{u}(t,s)+y,\widetilde{u}_t(t,s),\widetilde{u}_s(t,s),\widetilde{u}_{ts}(t,s)),$$
$$\varphi_1^0(t,\widetilde{u}(t,0)+y,z_1(t))\le(\widetilde{u}_t(t,0)|z_1(t))+\varphi_1(t,\widetilde{u}(t,0)+y,\widetilde{u}_t(t,0)),$$
$$\varphi_2^0(s,\widetilde{u}(0,s)+y,z_2(s))\le(\widetilde{u}_s(0,s)|z_2(s))+\varphi_2(s,\widetilde{u}(0,s)+y,\widetilde{u}_s(0,s))$$

is correct and $J_2(\widetilde{u})$ is finite, then it is easy to verify that for the functional the conditions of lemma 5 are fulfilled. We'll obtain the correctness of theorem 1 applying lemma 5.

Theorem 2 is proved analogously.

Theorem 2. *Let f,φ_1 and φ_2 be convex normal integrants and the function q be convex. In order to $\overline{u}$ be minimal point of the functional $J(u)$ on the space $A^n([0,T]\times[0,S])$ it is sufficient and if for $\widetilde{u}=\overline{u}$ the conditions of theorem 1 are fulfilled it is necessary that there we'll be find the functions for*

$$\overline{p}_1(\cdot)\in L_\infty^n([0,T]\times[0,S]),\ \overline{p}_2(\cdot)\in L_\infty^n([0,T]\times[0,S]),\ v(\cdot)\in A^n([0,T]\times[0,S]),$$

where $v(T,s)=v(t,S)$ *for* $t\in[0,T]$, $s\in[0,S]$, $v_1(\cdot)\in W_{1,1}^n[0,T]$, $v_2(\cdot)\in W_{1,1}^n[0,S]$ *such, that*

1) $$v_{ts}(t,s)\in\partial f^0\left(t,s,\overline{u}(t,s),\overline{p}_1(t,s),\overline{p}_2(t,s),v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}_1(t,v)dv+\right.$$
$$\left.+\int_0^t\overline{p}_2(\tau,s)d\tau-\int_0^T\overline{p}_2(\tau,s)d\tau\right),$$

2) $$v_t(t,0)-\dot{v}_1(t)\in\partial\varphi_1^0\left(t,\overline{u}(t,0),v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right),$$

3) $$v_s(0,s)-\dot{v}_2(s)\in\partial\varphi_2^0\left(s,\overline{u}(0,s),v_2(s)-\int_0^S\overline{p}_2(\tau,s)d\tau\right),$$

4) $$(v(0,0)-v_1(0)-v_2(0),v_1(T)-v(T,0),v_2(S)-v(0,S),v(T,S))\in$$
$$\in\partial q(\overline{u}(0,0),\overline{u}(T,0),\overline{u}(0,S),\overline{u}(T,S)),$$

5) $$f^0\left(t,s,\overline{u}(t,s),\overline{p}_1(t,s),\overline{p}_2(t,s),v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\right.$$
$$\left.-\int_0^T\overline{p}_2(\tau,s)d\tau\right)=(\overline{u}_t(t,s)|\overline{p}_1(t,s))+(\overline{u}_s(t,s)|\overline{p}_2(t,s))+\left(\overline{u}_{ts}(t,s)\middle|v(t,s)+\int_0^s\overline{p}_1(t,v)dv-\right.$$
$$\left.-\int_0^S\overline{p}_1(t,v)dv+\int_0^t\overline{p}_2(\tau,s)d\tau-\int_0^T\overline{p}_2(\tau,s)d\tau\right)+f(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s)),$$

6) $$\varphi_1^0\left(t,\overline{u}(t,0),v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)=\left(\overline{u}_t(t,0)\middle|v_1(t)-\int_0^S\overline{p}_1(t,v)dv\right)+\varphi_1(t,\overline{u}(t,0),\overline{u}_t(t,0)),$$

7) $$\varphi_2^0\left(s,\overline{u}(0,s),v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)=\left(\overline{u}_s(0,s)\middle|v_2(s)-\int_0^T\overline{p}_2(\tau,s)d\tau\right)+\varphi_2(s,\overline{u}(0,s),\overline{u}_s(0,s)).$$

Note 1. If in theorem 1 (or theorem 2) the function $f(t,s,u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))$ does not depended on $(u_t(t,s),u_s(t,s))$, then $\overline{p}_1(t,s)=p_2(t,s)=0$.

Note 2. If $M>0$ such that for $z(\cdot)\in A_0^n([0,T]\times[0,S])$, $\|z(\cdot)\|_{A_0^n}\le\alpha$ there exists the function $u_2(\cdot)\in A^n([0,T]\times[0,S])$, such that $\Phi(u_z,z)\le M$ then theorem 1 *is* also correct.

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