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ON NORMS OF INTERMEDIATE DERIVATIVES IN SPACES OF SMOOTH VECTOR FUNCTIONS ON ALL AXIS

Abstract

In the paper the values of norms of operators of intermediate derivatives in the Sobolev type spaces are found.

Let H be a separable Hilbert space, A be a positive-definite self-adjoint operator in H.

Denote by $L_2(R:H)$ a Hilbert space of the vector-functions $f(t)$ with values from H, measurable and quadratically integrable by Bochner with the norm

$$\|f\|_{L_2(R:H)}=\left(\int_{-\infty}^{+\infty}\|f(t)\|^2\,dt\right)^{1/2}.$$

Further we determine the space [1]

$$W_2^n(R:H)=\left\{u:u^{(n)}\in L_2(R:H),\ A^n u\in L_2(R:H)\right\},$$

where $n\geq 1$ is a natural number, and the derivatives are understood in sense of the theory of generalized functions. We determine the norm in $W_2^n(R:H)$ in the following form

$$\|u\|_{W_2^n(R:H)}=\left(\left\|u^{(n)}\right\|^2_{L_2(R:H)}+\left\|A^n u\right\|^2_{L_2(R:H)}\right)^{1/2}.$$

As it is known by the theorem on intermediate derivatives [1, p.29]. the operators

$$A^{n-j}\frac{d^j}{dt^j}:W_2^n(R:H)\to L_2(R:H),\quad j=1,\ldots,n-1, \tag{1}$$

are bounded. The norms of these operators are calculated in [2] and equal

$$N_{n,j}(R)=\sup_{\xi\in R}\left(\frac{\xi^{2j}}{\xi^{2n}+1}\right)^{1/2},\quad j=1,2,\ldots,n-1,$$

i.e. they have the exact inequality

$$\left\|A^{n-j}u^{(j)}\right\|_{L_2(R:H)}\leq N_{n,j}(R)\|u\|_{W_2^n(R:H)}. \tag{2}$$

Further from density theorem [1, p.29] it follows that $D_n(R:H)$ is a set of infinitely differentiable functions with compact carriers in R that are dense in the space $W_2^n(R:H)$.

Let $s>0$ be an integer number. Then it's easy to see that the operators $A^{n-j}\frac{d^j}{dt^j}:W_2^{n+s}(R:H)\to W_2^s(R:H)$ are continuous operators. Really, assuming that $u\in D_{n+s}(R:H)$ we obtain

$$\left\|A^{n-j}u^{(j)}\right\|^2_{W_2^s(R:H)}=\left\|A^{n+s}u^{(j)}\right\|^2_{W_2^s(R:H)}+\left\|A^{n-j}u^{(j+s)}\right\|^2_{L_2(R:H)}\leq$$
$$\leq\left(N^2_{n+s,j}(R)+N^2_{n+s,j+s}(R)\right)\|u\|^2_{W_2^{n+s}(R:H)}. \tag{3}$$

In the present paper we find the values of norms of operators of intermediate derivatives

$$M_{j,s}(R)=\sup_{0\neq u\in W_2^{n+s}(R:H)}\left\|A^{n-j}u^{(j)}\right\|_{W_2^s(R:H)}\|u\|_{W_2^{n+s}(R:H)}\|u\|^{-1}_{W_2^{n+s}(R:H)} \quad (4)$$

for the fixed natural $s>0$.

From the inequality (3) it follows that

$$M_{j,s}(R)\leq\left(N^2_{n+s,j}+N^2_{n+s,j+s}\right)^{1/2},$$

and the numbers $N^2_{n+s,j}$, $N^2_{n+s,j+s}$ are determined from the equality (2).

The following holds

Theorem. *The norms of operators of intermediate derivatives*

$$A^{n-j}\frac{d^j}{dt^j}:W_2^{n+s}(R:H)\to W_2^s(R:H),\ \ j=1,2,...,n-1$$

are equal to

$$M_{j,s}(R)=\sup_{\xi\in R}\left(\frac{\xi^{2j}+\xi^{2(j+s)}}{\xi^{2(n+s)}+1}\right)^{1/2},\quad j=1,2,..,n-1. \quad (5)$$

Proof. Let $u\in D_{n+s}(R:H)$. Then consider the functional

$$E_j(u;\beta)=\|u\|^2_{W_2^{n+s}(R:H)}-\beta\left\|A^{n-j}u^{(j)}\right\|^2_{W_2^s(R:H)},\ \ j=1,...,n-1 \quad (6)$$

for $\beta\in[0,M_{j,s}^{-2})$.

It's obvious that

$$E_j(u;\beta)=\left\|u^{(n+s)}\right\|^2_{L_2(R:H)}+\left\|A^{n+s}u\right\|^2_{L_2(R:H)}-$$
$$-\beta\left(\left\|A^{n+s-j}u^{(j)}\right\|^2_{L_2(R:H)}+\left\|A^{n-j}u^{(s+j)}\right\|^2_{L_2(R:H)}\right).$$

If we denote the Fourier transformation of the vector-function $u(t)$ by $\hat{u}(\xi)$, then from Plansharel theorem if follows that

$$E_j(u;\beta)=\left\|\xi^{n+s}\hat{u}(\xi)\right\|^2_{L_2(R:H)}+\left\|A^{n+s}\hat{u}(\xi)\right\|^2_{L_2(R:H)}-$$
$$-\beta\left(\left\|\xi^{j}A^{n+s-j}\hat{u}(\xi)\right\|^2_{L_2(R:H)}+\left\|\xi^{j}A^{n+s-j}\hat{u}(\xi)\right\|^2_{L_2(R:H)}\right)= \quad (7)$$
$$=\int_{-\infty}^{+\infty}\left(P_j(\xi,\beta:A)\hat{u}(\xi),\hat{u}(\xi)\right)d\xi,$$

where the polynomial operator bundle $P_j(\xi:\beta:A)$ with real arguments to ξ and β $\left(\xi\in R,\ \beta\in[0,M_{j,s}^{-2})\right)$ is determined in the following form:

$$P_j(\xi:\beta:A)=\xi^{2(n+s)}+A^{2(n+s)}-\beta\left(\xi^{2j}A^{2(n+s-j)}+\xi^{2(j+s)}A^{2(n-j)}\right). \quad (8)$$

Show that for $\xi \in R$ and $\beta \in [0, M_{j,s}^{-2}(R))$ the bundle $P_j(\xi:\beta:A)$ is positive. Really, for $\mu \in \sigma(A)$ $(\mu \ge \mu_0 > 0)$, $\xi \in R$ and $\beta \in [0, M_{j,s}^{-2}(R))$ we have

$$P_j(\xi;\beta:\mu) = \xi^{2(n+s)} + \mu^{2(n+s)} - \beta\left(\xi^{2j}\mu^{2(n+s-j)} + \mu^{2(n-j)}\xi^{2(j+s)}\right) =$$

$$= \left(\xi^{2(n+s)} + \mu^{2(n+s)}\right)\left(1 - \beta\frac{\xi^{2j}\mu^{2(n+s-j)} + \mu^{2(n-j)}\xi^{2(j+s)}}{\xi^{2(n+s)} + \mu^{2(n+s)}}\right) \ge$$

$$\ge \left(\xi^{2(n+s)} + \mu^{2(n+s)}\right)\left(1 - \beta\sup_{\xi\in R}\frac{\xi^{2j} + \xi^{2(j+s)}}{\xi^{2(n+s)} + 1}\right) =$$

$$= \left(\xi^{2(n+s)} + \mu^{2(n+s)}\right)\left(1 - \beta M_{j,s}^{-2}(R)\right) > 0 .$$

Consequently, from spectral expansion of the operator A it follows that $P_j(\xi:\beta:A) > 0$ for $\xi \in R$, $\beta \in [0, M_{j,s}^{-2}(R))$ and therefore from the equality (7) it follows that for $\beta \in [0, M_{j,s}^{-2}(R))$ and $u \in D_{n+s}(R:H)$

$$E_j(u;\beta) > 0,\ j = 1,2,...,n-1. \qquad (9)$$

Thus passing to the limit in the inequality (9) when $\beta \to M_{j,s}^{-2}(R)$ we get that for all $u \in D_{n+s}(R:H)$ the inequality

$$\|u\|^2_{W_2^{n+s}(R:H)} - M_{j,s}^{-2}(R)\left\|A^{n-j}u^{(j)}\right\|^2_{W_2^{n+s}(R:H)} \ge 0$$

holds.

It follows that

$$\left\|A^{n-j}u^{(j)}\right\|_{W_2^{n+s}(R:H)} \le M_{j,s}(R)\|u\|_{W_2^{n+s}(R:H)},\quad j = 1,2,...,n-1. \qquad (10)$$

Show that the inequality (10) is exact.

With this aim for the given $\varepsilon > 0$ we show the existence of such a vector function $u_\varepsilon(t) \in W_2^{n+s}(R:H)$ that

$$E_j\left(u_\varepsilon, M_{j,s}^{-2}(R) + \varepsilon\right) \equiv \|u\|^2_{W_2^{n+s}(R:H)} - \left(M_{j,s}^{2}(R) + \varepsilon\right)\left\|A^{n-j}u^{j}\right\|^2_{W_2^{n+s}(R:H)} < 0. \qquad (11)$$

We'll search the vector function $u_\varepsilon(t)$ in the form of: $u_\varepsilon(t) = g_\varepsilon(t)\varphi$ where $\varphi \in D\left(A^{2(n+s)}\right)$, $\|\varphi\| = 1$, and $g_\varepsilon(t)$ is a scalar function from the space $W_2^{n+s}(R;C)$, C is a complex plane. Using Plansherel theorem we write the inequality (11) in the equivalent form

$$E_j\left(u_\varepsilon, M_{j,s}^{-2}(R) + \varepsilon\right) = \int_{-\infty}^{+\infty}\left(P_j\left(\xi; M_{j,s}^{-2}(R) + \varepsilon, A\right)\varphi, \varphi\right)\left|\hat{g}_\varepsilon(\xi)\right|^2 d\xi < 0. \qquad (12)$$

Here $\hat{g}_\varepsilon(\xi)$ is Fourier transformation of the vector-function $g_\varepsilon(t)$ and the bundle $P_j\left(\xi, M_{j,s}^{-2}(R) + \varepsilon, A\right)$ is determined from the equality (8) substituting β by $M_{j,s}^{-2}(R) + \varepsilon$.

Let the operator A have if only one eigen value μ, then we take φ as a corresponding normed vector i.e. $A\varphi=\mu\varphi$, $\|\varphi\|=1$. In this case it's easy to see that

$$P_j\left(\xi;M_{\varphi,s}^{-2}(R)+\varepsilon,A\right)\varphi,\varphi)=P_j\left(\xi;M_{j,s}^{-2}(R)+\varepsilon;\mu\right)$$

Now at the point $\xi_0\in R$ such that

$$M_{j,s}(R)=\left(\frac{\xi_0^{2j}+\xi_0^{2(j+s)}}{\xi_0^{2(n+s)}+1}\right).$$

Then in the point $\xi=\xi_0\mu\in R$

$$P_j\left(\xi;M_{j,s}^{-2}(R)+\varepsilon,\mu\right)=\mu^{2(n+s)}\left(\xi_0^{2(j+s)}+1\right)\times$$
$$\times\left[1-\left(M_{j,s}^{-2}(R)+\varepsilon\right)\left(\frac{\xi_0^{2j}+\xi_0^{2(j+s)}}{\xi_0^{2(n+s)}+1}\right)^2\right]= \qquad (13)$$
$$=-\varepsilon\mu^{2(n+s)}\left(\xi_0^{2(n+s)}+1\right)M_{j,s}^{2}(R)<0.$$

If the operator A hasn't an eigen vector, then for any $\mu\in\sigma(A)$ and for any $\delta>0$ using the spectral expansion of the operator A we can construct the vector $\varphi_\delta\in D\left(A^{2(n+s)}\right)$, $\|\varphi_\delta\|=1$ such that

$$A^m\varphi_\delta=\mu^m\varphi_\delta+o(\delta),\ \ (\delta\to 0),\ m=1,2,...$$

by the inequality (13) $P_j\left(\xi;M_{j,s}^{-2}(R)+\varepsilon,A\right)\varphi_\delta,\varphi_\delta)<0$ for small $\delta>0$. Thus we always can construct the vector φ, $\|\varphi\|=1$ such that $P_j\left(\xi;M_{j,s}^{-2}(R)+\varepsilon,A\right)\varphi,\varphi)<0$ at some point $\xi=\xi_0\mu$. Since $\left(P_j\left(\xi;M_{j,s}^{-2}(R)+\varepsilon,A\right)\varphi,\varphi\right)$ is a continuous function of the argument ξ then the inequality (13) are satisfied in some interval (η_1,η_2) containing the point $\xi=\xi_0\mu$. Now we can construct the function $g_\varepsilon(t)$. Let $\hat{g}_\varepsilon(\xi)$ be an infinite differentiable function with carrier in the interval (η_0,η_1). Denote its inverse Fourier transformation by $g_\varepsilon(t)$

$$g_\varepsilon(t)=\frac{1}{\sqrt{2\pi}}\int_{\eta_1}^{\eta_2}\hat{g}(\xi)e^{i\xi t}d\xi,\ \ t\in R.$$

By Peli-Wiener theorem $g_\varepsilon(t)\in W_2^{n+s}(R:C)$ and for the vector-functions $u_\varepsilon(t)=g_\varepsilon(t)$ from the inequality (7) we obtain

$$E_j\left(u_\varepsilon,M_{j,s}^{-2}(R)+\varepsilon\right)=E_j\left(g_\varepsilon(t)\varphi,M_{j,s}^{-2}(R)+\varepsilon\right)=$$
$$=\int_{\eta_1}^{\eta_2}\left(P_j\left(\xi,M_{j,s}^{-2}(R)+\varepsilon;A\right)\varphi,\varphi\right|\hat{g}_\varepsilon(\xi)\Big\}^2d\xi.$$

Since for $\xi\in(\eta_1,\eta_2)$ $\left(P_j\left(\xi,M_{j,s}^{-2}(R)+\varepsilon;A\right)\varphi,\varphi\right)<0$, then $E_j\left(u_\varepsilon,M_{j,s}^{-2}+\varepsilon\right)<0$.

Consequently the inequality (10) is exact. The theorem is proved.

References

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