2001 VOL. XV(XXIII)

MAMEDOV O.M.

A REMARK ON THE TERNARY DIFFERENCE TERM FOR MODULAR VARIETIES

Abstract

We describe an easy extension of defining properties of the ternary difference term for modular varieties.

In general algebra, the first development of a commutator theory is due to J.Smith who succeeded in defining a commutator operation (as useful as group commutator) for congruences of algebras in permutable varieties; Smith's theory was soon generalized to modular varieties by J.Hagemann and C. Herrmann. The proofs of many fundamental theorems depend on difference term that is composed from Day's terms in a very nontrivial way. Recently, the class of varieties with a (weak) difference term was characterized, see [2]. All nondefined notions may be found in [1].

We remind the reader the two following results.

Theorem (A. Day). *A variety* $\mathbf{V}$ *is modular if and only if for some* n *there are 4-ary terms* $m_0(xyzu),\dots,m_n(xyzu)$ *such that* $\mathbf{V}$ *satisfies*

(i) $m_0(xyzu)=x$, $m_n(xyzu)=u$;

(ii) $m_i(xyyx)=x$, $i\le n$;

(iii) $m_i(xxyy)=m_{i+1}(xxyy)$ *for all even* $i<n$;

(iv) $m_i(xyyz)=m_{i+1}(xyyz)$ *for all odd* $i<n$.

Theorem (5.5(i,ii)[1]). *For each modular variety* $\mathbf{V}$ *there is a ternary term* d, *called a difference term, satisfying the following*

(i) $d(xxy)=y$ *is an identity of* $\mathbf{V}$, *and*

(ii) *if* $<a,b>\in\alpha\in ConA$, $A\in\mathbf{V}$, *then* $d(abb)[\alpha,\alpha]a$.

Our aim is to present the following

Proposition. *For each modular variety* $\mathbf{V}$ *there are ternary terms* $d_0,\dots,d_n$ *(for the same* n *as in Day's theorem) satisfying the following*

(i) $d_i(xxy)=y$, $i=0,\dots,n$ *is an identity of* $\mathbf{V}$, *and*

(ii) *if* $<a,b>\in\alpha, <a,c>\in\beta$, $\alpha,\beta\in ConA$, $A\in\mathbf{V}$ *and*

$$d_i(abc)[\alpha,\beta]m_i(cbba) \text{ for all odd } i,$$

then $d_i(abc)[\alpha,\beta]m_i(cbaa)$ *for all even* i *(here* m_i *is Day's term).*

In particular, $d_n(abc)[\alpha,\beta]a$.

Proof. Obviously, by Day's theorem there are Day's terms $m_0,\dots,m_n$ and so define inductively $d_0(xyz)=z$ and

$$d_{i+1}(xyz)=\begin{cases} m_{i+1}(d_i(xyz),y,x,d_i(xyz)), & i \text{ odd},\\ m_{i+1}(d_i(xyz),x,y,d_i(xyz)), & i \text{ even},\end{cases}$$

as in the proof of Theorem 5.5 (see [1]).

Day's terms imply that for every i $d_i(xxy)=y$ is an identity of $\mathbf{V}$, so (i) is proved.

To show (ii) assume $<a,b>\in\alpha$, $<a,c>\in\beta$ and for odd i

$$d_i(abc)[\alpha,\beta]m_i(cbba).$$

Obviously, $c = d_0(abc)[\alpha,\beta]m_0(cbaa) = c$, so the case $i = 0$ follows from Day's identities.

Suppose i is odd, then by (ii) we have

$$d_{i+1}(abc) = m_{i+1}(d_i(abc), b, a, d_i(abc))[\alpha,\beta]m_{i+1}(m_i(cbba), b, a, m_i(cbba)).$$

Now identifying here intermediate variables and using Day identities, we obtain

$$m_{i+1}(m_i(cbba), b, b, m_i(cbba)) = m_i(cbba) = m_{i+1}(cbba) = m_{i+1}(m_i(cbbc), b, b, m_i(abba)).$$

It is easy to see that the following matrix belongs to $M(\alpha,\beta)$

$$\begin{pmatrix} m_{i+1}(m_i(cbba), b, b, m_i(cbba)), & m_{i+1}(m_i(cbbc), b, b, m_i(abba)) \\ m_{i+1}(m_i(cbba), b, a, m_i(cbba)), & m_{i+1}(m_i(cbbc), b, a, m_i(abba)) \end{pmatrix}.$$

The $\alpha-\beta$-term condition gives

$$m_{i+1}(m_i(cbba), b, a, m_i(cbba))[\alpha,\beta]m_{i+1}(m_i(cbbc), b, a, m_i(abba)) = m_{i+1}(cbaa).$$

So

$$d_{i+1}(a,b,c)[\alpha,\beta]m_{i+1}(m_i(cbba), b, a, m_i(cbba))[\alpha,\beta]m_{i+1}(cbaa).$$

As $m_n(xyzu) = u$ then in particular

$$d_n(abc)[\alpha,\beta]a.$$

Note. If we put $b = c$ in proposition then we obtain:

if $<a,b> \in \alpha$ and simultaneously $<a,b> \in \beta$ then, as it is easy to see, $d_n(abb)[\alpha,\beta]a$.

References

[1]. Freese R., McKenzie R. *Commutator theory for congruence modular varieties.* London Mat.Soc.Lect.Note Series, 125, Cambridge, 1987, 228p.

[2]. Kearnes K.A., Szendrei A. *The relationship between two commutators.* Internat. J.Algebra Comput. 8, 1998, p.497-531.

Mamedov O.M.

Institute of Mathematics & Mechanics of NAS of Azerbaijan.

9, F.Agayeva str., 370141, Baku, Azerbaijan.

Tel.: 39-47-20(off.).

Received September 20, 2000; Revised March 16, 2001.

Translated by author.