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THE IMBEDDING THEOREMS IN THE LIZORKIN-TRIEBEL-MORREY TYPE SPACE WITH DOMINANT MIXED DERIVATIVES

Abstract

In the work are constructed the new space $S^{l}_{p,\theta,a,\chi,\tau}L(G)$ and receiving integral representation for these space the imbedding theorems are proved.

The Lizorkin-Triebel-Morrey type space with dominant mixed derivatives $S^{l}_{\mathbf{p},\theta,a,\chi,\tau}L(G)$, $\mathbf{p}\in(1,\infty)^n$; $a\in[0,1]^n$; $\theta\in(1,\infty)$; $\tau\in[1,\infty]$; $\chi,l\in(0,\infty)^n$ is constructed, new integral representations are obtained. Note that here the domain $G\subset R^n$ satisfies the flexible horn conditions introduced by O.V.Besov. The space of Sobolev and Nikolsky with dominant mixed derivative (difference) $S^{r}_{p}W$ and $S^{r}_{p}H$ where introduced and studied by S.M.Nikolsky. Besov's spaces with dominand mixed derivative $S^{r}_{p,q}B$ were introduced and studied by different methods by A.D.Jabrailov and T.I.Amirov. The spaces of Sobolev-Lioville with dominant mixed derivatives $S^{r}_{p}L$ were studied by P.I.Lizovkin and S.M.Nokolsky. The Lizovkin-Triebel spaces with dominant mixed derivative $S^{l}_{p,\theta}L(G)$ were introduced (in weight case) in the work [3], and the new descriptive norm was introduced in [4] and were obtained interpolational theorems of functions from these spaces. The space $L^{l}_{p,\theta,a,\chi,\tau}(G)$ was introduced by V.Guliyev and studied in [5], and the space $S^{l}_{\mathbf{p},\theta,a,\chi,\tau}B(G)$ was defined and studied in [6].

Let R^n-n be an n-dimensional Euchlidean space of points $x=(x_1,...,x_n)$, $G\subset R^n$, $\chi,t,h\in(0,\infty)^n$; $e_n=\{1,2,...,n\}$, $e\subseteq e_n$. The number of all possible subspaces e from e_n equal to 2^n. Let also $k=(k_1,...,k_n)$, $k^e=(k^e_1,...,k^e_n)$, $k^e_j=k_j$ for $j\in e$; $k^e_j=0$ for $j\in e_n\setminus e=e'$;

$$\Delta^{k^e}(t)f(x)=\left(\prod_{j\in e}\Delta^{k_j}_j(t_j)\right)f(x);\ \ \Delta^{k_j}_j(t_j)f(x)=\sum_{i=1}^{k_j-i}(-1)^{k_j-i}C^i_{k_j}f\left(x+it_je^j\right)$$

$$D^{k^e}f(x)=D^{k^e_1}_1D^{k^e_2}_2...D^{k^e_n}_nf(x);\ \ \int_{a^e}^{b^e}f(x)dx^e=\left(\prod_{j\in e}\int_{a_j}^{b_j}dx_j\right)f(x);$$

$$I_{t\chi}(x)=\left\{y:\left|y_j-x_j\right|<t_j^{\chi_j},\ j\in e_n\right\};$$

$$G_{t\chi}(x)=G\cap I_{t\chi}(x);\ \ \lfloor t_j\rfloor_1=\min\{1,t_j\},\ \ j\in e_n.$$

For $T\in(0,\infty)^n$ we consider for each $x\in G$ the path

$$\rho(t)=\rho(t,x)=(\rho_1(t_1,x),\rho_2(t_2,x),...,\rho_n(t_n,x)),\ 0\le t_j\le T_j,\ \ 1\le j\le n;$$

where for all $j,1\le j\le n$, $\rho_j(0,x)=0$, the functions $\rho_j(u_j,x)$ are absolutely continuous by u_j in $[0,T_j]$ and $\left|\rho'_j(u_j,x)\right|\le1$ almost for all $u_j\in[0,T_j]$, where

$\rho_j'(u_j,x)=\frac{\partial}{\partial u_j}\rho_j(u_j,x)$. For $\theta\in(0,1]^n$ each of sets $V(x,\theta)=\bigcup_{0\le t_j\le T_j, j\in e_n}[\rho(t,x)+t\theta I]$, $x+V(x,\theta)\subset G$ will be called a flexible horn, and the point x will be called the vertex of $x+V(x,\theta)$, where $t\theta I=\{(t_1\theta_1 y_1),...,(t_n\theta_n y_n): y\in I\}$. We shall assume that $x+V(x,\theta)\subset G$.

Let $m=(m_1,...,m_n)$, m_j be natural, $k=(k_1,...,k_n)$, $k_j\ge 0$ be entire nonnegative, $m_j>l_j>k_j\ge 0$, $j\in e_n$.

By $S^l_{p,\theta,a,\chi,\tau}L(G)$ we denote a Banach space of locally summable functions on G with finite norm

$$\|f\|_{S^l_{p,\theta,a,\chi,\tau}L(G)}=\sum_{e\subseteq e_n}\left\|\left\{\int_{0^e}^{1^e}\left[\prod_{j\in e}h_j^{k_j-l_j}\delta^{m^e-k^e}(h)D^{k^e}f(\cdot)\right]^{\theta}\prod_{j\in e}\frac{dh_j}{h_j}\right\}^{\frac{1}{\theta}}\right\|_{p,a,\chi,\tau}, \tag{1}$$

where $\delta^{m^e}(h)f(x)=\left(\prod_{j\in e}\delta_j^{m_j}(h_j)\right)f(x)$, $\delta_j^{m_j}(h_j)f(x)=\int_{-1}^{1}\Delta_j^{m_j}(h_ju,G_h)f(x)du$ and

$$\|f\|_{p,a,\chi,\tau}=\sup_{\chi\in G}\left\{\int_0^{t_1^0}\cdots\int_0^{t_n^0}\frac{1}{\prod_{j=1}^{n}[t_j]_1^{\chi_j\frac{a_j}{p_j}}}\left\{\int_{G_{t_n^{\chi_n}}(x_n)}\left[\cdots\left\{\int_{G_{t_2^{\chi_2}}(x_2)}\times\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\times\left(\int_{G_{t_1^{\chi_1}}(x_1)}|f|^{p_1}dy_1\right)^{p_2/p_1}dy_2\right\}^{p_3/p_2}\cdots\right]^{p_n/p_{n-1}}dy_n\right\}^{1/p_n}\right]^{\tau}\prod_{j\in e_n}\frac{dt_j}{t_j}\right\}^{1/\tau}. \tag{2}$$

For $\tau=\infty$, the Lizorkin-Triebel-Morrey space will be called a space $S^l_{p,\theta,a,\chi,\infty}L(G)\equiv S^l_{p,\theta,a,\chi}L(G)$. The properties of the space $S^l_{p,\theta,a,\chi,\tau}L(G)$:

1) $S^l_{p,\theta,a,\chi,\tau}L(G)\to S^l_{p,\theta,a,\chi}L(G)\to S^l_{p,\theta}L(G)$ and

$$\|f\|_{S^l_{p,\theta}L(G)}\le\|f\|_{S^l_{p,\theta,a,\chi}L(G)}\le C\|f\|_{S^l_{p,\theta,a,\chi,\tau}L(G)}.$$

2) For each real $C>0$

$$\|f\|_{S^l_{p,\theta,a,c\chi,\tau}L(G)}=\frac{1}{c^{\tau}}\|f\|_{S^l_{p,\theta,a,\infty,\tau}L(G)}.$$

3) $\|f\|_{S^l_{p,\theta,0,\chi,\infty}L(G)}=\|f\|_{S^l_{p,\theta}L(G)}$.

4) Let $1<\theta\le r\le s\le\sigma<\infty$, $\theta\le\min_{1\le j\le n}p_j\le\max_{1\le j\le n}p_j\le\sigma$.

Then

$$S^l_{p,\theta,a,\chi,\tau}B(G_h)\to S^l_{p,r,a,\chi,\tau}L(G)\to S^l_{p,s,a,\chi,\tau}L(G)\to S^l_{p,\sigma,a,\chi,\tau}B(G_h).$$

In particular, for $l\in N^n$, $r=s=2$

$$S^l_{\mathbf{p},\theta,a,\chi,\tau}B(G_h)\to S^l_{\mathbf{p},a,\chi,\tau}W(G)=S^l_{\mathbf{p},2,a,\chi,\tau}L(G)\to S^l_{\mathbf{p},\sigma,a,\chi,\tau}B(G_h).$$

Let the function $M_e\in C^\infty(R^n,R^n)$ and finitely uniformly with respect to z from arbitrary compactum be such that

$$S(M_e)=suppM_e(\cdot,z)\subset I_1.$$

Let $T=T(T_1,...,T_n)$, $0<T_j\le 1$, $j\in e_n$. Suppose

$$V=\bigcup_{0<t_j\le T_j}\left\{y:\left(\frac{y}{t^e+T^{e'}}\right)\in S(M_e)\right\}\subset I_T.$$

Let $U\subset G$ be an open set in R^n. We'll always consider that

$$U+V\subset G$$

$$G_{T\chi}(U)=\bigcup_{x\in U}G_{T\chi}(x)=(U+I_{T\chi}(x))\cap G$$

if $0<\chi_j\le 1, 0<T_j\le 1$, $j\in e_n$ then it follows that $I_T\subset I_{T\chi}\Rightarrow U+V\subset G_{T\chi}(U)$.

Lemma 1. *Let* $1\le\mathbf{p}\le\mathbf{q}\le\mathbf{r}\le\infty, 0<\chi_j\le 1, 0<t_j\le T_j\le 1, 0<\rho_j<\infty$, $j\in e_n$, $1\le\tau\le\infty$, $f\in L_{\mathbf{p},a,\chi,\tau}(G_{T\chi}(U))$

$$I_e(x,t^e+T^{e'})=\int_{R^n}M_e\left(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}}\right)f_e(x+y,t^e)dy, \tag{3}$$

where

$$\left|f_e(x,t^e)\right|\le C\int_{-1^e}^{1^e}\delta^{m^e}(\delta t)f(x+t_1^e u_1^e+...+t_n^e u_n^e)du^e.$$

Then the inequality

$$\sup_{x\in U}\left\|I_e(\cdot,t^e+T^{e'})\right\|_{q,U_{\rho^\chi}(\bar{x})}\le C\prod_{j\in e'}T_j^{1-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{r_j}\right)}\times$$

$$\times\prod_{j\in e}t_j^{l_j+1-(1-\chi_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)}\prod_{j\in e_n}[\rho_j]_1^{\chi_j a_j\frac{1}{r_j}}\prod_{j\in e_n}\rho_j^{\chi_j\left(\frac{1}{q_j}-\frac{1}{r_j}\right)}\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|_{p,a,\chi,\tau} \tag{4}$$

holds.

Proof. We first prove the lemma for $p_1=...=p_n=p$, $q_1=q_2=...=q_n=q$. By virtue of Holder inequality ($q\le r$) we have

$$\left\|I_e(\cdot,t^e+T^{e'})\right\|_{q,U_{\rho^\chi}(\bar{x})}\le\prod_{j\in e_n}\rho_j^{\chi_j\left(\frac{1}{q}-\frac{1}{r}\right)}\left\|I_e(\cdot,t^e+T^{e'})\right\|_{r,U_{\rho^\chi}(\bar{x})}. \tag{5}$$

Estimate the norm $\left\|I_e(\cdot,t^e+T^{e'})\right\|_{q,U_{\rho^\chi}(\bar{x})}$. For $1\le p\le r\le\infty, s\le r$ we represent the integrand

$$|M_e f_e|=\left(|f_e|^p|M_e|^e\right)^{\frac{1}{r}}\left(|f_e|^p\chi\right)^{\frac{1}{p}-\frac{1}{r}}\left(|M_e|^s\right)^{\frac{1}{r}}$$

and for $|I_e|$ apply the Holder inequality. Then

$$\left|I_e\left(x,t^e+T^{e'}\right)\right|\le\left(\int\limits_{R^n}\left|f_e\left(x+y,t^e\right)\right|^p\left|M_e\left(\frac{y}{t^e+T^{e'}},\frac{\rho(t,x)}{t^e+T^{e'}}\right)\right|^s dy\right)^{\frac{1}{r}}\times$$

$$\times\left(\int\limits_{R^n}\left|f_e\left(x+y,t^e\right)\right|^p\chi\left(y:t^e+T^{e'}\right)dy\right)^{\frac{1}{p}-\frac{1}{r}}\left(\int\limits_{R^n}\left|M_e\right|^e dy\right)^{\frac{1}{s}-\frac{1}{r}};$$

$$\left\|I_e\left(\cdot,t^e+T^{e'}\right)\right\|_{r,U_{\rho^\chi}(\bar{x})}\le\sup_{x\in U_{\rho^\chi}(\bar{x})}\left(\int\limits_{R^n}\left|f_e\left(x+y,t^e\right)\right|^p\chi\left(y:t^e+T^{e'}\right)dy\right)^{\frac{1}{p}-\frac{1}{r}}\times$$

$$\times\sup_{y\in V}\left(\int\limits_{U_{\rho^\chi}(\bar{x})}\left|f_e\left(x+y,t^e\right)\right|^p dx\right)\left(\int\limits_{R^n}\left|M\left(e\frac{y}{t^e+T^{e'}},\frac{\rho(t,x)}{t^e+T^{e'}}\right)\right|^s dy\right)^{\frac{1}{s}}. \quad (6)$$

Set $G_{T^\chi}(U)=Q$. Since $U+V\subset Q;\ 0<t\le T\le 1,\ \chi\le 1\Rightarrow Q_{t^e+T^{e'}}(x)\subset Q_{\left(t^\chi\right)^e+\left(T^\chi\right)^{e'}}(x)$, then $\forall x\in U$ we have

$$\int\limits_{R^n}\left|f_e\left(x+y,t^e\right)\right|^p\chi\left(y:t^e+T^{e'}\right)dy\le$$

$$\le\int\limits_{(U+V)_{t^e+T^{e'}}(\bar{x})}\left|f_e\left(y,t^e\right)\right|^p dy\le\int\limits_{Q_{t^e+T^{e'}}(\bar{x})}\left|f_e\left(y,t^e\right)\right|^p dy\le$$

$$\le\prod_j t_j^{l_j p}\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|^p_{p,Q_{\left(t^\chi\right)^e+\left(T^\chi\right)^{e'}}(\bar{x})}\le$$

$$\le\prod_{j\in e}t_j^{l_j p}\prod_{j\in e}t_j^{\chi_j a_j}\prod_{j\in e'}T_j^{\chi_j a_j}\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|^p_{p,a,\chi}. \quad (7)$$

For $y\in V$

$$\int\limits_{U_{\rho^\chi}(\bar{x})}\left|f_e\left(x+y,t^e\right)\right|^p dx\le\int\limits_{Q_{\rho^\chi}(\bar{x}+y)}\left|f_e\left(x,t^e\right)\right|^p dx\le$$

$$\le\prod_{j\in e}t_j^{l_j p}\prod_{j\in e_n}\left[\rho_j\right]_1^{\chi_j a_j}\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|^p_{p,a,\chi}. \quad (8)$$

$$\int\limits_{R^n}\left|M_e\left(\frac{y}{t^e+T^{e'}},\frac{\rho(t,x)}{t^e+T^{e'}}\right)\right|^s dy=\prod_{j\in e}t_j\prod_{j\in e'}T_j\|M\|_s^s. \quad (9)$$

We obtain from (6)-(9)

$$\left\|I_e\left(\cdot,t^e+T^{\chi'}\right)\right\|_{q,U_{\rho^\chi}(\bar{x})}\le c\prod_{j\in e'}T_j^{1-\left(1-\chi_j a_j\right)\left(\frac{1}{p}-\frac{1}{r}\right)}\times$$

$$\leq \prod_{j\in e} t_j^{l_j+1-(1-\chi_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)} \prod_{j\in e_n} [\rho_j]_1^{\chi_j a_j \frac{1}{r}} \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{p,a,\chi}. \qquad (10)$$

we get from (5)-(10)

$$\left\| I_e\left(\cdot, t^e + T^{e'}\right)\right\|_{q,U_{\rho^\chi}(\bar{x})} \leq c \prod_{j\in e'} T_j^{1-(1-\chi_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)} \times$$

$$\times \prod_{j\in e} t_j^{l_j+1-(1-\chi_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)} \prod_{j\in e_n} [\rho_j]_1^{\chi_j a_j \frac{1}{r}} \prod_{j\in e_n} \rho_j^{\chi_j\left(\frac{1}{q}-\frac{1}{r}\right)} \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{p,a,\chi}. \qquad (11)$$

Allowing for the inequality

$$\|\cdot\|_{p,a,\chi,G} \leq C \|\cdot\|_{p,a,\chi,\tau,G}, \quad 1 \leq \tau \leq \infty.$$

We obtain from (11)

$$\left\| I_e\left(\cdot, t^e + T^{e'}\right)\right\|_{q,U_{\rho^\chi}(\bar{x})} \leq C \prod_{j\in e'} T_j^{1-(1-\chi_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)} \times$$

$$\times \prod_{j\in e} t_j^{l_j+1-(1-\chi_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)} \prod_{j\in e_n} [\rho_j]_1^{\chi_j a_j \frac{1}{r}} \prod_{j\in e_n} \rho_j^{\chi_j\left(\frac{1}{q}-\frac{1}{r}\right)} \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{p,a,\chi,\tau}. \qquad (12)$$

By a consecutive use of the inequality (12) in each variable separately we obtain an inequality for vector $\mathbf{p} = (p_1, ..., p_n)$, $q = (q_1, ..., q_n)$, $r = (r_1, ..., r_n)$

$$\sup_{\bar{\chi}\in U'} \left\| I_e\left(\cdot, t^e + T^{e'}\right)\right\|_{q,U_{\rho^\chi}(\bar{x})} \leq C \prod_{j\in e'} T_j^{1-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{r_j}\right)} \times$$

$$\times \prod_{j\in e} t_j^{l_j+1-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{r_j}\right)} \prod_{j\in e_n} [\rho_j]_1^{\chi_j a_j \frac{1}{r_j}} \prod_{j\in e_n} \rho_j^{\chi_j\left(\frac{1}{q_j}-\frac{1}{r_j}\right)} \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{p,a,\chi,\tau}. \qquad (13)$$

Corollary 1. *Assuming in the inequality (13)* $r \equiv \infty$ *if* $0 < \vec{\rho} \leq \vec{1}$, *or* $r \equiv q$, *if* $\vec{\rho} > 1$ *(for* $\tau = \infty$*) we get*

$$\sup_{\bar{\chi}\in U'} \left\| I_e\left(\cdot, t^e + T^{e'}\right)\right\|_{q,U_{\rho^\chi}(\bar{x})} \leq c_1 \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{\mathbf{p},a,\chi,\tau} \prod_{j\in e_n} [\rho_j]_1^{\chi_j a_j \frac{1}{r_j}},$$

$$\left\| I_e\left(\cdot, t^e + T^{e'}\right)\right\|_{q,b,\chi,U} \leq c_2 \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{\mathbf{p},a,\chi,\tau}. \qquad (14)$$

For $1 \leq \tau_1 < \tau_2 \leq \infty$

$$\left\| I_e\left(\cdot, t^e + T^{e'}\right)\right\|_{q,b,\chi,\tau_2,U} \leq c_3 \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{\mathbf{p},a,\chi,\tau_1} \qquad (15)$$

Lemma 2. *Let all the conditions of lemma 1 be fulfilled,* $\eta=(\eta_1,...,\eta_n)$, $0<\eta_j\le T_j$, $\alpha=(\alpha_1,...,\alpha_n)$, $\alpha_j\ge 0$ *be entire and let* $j=1,2,...,n$

$$\varepsilon_j=l_j-\alpha_j-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{q_j}\right), \tag{16}$$

$$F_{\eta^e}(x)=\prod_{j\in e'}T_j^{-1-\alpha_j}\int_{0^e}^{\eta^e}\prod_{j\in e}t_j^{-2-\alpha_j}I_e\left(x,t^e+T^{e'}\right)dt^e, \tag{17}$$

$$F_{\eta T^e}(x)=\prod_{j\in e'}T_j^{-1-\alpha_j}\int_{\eta^e}^{T^e}\prod_{j\in e}t_j^{-2-\alpha_j}I_e\left(x,t^e+T^{e'}\right)dt^e. \tag{18}$$

Then the inequality

$$\sup_{\overline{x}\in U}\left\|F_{\eta^e}\right\|_{q,U_{\rho^\chi}(\overline{x})}\le C_1\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|_{\mathbf{p},a,\chi,\tau}\times$$

$$\times\prod_{j\in e'}T_j^{-\alpha_j-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\prod_{j\in e_n}\left[\rho_j\right]_1^{\chi_j\frac{a_j}{q_j}}\prod_{j\in e}\eta_j^{\varepsilon_j},\quad (\varepsilon_j>0). \tag{19}$$

$$\sup_{\overline{x}\in U}\left\|F_{\eta T^e}\right\|_{q,U_{\rho^\chi}(\overline{x})}\le C_2\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|_{\mathbf{p},a,\chi,\tau}\times$$

$$\times\prod_{j\in e'}T_j^{-\alpha_j-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\prod_{j=e_n}\left[\rho_j\right]_1^{\chi_j\frac{a_j}{q_j}}\begin{cases}\prod_{j\in e}\eta_j^{\varepsilon_j}, & \varepsilon_j>0\\ \prod_{j\in e}\ln\frac{T_j}{\eta_j}, & \varepsilon_j=0\\ \prod_{j\in e}\eta_j^{\varepsilon_j}, & \varepsilon_j<0\end{cases} \tag{20}$$

hold.

Lemma 3. *Let* $1\le p_j\le q_j\le\infty$, $0<\chi_j\le 1$, $0<t_j\le T_j\le 1$, $a=(a_1,a_2,...,a_n)$, $0\le a_j\le 1$, $\alpha=(\alpha_1,...,\alpha_n)$, $\alpha_j\ge 0$ *be entire* $j\in e_n$, $1\le\tau_1\le\tau_2\le\infty$, *and let* $\varepsilon_j>0$, $j\in e_n$ *(* ε_j *has been defined in lemma 2)*

$$\varepsilon_j^0=l_j-\alpha_j-(1-\chi_j a_j)\frac{1}{p_j}.$$

Then the inequality

$$\left\|F_{T^e}\right\|_{q,b,\chi,\tau_2,U}\le C\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|_{\mathbf{p},a,\chi,\tau_1} \tag{21}$$

is valid. Here b_j *is a number that satisfies the inequalities*

$0\le b_j\le 1$, *if* $\varepsilon_j^0>0$ *for* $j\in e$,

$0\le b_j<1$ *if* $\varepsilon_j^0=0$ *for* $j\in e$; $\qquad 0\le b_j\le a_j$, *for* $j\in e'$ (22)

$$0 \le b_j < 1 + \frac{\varepsilon_j^n q_j (1-a_j)}{1-\chi_j a_j} = a_j + \frac{\varepsilon_j q_j (1-a_j)}{1-\chi_j a_j} \ \ if \ \ \varepsilon_j^0 < 0, \ for\ j \in e.$$

Theorem 1. *Let an open set* $G \subset R^n$ *satisfy the condition of flexible horn,* $1 < \mathbf{p} \le \mathbf{q} \le \infty$, $l^1, l \in (0,\infty)^n$, $\frac{1}{c} = \max l_j \chi_j$; $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \ge 0$ *be entire* $j \in e_n$; $1 < \tau_1 < \tau_2 \le \infty, 1 < \theta \le \theta_1 < \infty$; $\varepsilon_j > 0, j \in e_n$ *(*$\varepsilon_j, \varepsilon_j^0$ *have been defined in lemma 1 and 2) and let* $f \in S^l_{\mathbf{p},\theta,a,\chi,\tau_1} L(G)$, *then*

$$D^\alpha : S^l_{\mathbf{p},\theta,a,\chi,\tau_1} L(G) \to L_{q,b,\chi,\tau_2}(G)$$

moreover, it is valid the inequality

$$\left\| D^\alpha f \right\|_{q,G} \le C_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{\varepsilon_{e,j}} \times$$

$$\times \left\| \left\{ \int_{0^e}^{1^e} \left[\prod_{j \in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right]^\theta \prod_{j \in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}} \right\|_{\mathbf{p},a,\chi,\tau_1}, \tag{23}$$

$$\left\| D^\alpha f \right\|_{q,b,\chi,\tau_2,G} \le C_2 \|f\|_{S^l_{\mathbf{p},\theta,a,\chi,\tau_1} L(G)}, \quad (\mathbf{p} \le q < \infty), \tag{24}$$

where

$$s_{e,j} = \begin{cases} \varepsilon_j, & j \in e \\ -\alpha_j - (1-\chi_j a_j)\left(\frac{1}{p_j} - \frac{1}{q_j}\right), & j \in e' \end{cases}$$

and if $\varepsilon_j - l_j^1 > 0$, $j \in e_n$, *then*

$$D^\alpha : S^l_{\mathbf{p},\theta,a,\chi,\tau_1} L(G) \to S^{l^1}_{q,\theta_1,b,\chi,\tau_2} L(G)$$

$$D^\alpha : S^l_{\mathbf{p},\theta,a,\chi,\tau_1} L(G) \to S^{l^1}_{q,\theta_1,b,\chi,\tau_2} B(G)$$

and the inequalities

$$\left\| D^\alpha f \right\|_{S^{l^1}_{q,\theta_1} L(G)} \le C_3 \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j} - l_j^1} \times$$

$$\times \left\| \left\{ \int_{0^e}^{1^e} \left[\prod_{j \in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right]^\theta \prod_{j \in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}} \right\|_{\mathbf{p},a,\chi,\tau_1} \quad (p \le q < \infty), \tag{25}$$

$$\left\| D^\alpha f \right\|_{S^{l^1}_{q,\theta_1,b,\chi,\tau_2} L(G)} \le C_4 \|f\|_{S^l_{\mathbf{p},\theta,a,\chi,\tau_1} L(G)}, \quad (\mathbf{p} \le q < \infty), \tag{26}$$

$$\left\|D^{\alpha}f\right\|_{S^{l^1}_{q\theta_1}B(G)} \le C_5 \sum_{e\subseteq e_n}\prod_{j\in e_n}T_j^{s_{e,j}-l_j^1}\left\|\left\{\int_{0^e}^{1^e}\left[\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right]^{\theta}\prod_{j\in e}\frac{dt_j}{t_j}\right\}^{\frac{1}{\theta}}\right\|_{\mathbf{p},a,\chi,\tau_1}, \quad (27)$$

$$\left\|D^{\alpha}f\right\|_{S^{l^1}_{q\theta_1,b,\chi,\tau_2}B(G)} \le C_4\|f\|_{S^{l}_{\mathbf{p},\theta,a,\chi,\tau_1}L(G)}, \quad (\mathbf{p}\le q<\infty) \quad (28)$$

are valid. Moreover $T\le\min\{1,T_0\}$ *and* C_1,C_2,C_3,C_4,C_5,C_6 *is a constant not depending on* f *and* C_1,C_3 *and* C_5 *also doesn't depend on* T .

In particular if $\varepsilon_j^0>0 \quad j\in e_n$, then $D^{\alpha}f$ is continuous on G, and

$$\sup_{x\in G}\left|D^{\alpha}f\right| \le C_1 \sum_{e\subseteq e_n}\prod_{j\in e_n}T_j^{s^0_{e,j}}\left\|\left\{\int_{0^e}^{1^e}\left[\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right]^{\theta}\prod_{j\in e}\frac{dt_j}{t_j}\right\}^{\frac{1}{\theta}}\right\|_{\mathbf{p},a,\chi,\tau_1}, \quad (29)$$

where

$$s^0_{e,j}=\begin{cases}\varepsilon_j^0, & j\in e\\ -\alpha_j-(1-\chi_j a_j)\dfrac{1}{p_j}, & j\in e'\end{cases}$$

Proof. First we pass to χ in $\overline{\chi}=c\chi$ (the greater χ, the greater ε). Let $f\in S^{l}_{\mathbf{p},\theta,a,\chi,\tau_1}L(G)$. It follows from the condition $\varepsilon>0$ that $l_j-\alpha_j>0,\ j\in e_n$, since $\mathbf{p}\le q,\ 0\le a_j\le 1,\ \ j\in e_n$

$$f\in S^{l}_{\mathbf{p},\theta,a,\chi,\tau_1}L(G)\to S^{l}_{\mathbf{p},\theta,a,\chi}L(G)\to S^{l}_{\mathbf{p},\theta}L(G)$$

It follows from theorem 1 in [4] that there exists $D^{\alpha}f\in L_p(G)$, then for almost each point $x\in G$ it is valid the equality

$$D^{\alpha}f(x)=\sum_{e\subseteq e_n}(-1)^{|\alpha|}\prod_{j\in e'}T_j^{-1-\alpha_j}\int_{0^e}^{T^e}\prod_{j\in e}\frac{dt}{t_j^{2+\alpha_j}}\times$$
$$\times\int M_e^{(\alpha)}\left(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}}\right)f_e(x+y,t^e)dy, \quad (30)$$

where $M_e(\cdot,z)\in C_0^{\infty}(R^n)$ and the carrier of (30) is contained in $x+V(x,\theta)\subset G$. On the basis of the Minkowsky inequality we obtain

$$\left\|D^{\alpha}f\right\|_{q,G}\le C\sum_{e\subseteq e_n}\left\|F_{T^e}\right\|_{q,G}. \quad (31)$$

By means of the inequality (19), for $U=G,\eta_j=T_j,\ \rho_j\to\infty,\ r_j=q_j \quad j\in e_n$

$$\left\|F_{T^e}\right\|_{q,G}\le C_1\left\|\prod_{j\in e}t_j^{-l_j}\delta^{m^e}(\delta t)f\right\|_{\mathbf{p},a,\chi,\tau_1}\prod_{j\in e'}T_j^{-\alpha_j-(1-\chi_j a_j)\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\prod_{j\in e}T_j^{e_j} \quad (32)$$

consequently

$$\left\|D^{\alpha} f\right\|_{q,G} \le C_2 \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}} \left\|\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f\right\|_{\mathbf{p},a,\chi,\tau_1}$$

hence for $1<\theta<\infty$ we get

$$\left\|D^{\alpha} f\right\|_{q,G} \le C_3 \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}} \left\{ \int_{0^e}^{1^e} \left\|\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f\right\|_{\mathbf{p},a,\chi,\tau_1}^{\theta} \prod_{j\in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}}$$

and if $p_j \le \theta,\ j\in e_n$

$$\left\|D^{\alpha} f\right\|_{q,G} \le C_4 \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}} \left\| \left\{ \int_{0^e}^{1^e} \left[\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f\right]^{\theta} \prod_{j\in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}} \right\|_{\mathbf{p},a,\chi,\tau_1}$$

To prove the inequalities (25)- and (27) we estimate the norm

$$\left\|\Delta^{M^e}(h,G_h) D^{\alpha} f\right\|_{q}.$$

After some transformations we obtain from the identity (30) the following inequalities

$$\left|\Delta^{M^e}(h,G_h) D^{\alpha} f\right| \le C \sum_{e\subseteq e_n} \prod_{j\in e'} T_j^{-1-\alpha_j} \int_{0^e}^{H^e} \frac{dt^e}{\prod_{j\in e} t_j^{2+\alpha_j}} \int \left|M_e^{(\alpha)}\right| \left|\Delta^{M^e}(h) f_e(x+y,t^e)\right| dy +$$

$$+ C \prod_{j\in e} h_j^{M_j} \sum_{e\subseteq e_n} \prod_{j\in e'} T_j^{-1-\alpha_j} \int_{H^e}^{T^e} \frac{dt^e}{\prod_{j\in e} t_j^{2+\alpha_j+M_j}} \int \left|M_e^{(\alpha+M^e)}\right| \times$$

$$\times \int \left|M_e^{(\alpha+M^e)}\right| \int_{0^e}^{1^e} f_e\left(x+y+M_1^e h_1^e u_1^e + \ldots + M_n^e h_n^e u_n^e, t^e\right) du^e dy = C \sum_{e\subseteq e_n} \left(L_e^1(x) + L_e^2(x)\right), \quad (33)$$

$$\left\|\Delta^{M^e}(h,G_h) D^{\alpha} f\right\|_{q} \le C \sum_{e\subseteq e_n} \left(\left\|L_e^1\right\|_{q,G} + \left\|L_e^2\right\|_{q,G} \right). \quad (34)$$

With the help of the inequality (19), for $\rho\to\infty$, $H=T$ it follows that

$$\left\|L_e^1\right\|_{q,G} \le C \prod_{j\in e_n} T_j^{s_{e,j}} \left\|\prod_{j\in e} t_j^{-l_j} \Delta^{M^e}(h) \delta^{m^e}(\delta t) f\right\|_{\mathbf{p},a,\chi,\tau_1} \le$$

$$\le C_1 \prod_{j\in e_n} T_j^{s_{e,j}} \left\|\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f\right\|_{\mathbf{p},a,\chi,\tau_1}$$

With the help of the inequality (20), for $\rho\to\infty$, $\left(l_j^1 \le M_j; j\in e\right)$

$$\left\|L_e^2\right\|_{q,G} \le C^1 \prod_{j\in e} h_j^{M_j} \prod_{j\in e_n} T_j^{s_{e,j}-M_j} \left\|\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f\right\|_{\mathbf{p},a,\chi,\tau_1} \le$$

$$\le C^2 \prod_{j\in e} h_j^{l_j^1} \prod_{j\in e_n} T_j^{s_{e,j}-l_j^1} \left\|\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f\right\|_{\mathbf{p},a,\chi,\tau_1}.$$

Consequently for $\theta < \theta_1$, we get from (34)

$$\left\{ \int_{0^e}^{1^e} \left[\frac{\left\| \Delta^{M^e}(h, G_h) D^\alpha f \right\|_q}{\prod_{j\in e} h_j^{l_j^1}} \right]^{\theta_1} \prod_{j\in e} \frac{dh_j}{h_j} \right\}^{\frac{1}{\theta_1}} \le$$

$$\le C^3 \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}-l_j^1} \left\{ \int_{0^e}^{1^e} \left\| \prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right\|_{\mathbf{p},a,\chi,\tau}^{\theta} \prod_{j\in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}}$$

hence, if $p_j \le \theta$, $j \in e_n$ we get (after summing $|e|$ times) that

$$\left\| D^\alpha f \right\|_{S_{q,\theta_2}^{l^1} B(G)} \le C^4 \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}-l_j^1} \times$$

$$\times \left\| \left\{ \int_{0^e}^{1^e} \left[\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right]^{\theta} \prod_{j\in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{p}} \right\|_{\mathbf{p},a,\chi,\tau}$$

and, if $\theta_1 \le \min q_j$, $j \in e_n$ then

$$\left\| D^\alpha f \right\|_{S_{q,\theta_1}^{l^1} L(G)} \le C \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}-l_j^1} \times$$

$$\times \left\| \left\{ \int_{0^e}^{1^e} \left[\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right]^{\theta} \prod_{j\in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}} \right\|_{\mathbf{p},a,\chi,\tau}$$

(24), (26), (28) are established on the basis of the inequality (21). Assume $\varepsilon_j^0 > 0$, $j \in e_n$. Show that $D^\alpha f$ is continuous in G. On the basis of the inequality (23) for $q \equiv \infty$, $\varepsilon_j = \varepsilon_j^0 > 0$, $j \in e_n$ we have

$$\left\| D^\alpha f - D^\alpha f_T \right\|_{\infty,G} \le \sum_{0\ne e\subseteq e_n} \left\| F_{T^e} \right\|_{\infty,G} \le$$

$$\le \sum_{0\ne e\subseteq e_n} \prod_{j=1}^{n} T_j^{s_{e,j}-l_j^1} \left\| \left\{ \int_{0^e}^{1^e} \left[\prod_{j\in e} t_j^{-l_j} \delta^{m^e}(\delta t) f \right]^{\theta} \prod_{j\in e} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}} \right\|_{\mathbf{p},a,\chi,\tau}$$

$\left\| D^\alpha f - D^\alpha f_T \right\|_{\infty,G} = 0$. Since $D^\alpha f_T$ is continuous in G, the convergence $L_\infty(G)$ in this case coincides with uniformity and consequently $D^\alpha f$ is continuous in G. The theorem is proved.

It is proved that a generalized derivative $D^{\alpha} f$ satisfies the multiple Holder condition in metrics L_q for f of the structured space.

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