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ON CONDITIONS OF CORRECT SOLVABILITY OF THE BOUNDARY-VALUE PROBLEM FOR ONE CLASS OF THE SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS ON FINITE SEGMENT

Abstract

In this paper the sufficient conditions providing correct solvability of boundary-value problem for one class of the second order operator-differential equation have been found in Hilbert space on finite segment.

In the present paper correct and one-valued solvability of one class of boundary-value problem for the second order operator differential equation of elliptic type on finite segment in a separable Hilbert space H is investigated.

The following boundary-value problem

$$-\left(\frac{d}{dt}-\omega_1 A\right)\left(\frac{d}{dt}-\omega_2 A\right)u(t)+\sum_{j=0}^{1}A_{2-j}u^{(j)}(t)=f(t),\ t\in(0,1), \qquad (1)$$

$$u(0)=0,\quad u(1)=0 \qquad (2)$$

is considered, where $f(t)$ and $u(t)$ are vector valued functions with values from H, and derivatives are understood in the sense of the theory of generalized functions [1]. The fulfilling of the following conditions:

1) ω_1 and ω_2 are real numbers such that $\omega_2>0, \omega_1<0$;
2) A is the positive defined self-adjoint operator in H;
3) the operator $B_j=A_jA^{-j}$ $(j=1,2)$ are bounded in H

are assumed with respect to the numbers ω_1, ω_2 and the linear operators A, A_1 and A_2.

As known the self-adjoint positive operator A generates scalar the scale of Hilbert spaces $H_\gamma=D(A^\gamma)(\gamma\ge 0)$ relatively to the norms $(x,y)=(A^\gamma x, A^\gamma y)$, $x,y\in D(A^\gamma)$. The Hilbert space of the measurable and square integrable by Bohner vector functions $f(t)$ with values from H_γ we'll denote by $L_2((0;1:H_\gamma))$. The scale product in $L_2((0;1);H_\gamma)$ is determined by the following way $(f,g)_{L_2((0;1)H_\gamma)}=\int_0^1 (f(t),g(t))_\gamma dt$.

Further determine the Hilbert spaces (see [1]):

$$W_2^2((0;1):H)=\{u:u\in L_2((0;1):H_2),\ u''\in L_2((0;1):H)\},$$

$$\mathring{W}_2^2((0;1):H)=\{u:u\in W_2^2((0;1):H),\ u(0)=u(1)=0\}$$

with scalar product

$$(u,\vartheta)_{W_2^2((0;1)H)}=(u'',\vartheta'')_{L_2((0;1)H)}+(u,\vartheta)_{L_2((0;1)H_2)}.$$

Definition 1. *If for any $f(t)\in L_2((0;1):H)$ there exists the vector function $u(t)\in W_2^2((0;1):H)$, which satisfies the equation (1) almost everywhere for $t\in(0:1)$ and the boundary conditions in the sense of $\lim_{t\to+0}\|u(t)\|_{3/2}=0$; $\lim_{t\to-1}\|u(t)\|_{3/2}=0$, then we'll call it as regular solution of the boundary-value problem (1), (2).*

The problem (1), (2) is called regular solvable, if the regular solution $u(t)$ of the problem (1), (2) satisfy the inequality

$$\|u\|_{W_2^2} \le const\|f\|_{L_2} .$$

Note that, for $\omega_2 = -\omega_1 = 1$ and when A is the normal operator with the spectrum in some corner sector the regular solvability of the problem (1), (2) has been investigated in paper [2]. In the case , when $\omega_2 = -\omega_1 = 1$ and A is the self- adjoint positive operator the correct solvability of the problem with additional hard conditions on A_1 and A_2 (i.e., when the norms of the operators B_1 and B_2 are sufficiently small numbers), has been investigated in S.S.Yakubov's and his disciples articles (see [3]). In this case when the operator A is strongly positive, and the operators A_1 and A_2 are relatively quite continuous, and A^{-1} is quite continuous the Fredholm property of the problem has been investigated in book [3] and so on.

In the present paper at first using the methods of paper [2] we'll estimate the norms of the intermediate operators in the space of solutions of the equation (1) when the operators $A_1 = A_2 = 0$ and then using these estimations we'll find the conditions of regular solvability of the problem (1), (2). Note that, the estimations of norms of the intermediate derivatives on the finite segment is an independent interesting mathematical problem.

Let's introduce some operators for the investigation of the problem (1), (2). Determine the operators

$$P_0 u = P_0(d/dt)\,u = -(d/dt - \omega_1 A)(d/dt - \omega_2 A)u\,, \quad u \in \overset{\circ}{W}{}_2^2((0;1):H),$$

$$P_1 u = \sum_{j=0}^{1} A_{2-j} u^{(j)}\,, \quad u \in \overset{\circ}{W}{}_2^2((0;1):H),$$

$$Pu = P_0 u + P_1 u\,, \quad u \in \overset{0}{W}{}_2^2((0;1):H)$$

in the space $\overset{\circ}{W}{}_2^2((0;1):H)$.

Now, prove some lemmas, which we'll use further.

Lemma 1. *The operator* P_0 *isomorphically maps the space* $\overset{\circ}{W}{}_2^2((0;1):H)$ *onto* $L_2((0;1):H)$.

Proof. The equation $P_0 u = 0$ has only zero solution from the space $\overset{\circ}{W}{}_2^2((0;1):H)$. Indeed the general solution of the equation $P_0(d/dt)u = 0$ from the space $W_2^2((0;1):H)$ has the form (see [3])

$$u_0(t) = e^{\omega_1 t A} g_0 + e^{\omega_2 (t-1) A} g_1\,, \qquad (3)$$

where $g_0, g_1 \in H_{3/2}$. From the condition $u(0) = 0$, $u(1) = 0$ it follows that

$$\begin{cases} g_0 + e^{-\omega_2 A} g_1 = 0, \\ e^{\omega_1 A} g_0 + g_1 = 0. \end{cases}$$

Hence we find that $g_0 = g_1 = 0$, i.e. $u_0(t) = 0$. Now let's show that the equation $P_0 u = f$ has the regular solution for any $f \in L_2((0;1):H)$.

It is easy to see that

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_0^{-}(i\xi)\left(\int_0^1 f(s) e^{-i\xi s} ds\right) e^{i\xi t} d\xi$$

almost everywhere satisfies the equation $P_0u=f$, $u_1(t)\in W_2^2(R:H)$, $(R=(-\infty,\infty))$.

Indeed, by Plancsharel theorem it suffices to prove that

$$\left\|\xi^2\hat{u}_1(\xi)\right\|^2_{L_2(R:H)}+\left\|A^2\hat{u}_1(\xi)\right\|^2_{L_2(R:H)}\le const\left\|\hat{f}(\xi)\right\|^2_{L_2(R:H)},$$

where $\hat{u}_1(\xi)$, $\hat{f}(\xi)$ are Fourier transformations of the vector-functions $u_1(t)$ andft, correspondingly.

Since

$$\left\|\xi^2\hat{u}_1(\xi)\right\|_{L_2(R:H)}=\left\|\xi^2P_0^{-1}(-i\xi)\right\|_{L_2(R:H)}\le\sup_{\xi\in R}\left\|\xi^2P_0^{-1}(-i\xi)\right\|_{H\to H}\cdot\left\|\hat{f}(\xi)\right\|_{L_2(R:H)},$$

then from the estimation

$$\left\|\xi^2P_0^{-1}(i\xi)\right\|\le\sup_{\mu\in\delta(A)}\left|\xi^2(i\xi-\omega_1\mu)^{-1}(-i\xi-\omega_{2\mu})^{-1}\right|\le 1$$

it follows that

$$\left\|\xi^2\hat{u}(\xi)\right\|_{L_2(R:H)}\le\left\|\hat{f}(\xi)\right\|_{L_2(R:H)}.$$

Analogously we obtain:

$$\left\|A^2\hat{u}(\xi)\right\|_{L_2(R:H)}=\left\|A^2P_0^{-1}(-i\xi)\hat{f}(\xi)\right\|_{L_2(R_+:H)}\le\sup_{\xi\in R}\left\|A^2P_0^{-1}(-i\xi)\right\|_{H\to H}\cdot\left\|\hat{f}(\xi)\right\|_{L_2(R:H)}\le$$

$$\le\frac{1}{|\omega_1\omega_2|}\left\|\hat{f}\right\|_{L_2(R_-:H)}.$$

Consequently $u_1(t)\in W_2^2(R:H)$.

Let's denote by $\omega(t)$ contraction of the vector-function $u_1(t)$ on $[0,1]$. Then from the theorem on traces follows that $\omega(0)\in H_{3/2}$ $\omega(1)\in H_{3/2}$. Now we'll search solution of the equation $P_0u=f$ in the form

$$u(t)=\omega(t)+u_0(t),$$

where $u_0(t)$ is determined by the equality (3) and the vectors g_0 and g_1 we'll determine from the condition $u(0)=0$, $u(1)=0$. Then for the determination of g_0 and g_1 we obtain the following system

$$\begin{cases}g_0+e^{-\omega_2A}g_1=\omega(0)\\ e^{\omega_1A}g_0+g_1=-\omega(1)\end{cases}$$

From this system we find

$$g_0=\left(E-e^{(\omega_1-\omega_2)A}\right)^{-1}\left(e^{\omega_1A}\omega(1)-\omega(0)\right),$$

$$g_1=\left(E-e^{(\omega_1-\omega_2)A}\right)^{-1}\left(e^{\omega_1A}\omega(0)-\omega(1)\right).$$

Obviously that when A is a self-adjoint operator the operator $E-e^{(\omega_1-\omega_2)A}$ is reversible to H, hence follows that the vectors g_0 and $g_1\in H_{3/2}$, consequently $u(t)\in \overset{\circ}{W}{}_2^2((0;1):H)$.

On the other hand by the theorem on intermediate derivatives it follows that

$$\left\|P_0u\right\|^2_{L_2}\le 2\left(\left\|u''\right\|^2_{L_2}+|\omega_1+\omega_2|^2\left\|Au'\right\|^2_{L_2}+\omega_1^2\omega_2^2\left\|A^2u\right\|^2_{L_2}\right)\le const\left\|u\right\|_{W_2^2},$$

i.e. the operator $P_0 : \overset{\circ}{W}{}_2^2((0;1):H) \to L_2((0:1):H)$ is bounded. Consequtntly, from the Banach theorem on reversible operator it follows that $P_0^{-1} : L_2((0;1):H) \to \overset{\circ}{W}{}_2^2((0;1):H)$ exists and is bounded. Lemma has been proved.

Lemma 2. *Let the conditions 1)-3) be fulfilled. Then the operator P is a continuous operator.*

Proof. Since $P = P_0 + P_1$, and the operator P_0 is a bounded operator (lemma 1) then it is sufiicient to prove, that the operator P_1 is bounded. The boundedness of the operator P_1 follows from the theorem on intermediate derivatives

$$\|P_1 u\| L_2 \le \sum_{j=0}^{1} \left\|A_{2-j} u^{(j)}\right\|_{L_2} \le \sum_{j=0}^{1} \|B_{2-j}\| \left\|A^{2-j} u^{(j)}\right\|_{L_2} \le$$

$$\le \sum_{j=0}^{1} \|B_{2-j}\| C_j \|u\|_{W_2^2} = const \|u\|_{W_2^2} .$$

Lemma 3. *For all $u \in \overset{\circ}{W}{}_2^2((0;1):H)$ the following inequalities*

$$\left\|A^2 u\right\|_{L_2} \le \frac{1}{|\omega_1 \omega_2|} \|P_0 u\|_{L_2} , \tag{4}$$

$$\left\|A \frac{du}{dt}\right\|_{L_2} \le \frac{1}{\sqrt{|\omega_1 \omega_2|}} \|P_0 u\|_{L_2} \tag{5}$$

holds.

Proof. To prove the inequality (4) we scalarly multiply the equation $P_0 u = f$ in the space $L_2((0:1):H)$ by $A^2 u \in L_2((0:1):H)$ and we find the real part of the obtained expression

$$\mathrm{Re}\left(P_0 u, A^2 u\right)_{L_2} = \mathrm{Re}\left(f, A^2 u\right)_{L_2} = \mathrm{Re}\left(-u'' + ({}_1 + \omega_2) A u' + |\omega_1 \omega_2| A^2 u, A^2 u\right)_{L_2} =$$

$$= -\mathrm{Re}\left(u'', A^2 u\right)_{L_2} + (\omega_1 + \omega_2) \mathrm{Re}\left(A u', A^2 u\right)_{L_2} + |\omega_1 \omega_2| \left\|A^2 u\right\|_{L_2}^2 . \tag{6}$$

Later allowing for $u(t) \in \overset{\circ}{W}{}_2^2((0;1):H)\,(u(0) = u(1) = 0)$ after integrating by parts we obtain:

$$-\mathrm{Re}\left(u'', A^2 u\right)_{L_2} = \|A u'\|_{L_2}^2 ,\ \mathrm{Re}\left(A u', A^2 u\right)_{L_2} = 0 .$$

Allowing for this equalities in (6) we find

$$\mathrm{Re}\left(P_0 u, {}^2 u\right)_{L_2} = \|A u'\|_{L_2}^2 + |\omega_1 \omega_2| \left\|A^2 u\right\|_{L_2}^2 . \tag{7}$$

Hence we obtain

$$|\omega_1 \omega_2| \left\|A^2 u\right\|_{L_2}^2 \le \mathrm{Re}\left(P_0 u, A^2 u\right)_{L_2} \le \|P_0 u\|_{L_2} \left\|A^2 u\right\|_{L_2} ,$$

i.e.

$$\left\|A^2 u\right\|_{L_2} \le \frac{1}{|\omega_1 \omega_2|} \|P_0 u\|_{L_2} . \tag{8}$$

Later after application of Young inequality, from the equality (7) we obtain:

$$\|A' u\|_{L_2}^2 + |\omega_1 \omega_2| \left\|A^2 u\right\|_{L_2}^2 \le \|P_0 u\|_{L_2} \left\|A^2 u\right\|_{L_2}^2 \le |\omega_1 \omega_2| \left\|A^2 u\right\|_{L_2}^2 + \frac{1}{4|\omega_1 \omega_2|} \|P_0 u\|_{L_2}^2$$

consequently

$$\|A'u\|_{L_2}^2 \le \frac{1}{2\sqrt{|\omega_1\omega_2|}}\|P_0u\|_{L_2}. \tag{9}$$

Lemma has been proved.

Now let's prove the main result of the paper.

Theorem. *Let the conditions 1)-3) be fulfilled and the inequality*

$$\alpha = \sum_{j=0}^{1} c_j \|B_{2-j}\| < 1 \tag{10}$$

holds, where

$$c_0 = \frac{1}{|\omega_1\omega_2|},\ c_1 = \frac{1}{2\sqrt{|\omega_1\omega_2|}}. \tag{11}$$

Then the problem (1), (2) is regular solvable.

Proof. Let's write the problem (1), (2) in the form of the equation

$$P_0u + P_1u = f,$$

where $u \in \overset{\circ}{W}{}_2^2((0:1):H)$, $f \in L_2((0:1):H)$. By theorem 1 there exists the bounded inverse operator $P_0^{-1}: L_2((0:1):H) \to \overset{\circ}{W}{}_2^2((0:1):H)$. After the substitution $P_0u = \vartheta$ we obtain the equation

$$\vartheta + P_1P_0^{-1}\vartheta = f$$

in the space $L_2((0:1):H)$. Since for any $\vartheta \in L_2((0:1):H)$

$$\|P_1P_0^{-1}\vartheta\|_{L_2} = \|P_1u\| \le \sum_{j=0}^{1}\|A_{2-j}u^{(j)}\|_{L_2} \le \sum_{j=0}^{1}\|B_{2-j}\|\cdot\|A^{2-j}u^{(j)}\|_{L_2},$$

then allowing for the inequality (4), (5) from lemma 3 in the latter inequality we obtain

$$\|P_1P_0^{-1}\vartheta\|_{L_2} \le \left(\sum_{j=0}^{2}c_j\|B_{2-j}\|\right)\|P_0u\|_{L_2} = \alpha\|P_0u\|_{L_2} = \alpha\|\vartheta\|_{L_2},$$

i.e. the operator $P_1P_0^{-1}$ has the norm less than $\alpha < 1$. Then we can find u:

$$u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1}f.$$

Hence the correctness of the inequality

$$\|u\|_{W_2^2} \le const\|f\|_{L_2}$$

follows. Theorem has been proved.

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