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ON SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER DEGENERATE ELLIPTICO-PARABOLIC EQUATIONS

Abstract

In the article the first boundary value problem for the second order elliptico-parabolic equations in divergent form is considered. The continuous differentiability of coefficients is assumed. The unique strong (almost everywhere) solvability of the formulated problem is proved.

Introduction. Let $\mathbf{R}_n$ be an n-dimensional Eucledian space of the points $x=(x_1,...,x_n)$, $\Omega\subset\mathbf{R}_n$ be a bounded n-dimensional domain with the boundary $\partial\Omega$, $B_R^{x^0}\subset\Omega$ be an n-dimensional open ball of the radius R with center at the points $x^0=(x_1^0,...,x_n^0)$, $Q_T=\{(x,t):x\in\Omega, 0<t<T<\infty\}$, $S_T=\{(x,t):x\in\partial\Omega, 0\le t\le T\}$, $\Gamma(Q_T)$ be a parabolic boundary of Q_T, i.e. $\Gamma(Q_T)=S_T\cup\{(x,t):x\in\Omega, t=0\}$, $Q_R^T=B_R^{x^0}\times[0,T]$, $A(Q_R^T)$ be a set of all functions $u(x,t)$ from $C^\infty(\overline{Q}_R^T)$ with support in $B_\rho^{x^0}\times[0,T]$, $\rho<R$, for which $u(x,0)=0$. Consider the following first boundary value problem in Q_T

$$Lu=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)+\frac{\partial}{\partial t}\left(\varphi(T-t)\frac{\partial u}{\partial t}\right)-\frac{\partial u}{\partial t}=f(x,t), \tag{1}$$

$$u\big|_{\Gamma(Q_T)}=0, \tag{2}$$

in assumption that $\|a_{ij}(x,t)\|$ is a real symmetric matrix, where for $(x,t)\in Q_T$, $\xi\in\mathbf{R}_n$

$$\gamma|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\gamma^{-1}|\xi|^2,\ \gamma\in(0,1]-const, \tag{3}$$

and besides

$$a_{ij}(x,t)\in C^{1,0}(\overline{Q}_T),\ i,j=1,...,n, \tag{4}$$

$$\varphi(0)=0,\ \varphi(z)>0,\ \varphi'(z)\ge0,\ \varphi''(z)\ge0, \varphi'(z)\ge\varphi(z)\varphi''(z);\ z\in(0,T). \tag{5}$$

The aim of the present article is the proof of the unique strong (almost everywhere) solvability of the boundary value problem (1)-(2) in corresponding Sobolev weight spaces for arbitrary $f(x,t)\in L_2(Q_T)$. Note that for the second order elliptic equations the analogous question is studied in [1-3] and for parabolic equations in [4-7]. As to the second order degenerate elliptico-parabolic equations we indicate papers [8-9], and also article [10] in which the strong solvability of the first boundary value problem for non-divergent structure equations with smooth coefficients was established.

1⁰. Auxiliary statements. Let $W_2^{1,0}(Q_T)$ and $W_{2,\varphi}^{2,2}(Q_T)$ be Banach spaces of measurable functions given on Q_T for which

$$\|u\|_{W_2^{1,0}(Q_T)}=\left(\int_{Q_T}\left(u^2+\sum_{i=1}^{n}u_i^2+\sum_{i,j=1}^{n}u_{ij}^2\right)dxdt\right)^{\frac{1}{2}}$$

and

$$\|u\|_{W^{2,2}_{2,\varphi}(Q_T)}=\left(\int_{Q_T}\left(u^2+\sum_{i=1}^{n}u_i^2+\sum_{i,j=1}^{n}u_{ij}^2+u_t^2+2\varphi(T-t)\sum_{i=1}^{n}u_{it}^2+\varphi^2(T-t)u_{tt}^2\right)dxdt\right)^{\frac{1}{2}}$$

are finite respectively, $\dot{W}^{2,2}_{2,\varphi}(Q_T)$ be a subsequence of $W^{2,2}_{2,\varphi}(Q_T)$, dense set in which it is totality of all functions from $C^\infty(\overline{Q_T})$ vanishing on $\Gamma(Q_T)$.

Here for $i,j=1,\dots,n$ $u_i=\dfrac{\partial u}{\partial x_i}$, $u_{ij}=\dfrac{\partial^2 u}{\partial x_i \partial x_j}$. Under the strong solution of the first boundary value problem (1)-(2) we'll understand the function $u(x,t)\in \dot{W}^{2,2}_{2,\varphi}(Q_T)$ satisfying a.e. in Q_T the equation

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x,t)u_{ij}+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x,t)u_i+\varphi(T-t)u_{tt}-(1+\varphi_t(T-t))u_t=f(x,t).$$

Lemma 1. *Let $u\in A(Q_R^T)$ and relative to the coefficients of the operator L the conditions (3)-(5) be satisfied. Then there exists the positive constants R_0 and T_0 depending only on coefficients of the operator L and n such that if $R\le R_0$, $T\le T_0$, then the following estimate is valid*

$$\|u\|_{W^{2,2}_{2,\varphi}(Q_R^T)}\le C_1\|Lu\|_{L_2(Q_R^T)}, \tag{6}$$

where $C_1=C_1(L,n)$ is a positive constant.

Proof. At first we consider the operator

$$L_0u=\Delta u+\varphi(T-t)u_{tt}-(1+\varphi_t(T-t))u_t.$$

Let's estimate norm L_2 by L_0u from below. We have

$$J=\int_{Q_R^T}(L_0u)^2dxdt=\int_{Q_R^T}\left[\Delta u+\varphi(T-t)u_{tt}-(1+\varphi_t(T-t))u_t\right]^2dxdt=$$

$$=\int_{Q_R^T}(\Delta u)^2dxdt+\int_{Q_R^T}\varphi^2(T-t)u_{tt}^2dxdt+\int_{Q_R^T}(1+\varphi_t(T-t))^2u_t^2dxdt+$$

$$+2\int_{Q_R^T}\varphi(T-t)u_{tt}\Delta u\,dxdt-2\int_{Q_R^T}(1+\varphi_t(T-t))u_t\Delta u\,dxdt-$$

$$-2\int_{Q_R^T}\varphi(T-t)(1+\varphi_t(T-t))u_tu_{tt}\,dxdt\equiv J_1+J_2+J_3+J_4+J_5+J_6.$$

For the integrals J_1,J_4,J_5,J_6 we have

$$J_1=\int_{Q_R^T}(\Delta u)^2dxdt=\sum_{i,j=1}^{n}\int_{Q_R^T}u_{ii}u_{jj}dxdt=-\sum_{i,j=1}^{n}\int_{Q_R^T}u_{iij}u_jdxdt=\sum_{i,j=1}^{n}\int_{Q_R^T}u_{ij}^2dxdt\ ,$$

$$J_4=2\sum_{i=1}^{n}\int_{Q_R^T}\varphi(T-t)u_{tt}u_{ii}dxdt=-2\sum_{i=1}^{n}\int_{Q_R^T}\varphi(T-t)u_iu_{itt}dxdt=$$

$$=2\sum_{i=1}^{n}\int_{Q_R^T}\left[\varphi(T-t)u_i\right]_tu_{it}dxdt=2\sum_{i=1}^{n}\int_{Q_R^T}\varphi(T-t)u_{it}^2dxdt-$$

$$-2\sum_{i=1}^{n}\int_{Q_R^T}\varphi_t(T-t)u_iu_{it}dxdt=2\sum_{i=1}^{n}\int_{Q_R^T}\varphi(T-t)u_{it}^2dxdt-$$

$$-\sum_{i=1}^{n}\int_{Q_R^T}\varphi_t(T-t)\left(u_i^2\right)_t dxdt=2\sum_{i=1}^{n}\int_{Q_R^T}\varphi(T-t)u_{it}^2dxdt-$$

$$-\sum_{i=1}^{n}\int_{Q_R^T}\varphi_{tt}(T-t)u_i^2dxdt,$$

$$J_5=-2\sum_{i=1}^{n}\int_{Q_R^T}(1+\varphi_t(T-t))v_{ii}u_tdxdt=2\sum_{i=1}^{n}\int_{Q_R^T}(1+\varphi_t(T-t))u_iu_{it}dxdt=$$

$$=\sum_{i=1}^{n}\int_{Q_R^T}(1+\varphi_t(T-t))\left(u_i^2\right)_tdxdt=\sum_{i=1}^{n}\int_{Q_R^T}\varphi_{tt}(T-t)u_i^2dxdt+\sum_{i=1}^{n}\int_{\Omega}u_i^2(x,T)dxdt\geq$$

$$\geq\sum_{i=1}^{n}\int_{Q_R^T}\varphi_{tt}(T-t)u_i^2dxdt,$$

$$J_6=-2\int_{Q_R^T}\varphi(T-t)(1+\varphi_t(T-t))u_tu_{tt}dxdt=-\int_{Q_R^T}\varphi(T-t)(1+\varphi_t(T-t))\left(u_t^2\right)_tdxdt=$$

$$=\int_{\Omega}\varphi(T)(1+\varphi_t(T))u_t^2(x,0)dx+\int_{Q_R^T}[\varphi(T-t)(1+\varphi_t(T-t))]_tu_t^2dxdt\geq$$

$$\geq\int_{Q_R^T}\left[-\varphi_t(T-t)-\varphi_t^2(T-t)-\varphi(T-t)\varphi_{tt}(T-t)\right]u_t^2dxdt.$$

Thus

$$J=J_1+J_2+J_3+J_4+J_5+J_6\geq\sum_{i,j=1}^{n}\int_{Q_R^T}u_{ij}^2dxdt+\int_{Q_R^T}\varphi^2(T-t)u_{tt}^2dxdt+$$

$$+\int_{Q_R^T}u_t^2dxdt+2\sum_{i=1}^{n}\int_{Q_R^T}\varphi(T-t)u_{it}^2dxdt.$$

Consequently,

$$\left(\int_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2+2\sum_{i=1}^{n}\varphi(T-t)u_{it}^2+\varphi^2(T-t)u_{tt}^2\right)dxdt\right)^{\frac{1}{2}}\leq\left(\int_{Q_R^T}(L_0u)^2dxdt\right)^{\frac{1}{2}}.\quad(7)$$

It's obvious that for the operator

$$L_0^{\left(x^0,t^0\right)}u=\sum_{i,j=1}^{n}a_{ij}\left(x^0,t^0\right)u_{ij}+\varphi(T-t)u_{tt}-(1+\varphi_t(T-t))u_t,$$

where $\left(x^0,t^0\right)\in Q_R^T$, the inequality (7) holds.

Consider the operator

$$L_1u=\sum_{i,j=1}^{n}a_{ij}(x,t)u_{ij}+\varphi(T-t)u_{tt}-(1+\varphi_t(T-t))u_t.$$

Let's estimate L_2 norm by L_1u from below. We have

$$\left\|L_0^{(x^0,t^0)}u\right\|_{L_2(Q_R^T)}=\left\|\left(L_0^{(x^0,t^0)}-L_1+L_1\right)u\right\|_{L_2(Q_R^T)}\le\|L_1u\|_{L_2(Q_R^T)}+$$

$$+\left\|\left(L_0^{(x^0,t^0)}-L_1\right)u\right\|_{L_2(Q_R^T)}\le\|L_1u\|_{L_2(Q_R^T)}+\sum_{i,j=1}^{n}\left|a_{ij}(x,t)-a_{ij}(x^0,t^0)\right|\|u_{ij}\|_{L_2(Q_R^T)}.$$

Since $a_{ij}(x,t)\in C^{1,0}(\overline{Q}_R^T)$, then for arbitrary $\varepsilon>0$ we can find $\delta>0$ such that $|a_{ij}(x,t)-a_{ij}(x^0,t^0)|<\varepsilon$, if as only $\rho[(x,t);(x^0,t^0)]<\delta$. Let $\varepsilon>0$ be chosen later. Then there exist R_1 and T_0 such that if $R\le R_1$ and $T\le T_0$, then

$$\left\|L_0^{(x^0,t^0)}u\right\|_{L_2(Q_R^T)}\le\|L_1u\|_{L_2(Q_R^T)}+\varepsilon\sum_{i,j=1}^{n}\|u_{ij}\|_{L_2(Q_R^T)}.$$

Taking into account this inequality in (7) we obtain

$$\int_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2+2\sum_{i=1}^{n}\varphi(T-t)u_{it}^2+\varphi^2(T-t)u_{tt}^2\right)dxdt\le$$

$$\le\left(\|L_1u\|_{L_2(Q_R^T)}+\varepsilon\sum_{i,j=1}^{n}\|u_{ij}\|_{L_2(Q_R^T)}\right)^2\le 2\left(\|L_1u\|^2_{L_2(Q_R^T)}+\varepsilon^2n^2\sum_{i,j=1}^{n}\|u_{ij}\|^2_{L_2(Q_R^T)}\right).$$

Hence assuming $\varepsilon=\dfrac{1}{2n}$ we obtain

$$\int_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2+2\sum_{i=1}^{n}\varphi(T-t)u_{it}^2+\varphi^2(T-t)u_{tt}^2\right)dxdt\le C_2\|L_1u\|^2_{L_2(Q_R^T)}.$$

Let $K_R^T=\{|x_i-x_i^0|<R\}\times(0,T)$. We extend the function $u(x,t)$ by zero in K_R^T. Let's fix $t'\in(0,T)$ and let $x'=(x_2,...,x_n)$. We have for $x_1\in(x_1^0-R,x_1^0+R)$

$$u(x_1,x',t')=u(x_1^0-R,x',t')+\int_{x_1^0-R}^{x_1}\frac{\partial}{\partial x_1}u(\tau,x',t')d\tau=\int_{x_1^0-R}^{x_1}\frac{\partial}{\partial x_1}u(\tau,x',t')d\tau.$$

Using the Hölder inequality we obtain

$$u^2(x_1,x',t')=\left(\int_{x_1^0-R}^{x_1}\frac{\partial}{\partial x_1}u(\tau,x',t')d\tau\right)^2\le\int_{x_1^0-R}^{x_1}d\tau\int_{x_1^0-R}^{x_1}\left(\frac{\partial}{\partial x_1}u(\tau,x',t')\right)^2d\tau\le$$

$$\le\int_{x_1^0-R}^{x_1^0+R}d\tau\int_{x_1^0-R}^{x_1^0+R}\frac{\partial}{\partial x_1}u(\tau,x',t')^2d\tau=2R\int_{x_1^0-R}^{x_1^0+R}\left(\frac{\partial u(\tau,x',t')}{\partial x_1}\right)^2d\tau.$$

Integrate both sides of the last inequality with respect to K_R^T

$$\int_{K_R^T}u^2(x_1,x',t')dxdt'\le\int_{K_R^T}\left[2R\int_{x_1^0-R}^{x_1^0+R}\left(\frac{\partial u(x_1,x',t')}{\partial x_1}\right)^2d\tau\right]dxdt\le 4R^2\int_{K_R^T}\left(\frac{\partial u(x_1,x',t')}{\partial x_1}\right)^2dxdt'.$$

Hence

$$\int_{Q_R^T}u^2(x,t)dxdt\le 4R^2\int_{Q_R^T}\left(\frac{\partial u(x_1,x',t)}{\partial x_1}\right)^2dxdt,$$

since $u(x,t)\equiv 0$ outside of Q_R^T.

Thus

$$\int_{Q_R^T} u^2 dxdt \le 4R^2 \int_{Q_R^T} \left(\frac{\partial u}{\partial x_1}\right)^2 dxdt \le 4R^2 \int_{Q_R^T} \sum_{i=1}^{n} u_i^2 dxdt . \qquad (8)$$

Analogous to (8) for U_i we obtain

$$\sum_{i=1}^{n} \int_{Q_R^T} u_i^2 dxdt \le 4R^2 \sum_{i,j=1}^{n} \int_{Q_R^T} u_{ij}^2 dxdt . \qquad (9)$$

Allowing for (8)-(9) we conclude

$$\int_{Q_R^T} \left(u^2 + \sum_{i=1}^{n} u_i^2 + \sum_{i,j=1}^{n} u_{ij}^2 +_t^2 + 2\varphi(T-t)\sum_{i=1}^{n} u_{it}^2 + \varphi^2(T-t)u_{tt}^2 \right) dxdt \le C_3 \int_{Q_R^T} (L_1 u)^2 dxdt, \ (10)$$

where $C_3 = C_3(R_1)$. We have further

$$\int_{Q_R^T} (L_1 u)^2 dxdt \le 2 \int_{Q_R^T} (L_1 u - Lu)^2 dxdt + 2 \int_{Q_R^T} (Lu)^2 dxdt$$

$$2 \int_{Q_R^T} (L_1 u - Lu)^2 dxdt = 2 \int_{Q_R^T} \left(\sum_{i=1}^{n} b_i(x,t) u_i \right)^2 dxdt \le$$

$$\le 2b_0^2 \int_{Q_R^T} \left(\sum_{i=1}^{n} u_i \right)^2 dxdt \le 2b_0^2 n^2 \int_{Q_R^T} \sum_{i=1}^{n} u_i^2 dxdt \le$$

$$\le 8b_0^2 n^2 R^2 \int_{Q_R^T} \sum_{i,j=1}^{n} u_{ij}^2 dxdt,$$

where

$$b_i(x,t) = \frac{\partial a_{ij}(x,t)}{\partial x_i}, \ b_0 \ge |b_i(x,t)|; \ i = 1,...,n.$$

Thus

$$\int_{Q_R^T} (L_1 u)^2 dxdt \le 8b_0^2 n^2 R^2 \int_{Q_R^T} \sum_{i,j=1}^{n} u_{ij}^2 dxdt + 2\int_{Q_R^T} (Lu)^2 dxdt . \qquad (11)$$

Allowing for (11) in (10) we obtain

$$\int_{Q_R^T} \left(u^2 + \sum_{i=1}^{n} u_i^2 + \sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 + 2\varphi(T-t)\sum_{i=1}^{n} u_{it}^2 + \varphi^2(T-t)u_{tt}^2 \right) dxdt \le$$

$$\le 8b_0^2 n^2 R^2 C_3 \int_{Q_R^T} \sum_{i,j=1}^{n} u_{ij}^2 dxdt + 2C_3 \int_{Q_R^T} (Lu)^2 dxdt .$$

we subordinate the number R_2 to the constraint $8b_0^2 n^2 R^2 C_3 < \frac{1}{2}$. Then if $R \le R_0 = \min\{R_1, R_2\}$, then

$$\int_{Q_R^T}\left(u^2+\sum_{i=1}^n u_i^2+\sum_{i,j=1}^n u_{ij}^2+u_t^2+2\varphi(T-t)\sum_{i=1}^n u_{it}^2+\varphi^2(T-t)u_{tt}^2\right)dxdt\le C_4\int_{Q_R^T}(Lu)^2\,dxdt,$$

where $C_4=C_4(L,n)$. From here the required estimate (6) with $C_1=C_4$ follows. The lemma is proved.

Lemma 2. *Let relative to the coefficients of the operator L the conditions (3)-(5) be satisfied and $R\le R_0, T\le T_0$. Let $u(x,t)\in C^\infty\left(\overline{Q_R^T}\right), u(x,0)=0$. Then for arbitrary $r\in(0,R)$ the estimate*

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_r^T)}\le C_5R^{-2}\left(1-\frac{r}{R}\right)^{-2}\left(\|Lu\|_{L_2(Q_R^T)}+\|u\|_{W_2^{1,0}(Q_R^T)}\right), \tag{12}$$

where $C_5=C_5(L,n)$ is a positive constant, is valid.

Proof. Let's consider the following function $\eta(x)=1$ at $x\in B_r^{x^0}$, $\eta(x)=0$, at $x\notin B_{\frac{R+r}{2}}^{x^0}$, $0\le\eta(x)\le1$, $\eta(x)\in C_0^\infty\left(B_R^{x^0}\right)$, where for $x\in B_R^{x^0}$

$$|\eta_i|\le\frac{C_6}{R-r},\quad |\eta_{ij}|\le\frac{C_6}{(R-r)^2},\ i,j=1,...,n; \tag{13}$$

where $C_6=C_6(n)$. Since $u(x,t)\eta(x)\in A\left(Q_R^T\right)$ then lemma 1 is applicable to $u(x,t)\eta(x)$ according to which

$$\|U\|_{W_{2,\varphi}^{2,2}(Q_r^T)}\le C_1\|L(U\eta)\|_{L_2(Q_R^T)}. \tag{14}$$

But on the other hand

$$L(u\eta)=\eta Lu+uL\eta+2\sum_{i,j=1}^n a_{ij}(x,t)u_i\eta_j.$$

Not losing generality we can assume that $R\le1$. The last equality subject to (13) implies the estimate

$$|L(u\eta)|\le|Lu|+\frac{C_7}{(R-r)^2}|u|+\frac{C_8}{R-r}\sum_{i=1}^n|u_i|, \tag{15}$$

where $C_7=C_7(L,n)$, $C_8=C_8(L,n)$.

Allowing for (15) in (14) we obtain

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_r^T)}\le C_9\|Lu\|_{L_2(Q_R^T)}+\frac{C_{10}}{(R-r)^2}\left(\|u\|_{L_2(Q_R^T)}+\sum_{i=1}^n\|u_i\|_{L_2(Q_R^T)}\right),$$

where the positive constants C_9 and C_{10} depend only on L and n.

By virtue of the last inequality

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_r^T)}\le\frac{C_{11}}{(R-r)^2}\left(\|Lu\|_{L_2(Q_R^T)}+\|u\|_{W_2^{1,0}(Q_R^T)}\right),$$

where $C_{11}=\max(C_9,C_{10})$. Hence the required estimate (12) with $C_5=C_{11}$ follows. The lemma is proved.

2^0. Basic coercive estimate.

Lemma 3. *Let relative to the coefficients of the operator L the conditions (3)-(5) be satisfied and $R\le R_0, T\le T_0$. Then for arbitrary $\rho\le1$ there exists a positive* *constant C_{12} depending only on L, n, ρ and the domain Ω such that for any function $u(x,t) \in W_{2,\varphi}^{2,2}(Q_T)$ the inequality*

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_T^\rho)} \le C_{12}\left(\|Lu\|_{L_2(Q_T)} + \|u\|_{L_2(Q_T)}\right), \tag{16}$$

where $Q_T^\rho = \Omega_\rho \times (0,T)$, $\Omega_\rho = \{x : x \in \Omega,\ dist(x, \partial\Omega) > \rho\}$, is valid.

Proof. Assume

$$A = \sup_{r \in (0,R)} \left\{ \left(1 - \frac{r}{R}\right)^2 \|u\|_{W_{2,\varphi}^{2,2}(Q_r^T)} \right\}.$$

Then there exists R_1, $0 < R_1 < R$ such that

$$A \le 2\left(1 - \frac{R_1}{R}\right)^2 \|u\|_{W_{2,\varphi}^{2,2}(Q_{R_1}^T)}. \tag{17}$$

Using lemma 2 for any $R_2 \in (R_1, R)$ from (17) we obtain

$$A \le 2\left(1 - \frac{R_1}{R}\right)^2 \|u\|_{W_{2,\varphi}^{2,2}(Q_{R_1}^T)} \le 2C_5 R_2^{-2}\left(1 - \frac{R_1}{R}\right)^2 \left(1 - \frac{R_1}{R_2}\right)^{-2} \times$$

$$\times\left(\|Lu\|_{L_2(Q_{R_2}^T)} + \|u\|_{W_2^{1,0}(Q_{R_2}^T)}\right) \le 2C_5 R_2^{-2}\left(1 - \frac{R_1}{R}\right)^2 \left(1 - \frac{R_1}{R_2}\right)^{-2} \times$$

$$\times\left(\|Lu\|_{L_2(Q_R^T)} + \|u\|_{W_2^{1,0}(Q_{R_2}^T)}\right). \tag{18}$$

Now we use the interpolational inequality according to which for arbitrary $\varepsilon > 0$

$$\|u\|_{W_2^{1,0}(Q_{R_2}^T)} \le \varepsilon \|u\|_{W_{2,\varphi}^{2,2}(Q_{R_2}^T)} + C_{13}\|u\|_{L_2(Q_{R_2}^T)},$$

where $C_{13} = C_{13}(\varepsilon, n)$.

Thus using the interpolational inequality we find from (18)

$$A \le 2C_5 R_2^{-2}\left(1 - \frac{R_1}{R}\right)^2 \left(1 - \frac{R_1}{R_2}\right)^{-2} \left(\|Lu\|_{L_2(Q_R^T)} + \varepsilon \|u\|_{W_{2,\varphi}^{2,2}(Q_{R_2}^T)} + C_{13}\|u\|_{L_2(Q_{R_2}^T)}\right) \le$$

$$\le 2C_5 R_2^{-2}\left(1 - \frac{R_1}{R}\right)^2 \left(1 - \frac{R_1}{R_2}\right)^{-2} \left(1 - \frac{R_2}{R}\right)^{-2} \varepsilon A + 2C_5 R_2^{-2}\left(1 - \frac{R_1}{R}\right)^2 \left(1 - \frac{R_1}{R_2}\right)^{-2} \|Lu\|_{L_2(Q_R^T)} +$$

$$+ 2C_5 R_2^{-2}\left(1 - \frac{R_1}{R}\right)^2 \left(1 - \frac{R_1}{R_2}\right)^{-2} C_{13}\|u\|_{L_2(Q_R^T)}. \tag{19}$$

Assume now $\delta = 1 - \frac{R_1}{R}$ and choose $R_2 \in (R_1, R)$ so that $1 - \frac{R_2}{R} = \frac{\delta}{2}$. We fix the chosen R_2. Since

$$1 - \frac{R_1}{R} = 2\left(1 - \frac{R_2}{R}\right),$$

then $1 - \frac{R_1}{R_2} = \frac{R}{R_2} - 1 > 1 - \frac{R_2}{R} = \frac{\delta}{2}$.

Therefore

$$2C_5R_2^{-2}\left(1-\frac{R_1}{R}\right)^2\left(1-\frac{R_1}{R_2}\right)^{-2}\left(1-\frac{R_2}{R}\right)<32C_5\delta^{-2}R_2^{-2}.$$

We choose and fix $\varepsilon=\dfrac{\delta^2R_2^2}{64C_5}$. Then from (19) we obtain

$$A\le C_{14}R_2^{-2}\left(\|Lu\|_{L_2(Q_R^T)}+\|u\|_{L_2(Q_R^T)}\right),$$

and further

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_{R_2}^T)}\le C_{14}\left(1-\frac{R_2}{R}\right)^{-2}R_2^{-2}\left(\|Lu\|_{L_2(Q_R^T)}+\|u\|_{L_2(Q_R^T)}\right),\tag{20}$$

where $C_{14}=C_{14}(L,n)$.

We can interpret the inequality (20) some differently. Let $\rho\in(0,1]$ be such that $\overline{B}_\rho^{x^0}\subset\Omega$. Then there exist $\rho_1=\rho_1(\rho)\in(0,\rho)$ and a positive constant C_{15} depending only on L,n,ρ for which the estimate

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_{\rho_1}^T)}\le C_{15}\left(\|Lu\|_{L_2(Q_\rho^T)}+\|u\|_{L_2(Q_\rho^T)}\right)\tag{21}$$

is valid.

Let the number ρ and by the same taken subdomain Ω_ρ be already given. We cover $\overline{\Omega_\rho}$ by a system of the balls $\left\{B_{\rho_1}^x\right\}$ and select from this covering the finite subcovering $\left\{B_{\rho_1}^{x^i}\right\}$, $i=1,\dots,N$. The number N obviously depends only on ρ,n and the domain Ω. Using now the estimate (21) for the cylinder $B_{\rho_1}^{x^i}\times(0,T)$ and summing by i from 1 to N we obtain

$$\|u\|^2_{W_{2,\varphi}^{2,2}(Q_T^\rho)}\le\sum_{i=1}^n\|u\|^2_{W_{2,\varphi}^{2,2}\left(B_{\rho_1}^{x^i}\times(0,T)\right)}\le 2C_{15}^2N\left(\|Lu\|^2_{L_2(Q_T)}+\|u\|^2_{L_2(Q_T)}\right),$$

where the required estimate (16) with $C_{12}=C_{15}\sqrt{2N}$ follows. The lemma is proved.

For the validity of the following theorem we need a condition on the domain Ω and namely we'll assume that the boundary $\partial\Omega\in C^2$.

Theorem 1. *Let relative to the coefficients of the operator L the conditions (3)-(5) be satisfied and $T\le T_0$, $\partial\Omega\in C^2$. Then there exists a positive constant C_{16} depending only on L,n,ρ and the domain Ω such that for any function $u(x,t)\in\dot{W}_{2,\varphi}^{2,2}(Q_T)$ the inequality*

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_T)}\le C_{16}\left(\|Lu\|_{L_2(Q_T)}+\|u\|_{L_2(Q_T)}\right)\tag{22}$$

is valid.

Proof. It's sufficient to prove the estimate (22) for smooth functions from $\dot{W}_{2,\varphi}^{2,2}(Q_T)$. We fix an arbitrary point $x^0\in\partial\Omega$. Since $\partial\Omega\in C^2$, then there exists non-degenerate transformation of the coordinates $x\leftrightarrow y$ such that if y^0 and $\partial\widetilde{\Omega}$ are correspondingly images of the point x^0 and the boundary $\partial\Omega$ for such transformation, then at some neighborhood of the point $y^0\partial\widetilde{\Omega}$ is given by the equation $y_n=0$ (in addition if $\widetilde{\Omega}$ is an image of the domain Ω, then for the points y belonging to the intersection of $\tilde{\Omega}$ with above mentioned neighbourhood, $y_n > 0$). Denote by $A_\rho(x^0) = \{x : |x - x^0| < 2\rho\}$ an open set which at such transformation passes to the semiball $\tilde{A}^{y^0}_{\rho,+} = \{y : |y - y^0| < 2\rho, y_n > 0\}$. Let $\tilde{A}^{y^0}_{\rho,-} = \{y : |y - y^0| < 2\rho, y_n > 0\}$, $C_{2\rho,+} = \tilde{A}^{y^0}_{\rho,+} \times (0,T)$, $C_{2\rho,-} = \tilde{A}^{y^0}_{\rho,-} \times (0,T)$, $C_{2\rho} = \{y : |y - y^0| < 2\rho\} \times (0,T)$.

Let $\tilde{u}(y,t)$ be a image of the function $u(x,t)$, and $\tilde{L}$ be an image of the operator L. It's obvious that the operator $\tilde{L}$ is an operator of such type as L. We extend the function $\tilde{u}(y,t)$ through the hyperplane $y_n = 0$ in an odd way and the coefficient of the operator $\tilde{a}_{ij}(y,t)$ in an even way to the semi-cylinder $C_{4\rho,-}$. It's obvious that $\tilde{u}(y,t) \in W^{2,2}_{2,\varphi}(C_{4\rho})$.

We use lemma 3

$$\|\tilde{u}\|_{W^{2,2}_{2,\varphi}(C_{2\rho})} \le C_{17}\left(\|\tilde{L}\tilde{u}\|^2_{L_2(C_{4\rho})} + \|\tilde{u}\|^2_{L_2(C_{4\rho})}\right),$$

where C_{17} depends only on $\tilde{L}, n, \rho$ and the domain Ω. Remembering the method of extension of the function $\tilde{u}(y,t)$ and the coefficient $\tilde{a}_{ij}(y,t)$ of the operator $\tilde{L}$ to the semicylinder $C_{4\rho,-}$ we conclude

$$\|\tilde{u}\|^2_{W^{2,2}_{2,\varphi}(C_{2\rho,+})} \le C_{17}\left(\|\tilde{L}\tilde{u}\|^2_{L_2(C_{4\rho,+})} + \|\tilde{u}\|^2_{L_2(C_{4\rho,+})}\right),$$

or in the variables x

$$\|u\|_{W^{2,2}_{2,\varphi}[A_\rho(x^0)\times(0,T)]} \le C_{18}\left(\|Lu\|^2_{L_2(Q_T)} + \|u\|^2_{L_2(Q_T)}\right), \tag{23}$$

where the constant C_{18} depends only on L, n, ρ and the domain Ω.

Extracting the inequality of the form (23) for the sets $A_\rho(x^i) \times (0,T)$, $i = 0,1,\dots,M$ and summing, we obtain

$$\|u\|^2_{W^{2,2}_{2,\varphi}[(\Omega\backslash\Omega_\rho)\times(0,T)]} \le (M+1)C_{18}\left(\|Lu\|^2_{L_2(Q_T)} + \|u\|^2_{L_2(Q_T)}\right). \tag{24}$$

On the other hand according to lemma 3

$$\|u\|^2_{W^{2,2}_{2,\varphi}(\Omega_\rho\times(0,T))} \le 2C^2_{12}\left(\|Lu\|^2_{L_2(Q_T)} + \|u\|^2_{L_2(Q_T)}\right). \tag{25}$$

From (24)-(25) it follows that

$$\|u\|^2_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{19}\left(\|Lu\|_{L_2(Q_T)} + \|u\|_{L_2(Q_T)}\right). \tag{26}$$

Whence the required estimate (22) with

$$C_{16} = C_{19} = \sqrt{(M+1)C_{18} + 2C^2_{12}}$$

follows.

The theorem is proved.

Corollary. *Let the conditions of theorem 1 be satisfied. There exists $T^0 = T^0(L,n)$ such that for arbitrary function $u(x,t) \in \dot{W}^{2,2}_{2,\varphi}(Q_T)$ the estimate*

$$\|u\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{20}\|Lu\|_{L_2(Q_T)}$$

is valid.

Proof. Let $t \in (0,T)$. We have

$$u(x,t) = \int_0^t \frac{\partial u(x,\tau)}{\partial t} d\tau .$$

Using the Hölder inequality we obtain

$$u^2(x,t) \le \int_0^t \left[\frac{\partial u(x,\tau)}{\partial t}\right]^2 d\tau \int_0^t dt \le T \int_0^t \left[\frac{\partial u(x,\tau)}{\partial t}\right]^2 d\tau .$$

Integrate the both sides with respect to Q_T

$$\int_{Q_T} u^2(x,t)dxdt \le T^2 \int_{Q_T} \left[\frac{\partial u(x,t)}{\partial t}\right]^2 dxdt .$$

Hence

$$\|u\|_{L_2(Q_T)} \le T\|u_t\|_{L_2(Q_T)} \le T\|u\|_{W^{2,2}_{2,\varphi}(Q_T)} .$$

On the other hand

$$\|u\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{19}\|Lu\|_{L_2(Q_T)} + C_{19}T\|u\|_{W^{2,2}_{2,\varphi}(Q_T)} .$$

Let $T_1 = C_{19}T_0 < \frac{1}{2}$. Then if $T \le T^0 = \min\{T_0, T_1\}$ then

$$\|u\|_{W^{2,2}_{2,\varphi}(Q_T)} \le 2C_{19}\|Lu\|_{L_2(Q_T)} .$$

Hence the corollary follows with $C_{20} = 2C_{19}$.

3^0. Solvability of the first boundary value problem for model equation.

By fulfilling the condition (5) consider the following first boundary value problem

$$L_0 u = \Delta u + \frac{\partial}{\partial t}\left[\varphi(T-t)\frac{\partial u}{\partial t}\right] - \frac{\partial t}{\partial t} = f(x,t),\ f(x,t) \in L_2(Q_T), \tag{27}$$

$$u\big|_{\Gamma(Q_T)} = 0 . \tag{28}$$

Theorem 2. *Let with respect to the function $\varphi(T-t)$ the condition (5) be satisfied, $T \le T^0$ and $\partial\Omega \in C^2$. Then for any $f(x,t) \in L_2(Q_T)$ the first boundary value problem (27)-(28) in unique strong (almost everywhere) solvable in $\dot{W}^{2,2}_{2,\varphi}(Q_T)$.*

Proof. At first we consider the case $f \in C^\infty(\overline{Q}_T)$. Introduce in consideration the function

$$\varphi_\varepsilon(z) = \begin{cases} \varphi(\varepsilon),\ z \le \varepsilon \\ \varphi(z),\ z > \varepsilon . \end{cases}$$

it's known that the boundary value problem

$$L_0^\varepsilon u^\varepsilon = \Delta u^\varepsilon + \frac{\partial}{\partial t}\left[\varphi_\varepsilon(T-t)\frac{\partial u^\varepsilon}{\partial t}\right] - \frac{\partial u^\varepsilon}{\partial t} = f(x,t),$$

$$u^\varepsilon\big|_{\Gamma(Q_T)} = 0,\ u^\varepsilon\big|_{t=T} = \vartheta(x,t),$$

where $\vartheta(x,t)$ is a solution of the problem

$$\Delta\vartheta - \frac{\partial\vartheta}{\partial t} = f$$

$$\vartheta\big|_{\Gamma(Q_T)}=0$$

has the unique solution u^{ε} belonging to the space $\dot{W}_2^{2,2}(Q_T)$. From the determination of φ_ε it follows that $u^{\varepsilon}\in\dot{W}_{2,\varphi_\varepsilon}^{2,2}(Q_T)$ and all the more $u^{\varepsilon}\in\dot{W}_{2,\varphi}^{2,2}(Q_T)$. From the corollary of theorem 1 we have

$$\left\|u^{\varepsilon}\right\|_{W_{2,\varphi}^{2,2}(Q_T)}\le C_{21}\left\|L_0^{\varepsilon}u^{\varepsilon}\right\|_{L_2(Q_T)}=C_{21}\|f\|_{L_2(Q_T)},$$

where C_{21} depends only on n,φ_ε and the domain Ω. By virtue of uniform boundedness of u^{ε} in the space $\dot{W}_{2,\varphi}^{2,2}(Q_T)$ we conclude that there exists a subsequence $\{u^{\varepsilon_k}(x,t)\}$, $\varepsilon_k\to 0$ for $k\to\infty$ weakly convergent to the function $u\in\dot{W}_{2,\varphi}^{2,2}(Q_T)$. Therefore for any $\omega(x,t)\in C_0^{\infty}(Q_T)$ the relation

$$\lim_{k\to\infty}\left(L_0u^{\varepsilon_k},\omega\right)=\left(L_0u,\omega\right) \tag{29}$$

is satisfied.

Thus

$$\left(L_0u^{\varepsilon_k},\omega\right)=\left(L_0^{\varepsilon_k}u^{\varepsilon_k},\omega\right)+\left[\left(L_0-L_0^{\varepsilon_k}\right)u^{\varepsilon_k},\omega\right]\to(f,\omega)\ (k\to\infty).$$

Then from (29) we obtain $(L_0u,\omega)=(f,\omega)$ for any $\omega\in C_0^{\infty}(Q_T)$. Consequently $L_0u=f$ a.e. in Q_T. Let now $f\in L_2(Q_T)$. Then its known that there exists the sequence $f^k\in C^{\infty}(\overline{Q}_T)$ such that $\left\|f^k-f\right\|_{L_2(Q_T)}\to 0\ (k\to\infty)$.

Let u_k be a solution of the problem

$$L_0u_k=f^k,\ u_k\big|_{\Gamma(Q_T)}=0.$$

Then

$$\|u_k\|_{W_{2,\varphi}^{2,2}(Q_T)}\le C_{22},$$

where C_{22} depends only on f,n,φ and the domain Ω.

Let the sequence $\{u_{k_l}(x,t)\}$ convergence to $u(x,t)$ weakly in $\dot{W}_{2,\varphi}^{2,2}(Q_T)$.

Then

$$\lim_{l\to\infty}\left(L_0u_{k_l},\omega\right)=\left(L_0u,\omega\right)$$

for any $\omega(x,t)\in C_0^{\infty}(Q_T)$. On the other side

$$L_0u_{k_l}=f^{k_l}\to f\ (l\to\infty).$$

Consequently we obtain $L_0u=f$ a.e. in Q_T.

Thus we obtain that the problem (27)-(28) has a strong solution in the space $\dot{W}_{2,\varphi}^{2,2}(Q_T)$. We show its uniqueness. Let u_1 and u_2 be the solutions of the same problem (27)-(28). Then on the basis of the corollary to theorem 1

$$\|u_1-u_2\|_{W_{2,\varphi}^{2,2}(Q_T)}=0.$$

Consequently $u_1(x,t)=u_2(x,t)$ a.e. in Q_T. The theorem is proved.

4^0. Solvability of a boundary value problem for a general equation.

Theorem 3. *Let the conditions (3)-(5) be satisfied $T\le T^0$ and $\partial\Omega\in C^2$. Then for any $f(x,t)\in L_2(Q_T)$ the first boundary value problem (1)-(2) is unique strong (almost* *everywhere) solvable in* $\dot{W}_{2,\varphi}^{2,2}(Q_T)$. *In addition for the solution* $u(x,t)$ *of the problem (1)-(2) the estimate*

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_T)} \le C_{23}\|f\|_{L_2(Q_T)}, \tag{30}$$

is valid, where the positive constant C_{23} *depends only on* L, n *and the domain* Ω.

Proof. For the proof of the theorem we use the continuation method by a parameter. Introduce for $t\in[0,1]$ in consideration a family of the operators

$$L^t = (1-t)L_0 + tL.$$

It's easy to see that $L^0 = L_0$, $L^1 = L$.

Show that the set E of those $t\in[0,1]$ for which the first boundary value problem

$$L^t u = f \quad \left((x,t)\in Q_T;\ u\in \dot{W}_{2,\varphi}^{2,2}(Q_T)\right) \tag{31}$$

is unique strong solvable, for any $f(x,t)\in L_2(Q_T)$ is non-empty and simultaneously open and closed relatively to the segment $[0,1]$.

Non-emptiness of the set E follows from that for $t=0$ the problem (31) coincides with the problem (27)-(28).

Now show that the set E is open with respect to the segment $[0,1]$. Let $t^0\in E$. Then from the corollary of theorem 1 we obtain

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_T)} \le C_{24}\left\|L^{t^0}u\right\|_{L_2(Q_T)} = C_{24}\|f\|_{L_2(Q_T)}. \tag{32}$$

Denote by M on operator which for any function $f(x,t)\in L_2(Q_T)$ associates the strong relation $u(x,t)$ of the problem (27)-(28) for $t=t^0$.

It's obvious that M is a linear operator from $L_2(Q_T)$ in $\dot{W}_{2,\varphi}^{2,2}(Q_T)$. From (32) it follows that the operator M is bounded, i.e.

$$\|Mf\|_{W_{2,\varphi}^{2,2}(Q_T)} \le C_{24}\|f\|_{L_2(Q_T)}. \tag{33}$$

On the other hand

$$L^t - L^{t^0} = (1-t)L_0 - (1-t^0)L_0 + (t-t^0)L = (t^0-t)(L_0-L). \tag{34}$$

Let $\delta>0$ be a number which will be selected later and $\left|t-t^0\right|<\delta$. We represent the problem (31) for such t in the equivalent form

$$L^{t^0}u = f + (t-t^0)(L_0-L)u,\ (x,t)\in Q_T,\ u\in \dot{W}_{2,\varphi}^{2,2}(Q_T). \tag{35}$$

Together with the problem (35) we consider the auxiliary problem

$$L^{t^0}u = f + (t-t^0)(L_0-L)z,\ (x,t)\in Q_T,\ u\in \dot{W}_{2,\varphi}^{2,2}(Q_T), \tag{36}$$

where $f(x,t)\in L_2(Q_T)$, $z(x,t)\in \dot{W}_{2,\varphi}^{2,2}(Q_T)$.

Denote by M_1 an operator which throws z in solution of the problem (36) to u

$$u = M_1 z.$$

We prove that the operator M_1 for the appropriately chosen number δ is contractive. Let

$$u_1 = M_1 z_1 \text{ and } u_2 = M_1 z_2,\ z_1, z_2 \in \dot{W}_{2,\varphi}^{2,2}(Q_T).$$

The difference $u_1 - u_2$ is a solution of the problem

$$L^{t^0}(u_1-u_2) = (t-t^0)(L_0-L)(z_1-z_2),\ (x,t)\in Q_T,\ u_1-u_2 \in \dot{W}_{2,\varphi}^{2,2}(Q_T).$$

According to the estimate (33)

$$\|u_1-u_2\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{24}\delta\|(L_0-L)(z_1-z_2)\|_{L_2(Q_T)}, \tag{37}$$

$$\|u_1-u_2\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{24}C_{25}\delta\|z_1-z_2\|_{L_2(Q_T)},$$

where the constant C_{25} depends only on L and n. Now we choose and fix $\delta=\dfrac{1}{2C_{24}C_{25}}$.

Then we have

$$\|u_1-u_2\|_{W^{2,2}_{2,\varphi}(Q_T)} \le \frac{1}{2}\|z_1-z_2\|_{W^{2,2}_{2,\varphi}(Q_T)}.$$

It remains to use the contracted mapping principle.

Now we show that the set E is closed relative to the segment $[0,1]$. Let $t_m \to t_0$ and $t_m \in E$. Let u_m $(m=1,2,...)$ be a solution of the problem

$$L^{t_m}u_m = f,\ u_m \in \overset{\circ}{W}{}^{2,2}_{2,\varphi}(Q_T),$$

where $f(x,t)$ is an arbitrary fixed function from $L_2(Q_T)$.

On the basis of the corollary of theorem 1

$$\|u_m\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{26}, \tag{38}$$

where C_{26} depends only on the functions f,n,L and the domain Ω.

Therefore from the sequence $\{u_m\}$ we can choose a subsequence weakly convergent to some function $u_0 \in \overset{\circ}{W}{}^{2,2}_{2,\varphi}(Q_T)$.

Therefore

$$\lim_{k\to\infty}\left(L^{t_0}u_{m_k},\omega\right)=\left(L^{t_0}u_0,\omega\right) \tag{39}$$

for any $\omega(x,t)\in C_0^\infty(Q_T)$.

For simplicity denote the subsequence $\{u_{m_k}(x,t)\}$ again by $\{u_m(x,t)\}$. Then from (39) we have

$$\lim_{m\to\infty}\left(L^{t_0}u_m,\omega\right)=\left(L^{t_0}u_0,\omega\right). \tag{40}$$

But on the other hand

$$\left(L^{t_0}u_m,\omega\right)=\left(\left(L^{t_0}-L^{t_m}\right)u_m,\omega\right)+\left(L^{t_m}u_m,\omega\right)=\left(\left(L^{t_0}-L^{t_m}\right)u_m,\omega\right)+(f,\omega)$$

therefore we obtain from (40)

$$\lim_{m\to\infty}\left(\left(L^{t_0}-L^{t_m}\right)u_m,\omega\right)+(f,\omega)=\left(L^{t_0}u_0,\omega\right). \tag{41}$$

Using (34) we have further for any fixed $\omega(x,t)\in C_0^\infty(Q_T)$

$$\left|\left(\left(L^{t_0}-L^{t_m}\right),\omega\right)\right| \le |t_m-t_0| \iint\limits_{Q_T}|(L_0-L)u_m||\omega|dxdt \le$$

$$\le |t_m-t_0|\left(\iint\limits_{Q_T}|(L_0-L)u_m|^2dxdt\right)^{\frac{1}{2}}\left(\int\limits_{Q_T}\omega^2dxdt\right)^{\frac{1}{2}} \le$$

$$C_{27}|t_m-t_0|\|u_m\|_{W^{2,2}_{2,\varphi}}\|\varphi\|_{L_2(Q_T)},$$

where C_{27} depends only on L,n. Allowing for (38) from the last inequality we conclude

$$\left|\left(L^{t_0}-L^{t_m}\right)u_m,\omega\right| \le C_{26}C_{27}|t_m-t_0|\|\varphi\|_{L_2(Q_T)}.$$

In other words

$$\lim_{m\to\infty}\left(\left(L^{t_0}-L^{t_m}\right)u_m,\omega\right)=0 \qquad (42)$$

using (42) in (41) we get

$$\left(L^{t_0}u_0,\omega\right)=\left(f,\omega\right). \qquad (43)$$

Since the equality (43) is valid for arbitrary function $\omega(x,t)\in C_0^\infty(Q_T)$ then it follows that $L^{t_0}u_0=f$, a.e. in Q_T. Thus the function $u_0(x,t)\in \overset{\circ}{W}{}^{2,2}_{2,\varphi}(Q_T)$ is a strong (moreover unique by virtue of the corollary of theorem 1) solution of the problem (31) for $t=t_0$, i.e. $t_0\in E$. The closure of E is proved and the proof of the existence of a solution of the problem (1)-(2) is completed. The estimate (30) with $C_{23}=C_{20}$ immediately follows from the corollary of theorem 1.

The proof of the theorem is completed.

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References

[1]. Ladizhenskaya O.A., Uraltceva N.N. *Linear and quasilinear elliptic type equations.* M.: Nauka, 1973, 576 p. (in Russian)

[2]. Talenti G. *Sopra una classe di equazioni ellitiche a coeeficienti misurableili.* –Ann.mat.pura appl., 1965, v.69, p.285-304.

[3]. Chicco M. *Solvability of the Dirichlet problem in* $H^{2,p}(\Omega)$ *for a class of linear second order elliptic partial differential equations.* Boll.Un.Mat.Ital., v.4, №4, 1971, p.374-387.

[4]. Ladizhenskaya O.A. *On the solvability of main boundary value problems for parabolic and hyperbolic type equations.* DAN SSSR, 97,1954, p.395-398. (in Russian)

[5]. Alxutov Yu.A., Mamedov I.T. *Some properties of solutions of the first boundary value problem for parabolic equations with discontinuous coefficients.* DAN SSSR, 1985, v.284, №1, p.11-16. (in Russian)

[6]. Alxutov Yu.A., Mamedov I.T. *The first boundary value problem for the second order non-divergent parabolic equations with discontinuous coefficients.* Matem.sb., 1986, v.131(173), №4(12), p.477-500 (in Russian).

[7]. Wen G.C. *Initial-mixed boundary value problems for parabolic equations of second order with measurable coefficients in a higher dimensional domain.* Proceedings of the second ISAAC Congress 2000, v.1, p.185-192.

[8]. Fikera G. *To the one theory of boundary value problem for the second order elliptico-parabolic equations.* Matem.per.sb……, 1963, v.7, №6, p.99-121.

[9]. Oleynik O.A., Radkevich E.V. *The second order equations with negative characteristic form.* VINITI, ser. "itogi nauki", matem. Analiz, 1971, p.7-252.

[10]. Franciosi M. *Un teorema di esistenza ed Unicita per la soluzione di un'equazione elliptico-parabolica, a coefficienti discontinui, in forma non divergenza.* –Boll.Un.Mat., Ital., 1985, v.6, №4-B, p.253-563.

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