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ON THE RATE OF EQUICONVERGENCE OF ORTHOGONAL EXPANSION IN EIGENFUNCTIONS OF STURM-LIOUVILLE OPERATOR

Abstract

At the paper Sturm-Liouville operator with summable on the interval $(0,1)$ coefficient is considered. The rate of equiconvergence of orthogonal expansion of absolutely continuous function by the system of eigenfunctions of Sturm-Liouville operator with the expansion in Fourier trigonometric series of the same function is studied. The estimation for the rate of equiconvergence on the compact of the interval $(0,1)$ expressed by the continuity modulus of operator's coefficient is obtained.

Let's consider on the interval $G=(0,1)$ the Sturm-Liouville operator

$$Lu=-u''+q(x)u \tag{1}$$

with the real coefficient $q(x)\in L_p(G),\ p\geq 1$. Let's suppose that the operator (1) allows the existence of full orthonormalized in $L_2(G)$ system of eigenfunctions $\{u_n(x)\}_{n=1}^{\infty}$ and the system of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n\geq 0,\ \lambda_n\to\infty,\ n\to\infty\ (Lu_n=\lambda_n u_n)$.

Let's introduce a partial sum of orthogonal expansion of the function $f(x)\in L_1(G)$ in eigenfunctions $\{u_n(x)\}_{n=1}^{\infty}$

$$\sigma_\nu(x,f)=\sum_{\mu_n\leq\nu}(f,u_n)u_n(x),\quad \nu>0,\quad \mu_n=\sqrt{\lambda_n}$$

and a partial sum $S_\nu(x,f)$ of Fourier trigonometrically series of the same function $f(x)$. Let's denote

$$R_\nu(x,f)=\sigma_\nu(x,f)-S_\nu(x,f).$$

It is known that the rate of equiconvergence of the partial sums $\sigma_\nu(x,f)$ and $S_\nu(x,f)$ at first were studied by V.A. Il'in and I.Joo [1] in the case $q(x)\in L_p(G),\ p>1$ for the absolutly continuous function $f(x)$ and a uniform on any compact $K\subset G$ exact by order estimate

$$|R_\nu(x,f)|=O(\nu^{-1}),\quad \nu\to\infty$$

was established.

In further the rate of equiconvergence of partial sums $\sigma_\nu(x,t)$ and $S_\nu(x,f)$ for the functions $f(x)$, from the class $H_1^{\omega}(G)$ and $H_r^{\alpha}(G)$, $r>1,\ \alpha\in(0,1]$ (Nikolskiy class) for this operator $(p>1)$ were studied in Sh.A. Alimov, I. Joo [2] and N. Lozhetich [3]. For the differential operators of arbitrary order the questions on rate on equiconvergence were investigated in papers [4-5]. In these papers we can find extensive lists of references on the given theme.

At the given paper the dependence of rate of equiconvergence from the modulus of the continuity of the coefficient $q(x)\in L_1(G)$ is studied.

Let's denote

$$\psi(\nu)=\nu^{-1}\inf_{m\geq 2}\left\{\omega_1\left(q,m^{-1}\right)\ln\nu+m\right\},\quad \nu\geq 2,$$

where $\omega_1(\cdot,\delta)$ is a modulus of continuity in $L_1(G)$.

Theorem. *Let $q(x)\in L_1(G)$ and the function $f(x)$ be absolutely continuous on the segment $[0,1]$. Then the orthogonal expansion of the function $f(x)$ by the system of eigenfunctions $\{u_n(x)\}$ of the operator (1) uniformly equiconvergens with its trigonometrical expansion on any compact $K\subset G$ and estimation*

$$\sup_{x\in K}\left|R_\nu(x,f)\right|\leq C\left(K,\|q\|_1\right)\psi(\nu)\|f\|_{W_1^1(G)} \tag{2}$$

is true, where $C\left(K,\|q\|_1\right)$ is a constant depending on the compact K and L_1 are the norms of the coefficient $q(x)$.

Corollary. *At $\nu\to\infty$ the estimation*

$$\sup_{x\in K}\left|R_\nu(x,f)\right|=o\left(\nu^{-1}\ln\nu\right)$$

is fulfilled.

Give some lemmas to prove the theorem.

Let's denote

$$\gamma(r,R,\mu_k,\nu)=\int_r^R\frac{\sin\nu t}{t}\sin\mu_k(t-r)dt\,,\qquad 0<r<R<\infty,\quad \mu_k>0\,.$$

Lemma 1. *For the quantities $\gamma(r,R,\mu_k,\nu)$* the estimations

$$\gamma=\begin{cases}O\left(\min\left\{\mu_k\nu^{-1},\nu\mu_k^{-1}\right\}\right) & when\quad |\mu_k-\nu|\geq\nu/2\,,\\ O\left(\ln\left(\nu|\mu_k-\nu|^{-1}\right)\right) & when\quad 1\leq|\mu_k-\nu|\leq\nu/2,\\ O(\ln\nu) & when\quad |\mu_k-\nu|\leq 1\end{cases} \tag{3}$$

are true.

Proof. The estimation (3) follows from paper [6]. But we'll consider a shorter proof of these estimations.

Let's represent γ in the form

$$\gamma=\cos\mu_k r\int_r^R\frac{\sin\nu t}{t}\sin\mu_k t dt-\sin\mu_k r\int_r^R\frac{\sin\nu t}{t}\cos\mu_k t dt \tag{4}$$

The first integral at the right part (4) we'll represent in the form

$$\int_r^R\frac{\sin\nu t}{t}\sin\mu_k t dt=\frac{1}{2}\left[\int_{r|\nu-\mu_k|}^{R|\nu-\mu_k|}\frac{\cos t}{t}dt-\int_{r(\nu+\mu_k)}^{R(\nu+\mu_k)}\frac{\cos t}{t}dt\right]=$$

$$=\frac{1}{2}\left[\int_{r|\nu-\mu_k|}^{r(\mu_k+\nu)}\frac{\cos t}{t}dt-\int_{R|\nu-\mu_k|}^{R|\nu-\mu_k|}\frac{\cos t}{t}dt\right].$$

Hence it follows that

$$\int_r^R\frac{\sin\nu t}{t}\sin\mu_k t dt=O\left(\ln\frac{\nu+\mu_k}{|\nu-\mu_k|}\right). \tag{5}$$

When $0<\mu_k\leq\nu/2$ from (5) we'll get $\left(\ln(1+x)=O(x),\ x>0\right)$

$$\int_r^R \frac{\sin \nu t}{t} \sin \mu_k t dt = O\left(\frac{\mu_k}{\nu - \mu_k}\right) = O\left(\frac{\mu_k}{\nu}\right). \tag{6}$$

The second integral at the right part (4) we'll represent in the form

$$\int_r^R \frac{\sin \nu t}{t} \cos_k t dt = -\operatorname{sgn}(\nu - \mu_k) si(|\nu - \mu_k| r) + \frac{1}{2}\operatorname{sgn}(\nu - \mu_k) si(|\nu - \mu_k| R) +$$

$$+\frac{1}{2} si((\nu + \mu_k)R) - \frac{1}{2} si((\nu + \mu_k)r), \tag{7}$$

where $si\, x = \int_x^\infty \frac{\sin t}{t} dt$.

Allowing for the inequality $|si\, x| \le const x^{-1}$, $x > 0$ from the relation (7) we'll get

$$\int_r^R \frac{\sin \nu t}{t} \cos \mu_k t dt = O\left(\frac{1}{|\mu_k - \nu| r}\right).$$

Consequently

$$\sin \mu_k r \int_r^R \frac{\sin \nu t}{t} \cos \mu_k t dt = O\left(\frac{\mu_k}{|\nu - \mu_k|}\right). \tag{8}$$

By virtue of the estimations (6) and (8) from (4) it follows the estimation

$$\gamma = O\left(\mu_k \nu^{-1}\right) \text{ when } 0 < \mu_k \le \nu/2.$$

By virtue of the relation $si\, x = O(1)$, $x > 0$ from (7) it follows

$$\int_r^R \frac{\sin \nu t}{t} \cos \mu_k t dt = O(1).$$

Since for $1 \le |\mu_k - \nu| \le \nu/2$ the inequality $\ln((\nu + \mu_k)/|\mu_k - \nu|) \ge \ln 3$ is fulfilled then from the last relation we'll get

$$\int_r^R \frac{\sin \nu t}{t} \cos \mu k t dt = O\left(\ln \frac{\nu + \mu_k}{|\mu_k - \nu|}\right) = O\left(\frac{\nu}{|\mu_k - \nu|}\right), \tag{9}$$

when $1 \le |\mu_k - \nu| \le \nu/2$.

Consequently by virtue of (5) and (9) the second part of the estimation (3) follows from (4).

When $\mu_k \ge 3\nu/2$ from the representation

$$\int_r^R \frac{\sin \nu t}{t} \cos \mu_k t dt = \frac{1}{2}\left[\int_{R(\mu_k - \nu)}^{R(m_k + \nu)} \frac{\sin t}{t} - \int_{r(\mu_k - \nu)}^{r(\mu_k + \nu)} \frac{\sin t}{t} dt\right]$$

we'll get that

$$\int_r^R \frac{\sin \nu t}{t} \cos \mu t dt = O\left(\ln \frac{\nu + \mu_k}{\mu_k - \nu}\right).$$

Hence and from the (4), (5) it follows the estimation

$$\gamma = O\left(\ln \frac{\mu_k + \nu}{\mu_k - \nu}\right) = O\left(\nu \mu_k^{-1}\right) \text{ when } \mu_k \ge 3\nu/2.$$

Consequently the first part of the estimation (3) is established (subject to (6)). But the third part of the estimation (3) follows from the inequality $|\sin x| \le x, x \ge 0$ and the representations

$$\gamma = \int_r^{1/\nu} \frac{\sin \nu t}{t} \sin \mu_k (t-r) dt + \int_{1/\nu}^{R} \frac{\sin \nu t}{t} \sin \mu_k (t-r) dt .$$

Lemma 1 is proved.

Let's denote that in case when $\mu_k = 0$ instead of the integral γ it should be considered the integral

$$\int_r^R \frac{\sin \nu t}{t} (t-r) dt .$$

This integral has the order $O(\nu^{-1})$.

Let's suppose

$$\Theta(x,y,z) = \sum_{\mu_n \le \nu} u_n(x) u_n(y),$$

$$D(x-y,\nu) = \frac{\sin((2[\nu]+1)\pi(x-y))}{\sin \pi(x-y)} \text{ (Dirichle kernel).}$$

Lemma 2. *If* $q(x) \in L_1(G)$ *then the estimation*

$$\left| \int_{x_1}^{x_2} [\Theta(x,y,\nu) - D(x-y,\nu)] dy \right| \le C(k, \|q\|_1) \psi(\nu), \qquad (10)$$

$$x_1, x_2 \in \overline{G}, \quad x_1 \le x_2$$

is true uniformly with respect to x *on any compact* $K \subset G$.

Proof. We'll fix any compact $K \subset G$ and any number R_0 for which $0 < 2R_0 < \rho(K, \partial G)$. For any number $R \in [R_0, 2R_0]$ let's introduce the function

$$\vartheta(x,y,\nu,R) = \begin{cases} \dfrac{1}{\pi} \dfrac{\sin \nu (x-y)}{x-y} & \text{when } |x-y| < R \\ 0 & \text{when } |x-y| \ge R \end{cases},$$

where $x \in K,\ y \in G$.

By $\hat{\vartheta}(x,y,\nu,R_0)$ we'll denote the averaging of the function

$$\hat{\vartheta}(x,y,\nu,R_0) = \frac{2}{R_0} \int_{R_0/2}^{R_0} \vartheta dR = S_{R_0}[\vartheta].$$

The Fourier's coefficients of the function $\hat{\vartheta}(x,y,\nu,R_0)$ by the system $\{u_k(y)\}$ has the form [1,6]

$$\hat{\vartheta}_k(x,\nu,R_0) = u_k(x) \delta^{\nu}_{\mu_k} + I(\mu_k,\nu,R_0) u_k(x) + \frac{2}{\pi} S_{R_0} \left[\frac{1}{\mu_k} \int_0^R \gamma(r,R,\mu_k,\nu) \times \right.$$
$$\left. \times [q(x+r) u_k(x+r) + q(x-r) u_k(x-r)] dr \right],$$

where

$$\delta_{\mu_k}^{\nu}=\begin{cases}1 & \text{when } \mu_k<\nu, \\ \frac{1}{2} & \text{when } \mu_k=\nu, \\ 0 & \text{when } \mu_k>\nu\end{cases} \quad I(\mu_k,\nu,R_0)=O\left(\frac{1}{1+|\mu_k-\nu|^2}\right). \tag{11}$$

From this equality follows in $L_2(G)$ by y the equality

$$\hat{\vartheta}(x,y,\nu,R_0)-\sum_{\mu_k\le\nu}u_k(x)u_k(y)=-\frac{1}{2}\sum_{\mu_k=\nu}u_k(x)u_k(y)+$$

$$+\sum_{k=1}^{\infty}I(\mu_k,\nu,R_0)u_k(x)u_k(y)+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{\mu_k}S_{R_0}\left[\int_0^R\gamma(r,R,\mu_k,\nu)\times\right.$$

$$\left.\times[q(x+r)u_k(x+r)+q(x-r)u_k(x-r)]\vphantom{\int_0^R}\right]u_k(y). \tag{12}$$

For the establishing the equality (12) it is enough to take into account the known facts (see [1])

$$|u_k(x)|\le C,\quad k=1,2,\ldots,\qquad \sum_{\tau\le\mu_k\le\tau+1}1\quad\le const,\quad \forall\tau\ge 0 \tag{13}$$

and let's apply the estimations (3) and (11).

Integrating every side of the equality (12) by y from the x_1 to the x_2 and using the rule of termwise integration of convergence in $L_2(G)$ series we'll get:

$$\int_{x_1}^{x_2}\hat{\vartheta}(x,y,\nu,R_0)dy-\int_{x_1}^{x_2}\Theta(x,y,\nu)dy=-\frac{1}{2}\sum_{\mu_k=\nu}u_k(x)\int_{x_1}^{x_2}u_k(y)dy+$$

$$+\sum_{k=1}^{\infty}I(\mu_k,\nu,R_0)u_k(x)\int_{x_1}^{x_2}u_k(y)dy+\frac{2}{p}\sum_{k=1}^{\infty}\frac{1}{\mu_k}S_{R_0}\left[\int_0^R\gamma(r,R,\mu_k,\nu)\times\right.$$

$$\left.\times[q(x+r)u_k(x+r)+q(x-r)u_k(x-r)]dr\vphantom{\int_0^R}\right]\cdot\int_{x1}^{x_2}u_k(y)dy=S_1+S_2+S_3.$$

From the formula of the value [7]

$$\frac{u_k(y-t)+u_k(y+t)}{2}=u_k(y)\cos\mu_k t+\frac{1}{2\mu_k}\int_{y-r}^{y+r}q(\xi)u_k(\xi)\sin\mu_k(|\xi-y|-t)d\xi$$

and from the estimation (13) it follows that

$$\left|\int_{x_1}^{x_2}u_k(y)dy\right|\le\frac{C}{\mu_k},\quad 0\le x_1\le x_2\le 1,\quad \mu_k\ne 0. \tag{14}$$

Using the estimations (13) and (14) for the sum S_1 we get the following estimation

$$|S_1|\le\frac{1}{2}\sum_{\mu_k=\nu}|u_k(x)|\left|\int_{x_1}^{x_2}u_k(y)\right|\le C\sum_{\mu_k=\nu}\mu_k^{-1}=C\nu^{-1}.$$

Subject to the estimations (11), (13) and (14) the sum S_2 also doesn't exceed the value $C\cdot\nu^{-1}$.

Let's estimate the sum S_3. For this let's represent it in the form

$$S_3 = \sum_{\mu_k \le 1} + \sum_{1 \le \mu_k \le \nu/2} + \sum_{1<|\mu_k - \nu|<\nu/2} + \sum_{|\mu_k - \nu| \le 1} + \sum_{\mu_k \ge 3\nu/2} = \sum_{i=1}^{5} S_3^i .$$

Applying lemma 1, the estimations (13) and (14) for the sum S_3^i $i \ne 2$ we'll get

$$\left|S_3^1\right| \le C\|q\|_1 \nu^{-1} \sum_{\mu_k \le 1} 1 \le C\|q\|_1 \nu^{-1};$$

$$\left|S_3^3\right| \le C\|q\|_1 \sum_{1\le|\mu_k - \nu| \le \nu/2} \mu_k^{-2} \ln\left(\frac{\nu}{|\mu_k - \nu|}\right) \le C\|q\|_1 \nu^{-2} \sum_{n=1}^{[\nu/2]} \ln\frac{\nu}{n} \le C\|q\|_1 \nu^{-1};$$

$$\left|S_3^4\right| \le C\|q\|_1 \sum_{|\mu_k - \nu| \le 1} \mu_k^{-2} \ln\nu \le C\|q\|_1 \nu^{-2} \ln\nu \le C\|q\|_1 \nu^{-1};$$

$$\left|S_3^5\right| \le C\nu\|q\|_1 \sum_{\mu_k \ge 3\nu/2} \mu_k^{-2} \le C\|q\|_1 \nu^{-1}.$$

For the estimation S_3^2 let's represent it in the form

$$S_3^2 = \frac{2}{\pi} \sum_{1<\mu_k<\nu/2} \mu_k^{-1} \left\{ S_{R_0}\left[\int_{x-R}^{x+R} \{q(\xi) - T_m(\xi)\} u_k(\xi)\gamma(|\xi - x|, R, \mu_k, \nu) d\xi \right] + \right.$$

$$\left. + S_{R_0}\left[\int_{x-R}^{x+R} T_m(\xi) u_k(\xi)\gamma(|\xi - x|, R, \mu_k, \nu) d\xi \right] \right\} \int_{x_1}^{x_2} u_k(y)dy = S_3^2(q - T_m) + S_3^2(T_m),$$

where $T_m(x)$ is a trigonometrical polynomial of the best approximation of the function $q(x)$ in the metric $L_1(G)$ of order m.

Using lemma 1, the estimations (13) and (14) we find

$$\left|S_3^2(q - T_m)\right| \le C|q - T_m|_1 \nu^{-1} \sum_{1 \le \mu_k \le \nu/2} \mu_k^{-1} \le C\|q - T_m\|_1 \nu^{-1} \ln\nu . \qquad (15)$$

Applying the estimations (13) and (14) and (see [4])

$$|\gamma(|x-\xi|, R, \mu_k, \nu)| \le C(R_0) \frac{\mu_k^{1-\varepsilon}}{\nu|x-\xi|^{\varepsilon}}, \quad 0<\varepsilon \le 1, \ 1 \le \mu_k \le \nu/2, \ R_0/2 \le R \le R_0$$

we'll get $\left(0 < \varepsilon < \dfrac{1}{2}\right)$

$$\left|S_3^2(T_m)\right| \le C\|T_m\|_2 \nu^{-1} \sum_{1 \le \mu_k \le \nu/2} \mu_k^{-(1+\varepsilon)} \le C\nu^{-1}\|T_m\|_2 . \qquad (16)$$

Taking into account the known inequalities

$$\|q - T_m\|_1 \le const\,\omega_1(q, m^{-1}), \text{ (see [8])}$$

$$\|T_m\|_2 \le const\, m\|T_m\|_1 \text{ (see [9])}$$

from (15) and (16) we'll get

$$\left|S_3^2(q - T_m)\right| \le C\nu^{-1} \ln\nu\, \omega_1(q, m^{-1}), \quad m \ge 2,$$

$$\left|S_3^2(T_m)\right| \le C\nu^{-1} m\|T_m\|_1 \le C\nu^{-1} m\|q\|_1 .$$

Consequently,

$$\left|S_3^2\right| \le C\left(k,\|q\|_1\right)\nu^{-1} \inf_{m\ge 2}\left\{\omega_1\left(q,m^{-1}\right)\ln\nu + m\right\}.$$

From the estimation for S_1, S_2, S_3^i, $i=\overline{1,5}$ follows

$$\left|\int_{x_1}^{x_2}\left\{\theta(x,y,\nu)-\hat{\vartheta}(x,y,\nu,R_0)dy\right\}\right| \le C\left(k,\|q\|_1\right)\psi(\nu),\ 0\le x_1\le x_2\le 1. \qquad (17)$$

If we consider the system of orthonormalized in $L_2(G)$ eigenfunctions of the operator $l_0 u=-u''$, $u^{(j)}(0)=u^{(j)}(1)$, $j=\overline{0,1}$ then we find the estimation

$$\left|\int_{x_1}^{x_2}\left\{D(x-y,\nu)-\vartheta(x,y,\nu,R_0)\right\}dy\right| \le C(K)\nu^{-1} \qquad (18)$$

for in this case $q(x)\equiv 0$.

The estimation (10) follows from (17) and (18). Lemma 2 is proved.

The proof of the theorem is led as in paper [1]. Let's consider it in short.

Let $f(x)$ be any absolutely continuous function on $[0,1]$. Let's represent it as a sum of two functions: $f(x)=f_1(x)+f_2(x)$, where $f_1(x)$ is linear and satisfies the condition $f_1(0)=0$, $f_1(1)=f(1)$. Such expantion provides the convergence to zero of substitutions by integrating by parts. It is evident that it is enough to proof separately estimation (2) for the function $f_1(x)$ and $f_2(x)$. Let's prove the estimation (2) for $f_1(x)$:

$$\left|R_\nu(x,f_1)\right| = \left|\int_0^1[\theta(x,y,\nu)-D(x-y,\nu)]f_1(y)dy\right| = \left|\int_0^1 f_1'(t)dt\int_0^1[\theta(x,\xi,\nu)-D(x-\xi,\nu)]d\xi - \right.$$

$$\left. -\int_0^1\int_0^y[\theta(x,\xi,\nu)-D(x-\xi,\nu)]d\xi f_1'(y)dy\right| \le C\left(k,\|q\|_1\right)\psi(\nu)\|f\|_{W_1^1(0,1)}.$$

The proof for the $f_2(x)$ is led absolutely analogously.

The theorem is proved.

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