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MAMEDOV I.T., MIRZOYEVA K.S.

PHRAGMEN-LINDELÖF TYPE THEOREM FOR THE SECOND ORDER ELLIPTIC EQUATIONS CONTAINING MINOR MEMBERS

Abstract

The second order non-divergent structure elliptic equation containing minor members is considered. The conditions are found on minor coefficients by fulfilling of which Phrafmen-Lindelöf type theorem is valid for solutions of equations.

Introduction. Let E_n be n- dimensional Euclidean space of the points $x=(x_1,...,x_n)$, $(n\ge 2)$, D be an unbounded domain with the boundary ∂D located inside of the cone $K=\left\{x:\left(\sum_{i=1}^{n-1}x_i^2\right)^{1/2}<kx_n, 0<x_n<\infty\right\}$. Consider the following Dirichlet problem in D

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=0;\ \ x\in D, \tag{1}$$

$$u\big|_{\partial D}=0 \tag{2}$$

in assumption that $\|a_{ij}(x)\|$ is real symmetric matrix, where for all $x\in D, \xi\in E_n$

$$\gamma|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\le\gamma^{-1}|\xi|^2;\ \gamma\in(0,1]-const. \tag{3}$$

The aim of our paper is the determination of conditions on the minor coefficients $b_i(x)\,(i=1,...,n)$ and $c(x)$, by realization of which for solutions of the problem (1)-(2) the Phragmen- Lindelöf type theorem is valid. In addition under the solution of indicated problem we'll understand its classical solution, i.e. the function $u(x)\in C^2(D)$ continuous up to ∂D, satisfying the condition (2) and reducing the equation (1) to identity.

For the second order non-divergent elliptic equations not containing the minor members , the analogous results were obtained in papers [1-4]. Concerning divergent elliptic equations we refer to papers [5-6]. Note also papers [7-10] in which Phragmen-Lindelöf type theorems for quasilinear elliptic equations were obtained. In [11-12] the analogous theorems were proved for degenerating in infinity elliptics without minor members. In the case when in elliptic equation the minor members exist, and the main part of the operator L has divergent form, in [13] Phragmen- Lindelöf type theorem is establisheb under the conditions $divB(x)\ge 0, (B(x),x)\le 0, c(x)\le 0$ for all $x\in D$ where $B(x)=(b_1(x),...,b_n(x))$. The result obtained in the present paper is generalization of the result of [4], where the equation of the form (1) for the $c(x)\equiv 0$ is considered.

1^0. Theorem on increase of positive solutions.

We'll denote the ball $\{x:|x|<R\}$ for $R>0$ by Q_R, and by S_R the sphere $\{x:|x|=R\}$. Everywhere later on the record $C(...)$ means that the positive constant C depends only on contents of parenthesis. Impose now the following conditions on minor coefficients of the operators L. We'll assume that the functions $b_1(x),...,b_n(x)$ and $c(x)$ are the elements of the space $L_\infty^{loc}(D)$ and there exists the ray $\bar{l}$ originating from the origin of coordinates and located outside of the cone K such that for all $x \in D \cap Q_{4R}$ and $x^0 \in \bar{l} \cap S_R$ for any $R \ge 1$

$$\left(B(x), x - x^0\right) \le 0, \tag{4}$$

where the vector $B(x)$ has the same sense as above. Besides we'll assume that for $x \in D$

$$c(x) \le 0. \tag{5}$$

Consider along with the operator L the "shortened" operator

$$N = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}$$

and let for $R \ge 1$ $D_R = D \cap Q_R$.

Lemma. *If relative to the coefficients of the operator N the conditions (3)-(4) are satisfied and $g(x) = \left|x - x^0\right|^{-S}$, where $x \in D_{4R}$, $x^0 \in \bar{l} \cap S_R$ $(R \ge 1)$ then there exists $S(\gamma, n)$ such that*

$$Ng(x) \ge 0. \tag{6}$$

Proof. Denote $\left|x - x^0\right|$ by ρ. Subject to (3) and (4) we have

$$Ng(x) = S\rho^{-S-2}\left[(S+2)\rho^{-2}\sum_{i,j=1}^{n} a_{ij}(x)(x_i - x_i^0)(x_j - x_j^0) - \right.$$

$$\left. - \sum_{i=1}^{n} a_{ii}(x) - \left(B(x), (x - x^0)\right)\right] \ge S\rho^{-S-2}\left[\gamma(S+2) - \gamma^{-1} n\right].$$

Now it's sufficient to choose

$$S = \max\{3, \gamma^{-2} n\} - 2, \tag{7}$$

and we obtain the required inequality (6).

Everywhere further not specifying this particularly we'll assume that the constant S is chosen according to the equality (7).

Theorem 1. *Let relative to the coefficients of the operator L the conditions (3)-(5) be satisfied. Then if $u(x)$ is a positive solution of the problem (1)-(2) in D, then there exists $\eta(\gamma, n, k, \bar{l})$ such that for all $R \ge 1$*

$$\sup_{D_{4R}} u \ge (1+\eta) \sup_{D_R} u. \tag{8}$$

Proof. At first we show that if the function $u(x)$ is a solution of the equation (1) then

$$Nu^2(x) \ge 0. \tag{9}$$

Really,

$$Nu^2(x) = 2u\left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}\right) +$$

$$+ 2\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \ge -2u^2 c(x) + 2\gamma |\nabla u|^2,$$

whence (9) follows.

Denote by φ the minimum angle between the ray $\bar{l}$ and generator the cone K and let

$$\beta = \begin{cases} \sin\varphi, & if \quad \varphi \le \dfrac{\pi}{2} \\ 1, & if \quad \varphi > \dfrac{\pi}{2} \end{cases}$$

Then $|x - x^0| \ge \beta R$ for $x \in D_{4R}$. Therefore if $G(x) = (\beta R)^S g(x)$, then $G(x) \le 1$ for $x \in D_{4R}$. Consider the following auxiliary function in $x \in D_{4R}$

$$V(x) = M\left[1 - G(x) + \sup_{S_{4R}} G(x)\right] - u^2(x),$$

where $M = \sup_{D_{4R}} u^2$. According to the lemma and (9) $NV(x) \le 0$ for $x \in D_{4R}$. On the other side $V(x)\big|_{S_{4R} \cap D} \ge 0$. By the maximum principle $V(x) \ge 0$ in D_{4R} and in particular

$$\sup_{D_R} u^2 \le M\left[1 - \left(\inf_{S_R} G(x) - \sup_{S_{4R}} G(x)\right)\right]. \qquad (10)$$

But as it's easy to see, if $x \in S_{4R}$ then $|x - x^0| \ge |x| - |x^0| = 3R$.

If $x \in S_R$ then $|x - x^0| \le |x| + |x^0| = 2R$.

Therefore

$$\sup_{S_{4R}} G(x) \le \beta^S 3^{-S}, \quad \inf_{S_R} G(x) \ge \beta^S 2^{-S}. \qquad (11)$$

Allowing for (11) in (10) we obtain

$$\sup_{D_R} u^2 \le M(1 - \eta_1), \qquad (12)$$

where $\eta_1 = \beta^S (2^{-S} - 3^{-S})$. Now it's sufficient to choose $\eta = \sqrt{1 + \eta_1} - 1$ and from (12) the required estimation (8) follows. The theorem is proved.

2^0. Phragmen- Lindelöf type theorem.

Theorem 2. *Let the solution $D \subset K$ of the problem (1)-(2) be determined in the domain $u(x)$, where relative to the coefficients of the operator L the conditions (3)-(5) are satisfied. Then either $u(x) \equiv 0$ in D or*

$$\lim_{r \to \infty} \frac{M(r)}{r^{\delta}} > 0, \qquad (13)$$

where $M(r) = \sup_{D \cap S_R} \dfrac{M(r)}{r^{\delta}} > 0$ *and* $\delta = \delta(\gamma, n, k, \bar{l})$.

Proof. Let $u(x) \ne 0$. Then there exists a point $y \in D$, in which $u(y) = a \ne 0$. Assume that $a > 0$. Denote by D^+ the set $\{x : x \in D, u(x) > 0\}$, and by D' - the connected component D^+ containing the point y. By the maximum principle D' is an unbounded domain.

Let m_0 be the least natural number for which $y \in Q_{4m_0}$. Let's fix arbitrary sufficiently large r. Denote by m a natural number satisfying the inequalities

$$4^m \le r < 4^{m+1}. \qquad (14)$$

From (14) it follows that

$$m > \frac{\ln r}{\ln 4} - 1.$$

We'll assume r as much large that $m > m_0$ and $\frac{\ln r}{\ln 4} - 1 \geq \frac{\ln r}{2\ln 4}$. Then

$$m > \frac{\ln r}{2\ln 4}. \quad (15)$$

Let $M(r) = \sup\limits_{D'\cap S_r} u$. Using successively the inequality (8) and allowing for (14)-(15) we obtain

$$M(r) \geq M(4^m) \geq (1+\eta)M(4^{m-1}) \geq \ldots \geq (1+\eta)^{m-m_0} M(4^{m_0}) \geq$$
$$\geq (1+\eta)^m \frac{a}{(1+\eta)^{m_0}} \geq (1+\eta)^{\frac{\ln r}{r \ln 4}} \frac{a}{(1+\eta)} m_0 = \eta_2^{\ln r} a_1, \quad (16)$$

where $\eta_2 = (1+\eta)^{\frac{1}{2\ln 4}}$, $a_1 = \frac{a}{(1+\eta)^{m_0}}$. From (16) we canceled that

$$M(r) \geq a_1 r^\delta,$$

where $\delta = \ln \eta_1$.

If $a < 0$ then we multiply the solution $u(x)$ by -1 and lead the analogous reasonings. Thus we show that for the sufficiently large r

$$M(r) \geq a_1 r^\delta,$$

where a_1 is a positive constant independent of r. Hence the required limit equality (13) follows. The theorem is proved.

3^0. Generalization in the case of degenerate equations.

Again consider the Dirichlet problem (1)-(2) in assumption that the operator L allows noninuform degeneration on infinity. In other words we'll assume that instead of the condition (3) the condition

$$\gamma \sum_{i=1}^n \lambda_i(x)\xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \gamma^{-1} \sum_{i=1}^n \lambda_i(x)\xi_i^2 \quad (17)$$

is satisfied, where γ has the same sense that in (3), $\xi \in E_n$, $x \in D$, $\lambda_i(x) = (1+|x|_\alpha)^{-\alpha_i}$, $|x|_\alpha = \sum_{i=1}^n |x_i|^{\frac{2}{2-\alpha_i}}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in [0,2)$; $i = 1, \ldots, n$.

Besides in determination of the cone K we assume that $k \in (0, 2^{-4}]$ and if $\alpha^+ = \max\{\alpha_1, \ldots, \alpha_n\}$, then

$$\alpha_n = \alpha^+. \quad (18)$$

Before to write the condition on the coefficients $b_1(x), \ldots, b_n(x)$, let's agree in the following designations. For $\xi \in E_n$, $R > 0$, $\mu > 0$ $\mathcal{E}_R^z(\mu)$ is an ellipsoid $\left\{x : \sum_{i=1}^n R^{-\alpha_i}(x_i - z_i)^2 < (\mu R)^2\right\}$. Let $z \in \mathcal{E}_R^0(5) \cap D$ and the point $x^0(z)$ is such that $x_1^0(z) > 0$, $x_2^0(z) = \ldots = x_{n-1}^0(z) = 0$, $x_n^0(z) > 0$. (By our constraints on k we can show that $x^0(z) \notin K_R$). Denote $\bigcup\limits_{z \in D \cup \partial\mathcal{E}_R^0} x^0(z)$ by A_R, $D \cap (\mathcal{E}_R^0(9) \setminus \mathcal{E}_R^0(1))$ by D and the vector

$\left(R^{\alpha_1}b_1(x),...,R^{\alpha_n}b_n(x)\right)$ by $B^{(R)}(x)$. We'll assume that for $x \in D^R, x^0 \in A_R$ for the sufficiently large R

$$\left(B^R(x), x - x^0\right) \le 0. \tag{19}$$

Theorem 3. *Let the coefficients of the operator L satisfying the conditions (5) and (17)-(19) be determined in the domain $D \subset K$. Then if $u(x)$ is a solution of the Dirichlet problem (1)-(2), then either $u(x) \equiv 0$ in D or*

$$\varliminf_{r\to\infty} \frac{M_1(r)}{r^{\delta_1}} > 0,$$

where $M_1(r) = \sup\limits_{D \cap \partial \mathcal{E}_r^u(1)} |u(x)|$ *and* $\delta_1 = \delta_1(\gamma, n, \alpha, k)$.

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Mamedov I.T., Mirzoyeva K.S.
Institute of Mathematics & Mechanics of NAS of Azerbaijan.
9, F.Agayeva str., 370141, Baku, Azerbaijan.
Tel.: 39-39-24(off.).

E-mail: ilham@lan.ab.az

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