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APPLICATION OF FINITE DIFFERENCES METHOD TO THE SOLUTION OF ONE PROBLEM ON BALANCE OF SPRING BEAM

Abstract

At the paper by the numerical method the some boundary value problem for differential equation of the bourth order is investigated. In addition the special difference scheme is constructed and the error method [2] is estimated.

Let's consider the following boundary value problem

$$\frac{d^4x(t)}{dt^4}+a(t)x(t)=f(t), \qquad (1)$$

$$x(1)=0,\ \frac{dx(0)}{dt}=0\,,\ \frac{d^2x(1)}{dt^2}=0\,,\ \frac{d^3x(0)}{dt^3}=0\,, \qquad (2)$$

where $a(t), f(t), (0\le t\le 1)$ are given, continuous functuions. It is known that (see, ex. [1]) the problem on balance of spring beam is reduced to the solution of the problem (1)-(2).

At this paper for the problem (1), (2) a special difference scheme is constructed and the error of the method ([2]) is estimated.

1. The construction of difference scheme.

Let's construct the uniform net

$$\overline{w}_h=\{t_i=ih,\ i=0,1,\ldots,n\}$$

with the step $h=1/n$, where n is a natural number.

Let's denote by $\overset{\vee}{w}_h$ the set of the following points of the net:

$$\overset{\vee}{w}=\{t_{i+1/2}=(i+1/2)h,\ i=0,1,\ldots,n-1\}.$$

Let's introduce the following notations

$\overset{\vee}{x}_t=1/2[x(t+h/2)+x(t-h/2)]$,

$\overset{\vee}{x}_t=1/h[x(t+h/2)-x(t-h/2)]$,

$\overset{\circ}{x}_t=1/(2h)[x(t+h)-x(t-h)]$, $a_i=a(t_i)$, $f_i=f(t_i)$, $x_i=x(t_i)$.

Let's introduce the difference operators

$L_4^{(h)}x(t)\equiv x_{\bar{t}\bar{t}\bar{t}\bar{t}}(t)=1/h^4\,[x(t+2h)-4x(t+h)+6x(t)-4x(t-h)+x(t-2h)]$

for $t\in \overset{\vee}{w}_h$

$l_h^{(0)}x(t)\equiv x_{\overset{\vee}{t}}(t)=1/2[x(t+h/2)+x(t-h/2)]$,

$l_h^{(1)}x(t)\equiv x_{\overset{\vee}{t}\overset{\circ}{t}}(t)=1/(4h)[x(t+3h/2)+x(t+h/2)-x(t-h/2)-x(t-3h/2)]$

$l_h^{(2)}x(t)\equiv x_{\overset{\vee}{t}\bar{t}\bar{t}}(t)=1/(8h^2)[x(t+5h/2)+x(t+3h/2)-2x(t+h/2)-2x(t-h/2)+$

$+x(t-3h/2)+x(t-5h/2)]$,

$l_h^{(3)}x(t)\equiv x_{\overset{\vee}{t}\,\overline{iii}}(t)=1/\left(16h^3\right)[x(t+7h/2)+x(t+5h/2)-3x(t+3h/2)-3x(t+h/2)+$

$+3x(t-h/2)+3x(t-3h/2)-x(t-5h/2)-x(t-7h/2)]$, for $t\in\overline{w}_h$.

Using the evident formulas

$$x(t+\alpha h)-x(t-\alpha h)=2x'(t)\alpha h+x'''(t)\left(\alpha^3/3\right)h^3+x^V(t)\left(\alpha^5/60\right)h^2+o\left(h^7\right),$$

$$x(t+\alpha h)+x(t-\alpha h)=2x(t)+x''(t)\alpha^2h^2+x^{IV}(t)\left(\alpha^4/12\right)h^4+o\left(h^6\right)$$

it is easy to note that

$$\frac{d^4x(t)}{dt^4}=L_h^{(4)}x(t)+a(t)x(t)+o\left(h^2\right),\quad \left(t\in\overset{\vee}{w}_h\right) \tag{3}$$

and for $t\in\overline{w}_h$

$$x(t)=l_h^{(0)}x(t)-\left(x''(t)/8\right)h^2-x^{(IV)}(t)h^4/(16\cdot 24)+o\left(h^6\right),$$

$$\frac{dx(t)}{dt}=l_h^{(1)}x(t)-7\left(x'''(t)/24\right)h^2-61h^4x^{(V)}(t)/(60\cdot 32)+o\left(h^6\right),$$

$$\frac{d^2x(t)}{dt^2}=l_h^{(2)}x(t)-\left(5^4+3^4-2\right)x^{(IV)}(t)h^2/(8\cdot 16\cdot 12)+o\left(h^4\right),$$

$$\frac{d^3x(t)}{dt^3}=l_h^{(3)}x(t)-\left(7^5+5^5-3^6-3\right)x^{(V)}(t)h^2/\left(60\cdot 2^5\cdot 16\right)+o\left(h^4\right).$$

Taking into account these formulas we can substitute the problem (1), (2) by the following problem

$$\left.\begin{aligned}
&L_h^4x(t)+a(t)x(t)=f(t)+o\left(h^2\right),\\
&l_h^{(0)}x(1)=x^{(IV)}(1)h^4/(16\cdot 24)+o\left(h^6\right),\quad l_h^{(1)}x(0)=61h^4x^{(V)}(0)/(60\cdot 32)+o\left(h^6\right)\\
&l_h^{(2)}x(1)=\left(5^4+3^4-2\right)x^{(IV)}(1)h^2/(8\cdot 16\cdot 12)+o\left(h^4\right),\\
&l_h^{(3)}x(0)=\left(7^5+5^5-3^6-3\right)x^{(V)}(0)h^2/\left(60\cdot 2^5\cdot 16\right)+o\left(h^4\right),\quad \left(t\in\overline{w}_h\right).
\end{aligned}\right\} \tag{4}$$

Note let's suppose that the functions $\Re$ satisfy one of the following conditions:

1) they are continuous on $[0,1]$;
2) they are continuous on $[0,1]$ and $f(1)=0$;
3) they are differentiable on $[0,1]$ and $a'(0)=0,\ f'(0)=0,\ f(1)=0$.

Then the system (4) is substituted by the following system:

$$x_{i+5/2}-4x_{i+3/2}+\left(6+h^4a_{i+1/2}\right)x_{i+1/2}-4x_{i-1/2}+x_{i-3/2}=h^4f_{i+1/2}+o\left(h^6\right)$$
$$(i=0,1,\dots,n-1), \tag{5}$$

$$x_{-1/2}=x_{1/2}+o\left(h^{r_1}\right),\ x_{-3/2}=x_{3/2}+o\left(h^{r_1}\right),\ x_{n+1/2}=-x_{n-1/2}+o\left(h^{r_2}\right),$$
$$x_{n+3/2}=-x_{n-3/2}+o\left(h^{r_2}\right). \tag{6}$$

Moreover if the condition 1) is fulfilled, then $r_1=5,\ r_2=4$;

if the condition 2) is fulfilled then $r_1=5,\ r_2=6$ and finally,

if the condition 3) is fulfilled then $r_1=7,\ r_2=6$.

Let's substitute the system (5), (6) by the difference scheme

$$\left.\begin{array}{l} X_{i+5/2}-4X_{i+3/2}+\left(6+h^4a_{i+1/2}\right)X_{i+1/2}-4X_{i-1/2}+X_{i-3/2}=h^4 f_{i+1/2}, \\ \qquad (i=0,1,\ldots,n-1), \\ X_{-1/2}=X_{1/2},\ X_{-3/2}=X_{3/2},\ X_{n+1/2}=-X_{n-1/2},\ X_{n+3/2}=-X_{n-3/2}\,. \end{array}\right\} \quad (7)$$

We'll assume the solution of the system (7) as approximate solution of the problem (1), (2) at the points $\overset{\vee}{w}_h$.

2. Let's show the boundedness of the solution of difference problem.

Let's multiply the equation (7) by $hX_{i+1/2}$ and sum

$$\sum_{i=0}^{N-1} h\left(X_{i+5/2}-4X_{i+3/2}+\left(6+h^4a_{i+1/2}\right)X_{i+1/2}-4X_{i-1/2}+X_{i-3/2}\right)X_{i+1/2}=$$

$$=\sum_{i=0}^{N-1} hh^4 f_{i+1/2}X_{i+1/2}\,.$$

It is easy to check that

$$\sum_{i=0}^{N-1} h\left(X_{i+5/2}-4X_{i+3/2}+6X_{i+1/2}-4X_{i-1/2}+X_{i-3/2}\right)X_{i+1/2}=$$

$$=\sum_{i=0}^{N-1} h\left(X_{i+3/2}-2X_{i+1/2}+X_{i-1/2}\right)^2+h\delta_h\,. \quad (8)$$

Really from the left we have:

$X_{5/2}X_{1/2}-4_{3/2}X_{1/2}+6X_{1/2}^2-4X_{-1/2}X_{1/2}+X_{-3/2}X_{1/2}+X_{7/2}X_{3/2}-4X_{5/2}X_{3/2}+$
$+6X_{3/2}^2-4X_{1/2}X_{3/2}+X_{-1/2}X_{3/2}+X_{9/2}X_{5/2}-4X_{7/2}X_{5/2}+6X_{5/2}^2-4X_{3/2}X_{5/2}+$
$X_{1/2}X_{5/2}+\ldots$
$\ldots+X_{N-1/2}X_{N-5/2}-4X_{N-3/2}X_{N-5/2}+6X_{N-5/2}^2-4X_{N-7/2}X_{N-5/2}+$
$X_{N-9/2}X_{N-5/2}+X_{N+1/2}X_{N-3/2}-4X_{N-1/2}X_{N-3/2}+6X_{N-3/2}^2-4X_{N-5/2}X_{N-3/2}+$
$+X_{N-7/2}X_{N-3/2}+X_{N+3/2}X_{N-1/2}-4X_{N+1/2}X_{N-1/2}+6X_{N-1/2}^2-4X_{N-3/2}X_{N-1/2}+$
$+X_{N-5/2}X_{N-1/2}$,

from the right we have

$X_{3/2}^2+4X_{1/2}^2+X_{-1/2}^2-4X_{3/2}X_{1/2}+2X_{3/2}X_{-1/2}-4X_{1/2}X_{-1/2}-$
$+X_{5/2}^2+4X_{3/2}^2+X_{1/2}^2-4X_{5/2}X_{3/2}+2X_{5/2}X_{1/2}-4X_{3/2}X_{1/2}+X_{7/2}^2+4X_{5/2}^2+$
$+X_{3/2}^2-4X_{7/2}X_{5/2}+2X_{7/2}X_{3/2}-4X_{5/2}X_{3/2}+\cdots$
$\ldots+X_{N-3/2}^2+4X_{N-5/2}^2+X_{N-7/2}^2-4X_{N-3/2}X_{N-5/2}+2X_{N-3/2}X_{N-7/2}-$
$-4X_{N-5/2}X_{N-7/2}+X_{N-1/2}^2+4X_{N-3/2}^2+X_{N-5/2}^2-4X_{N-1/2}X_{N-3/2}+$
$+2X_{N-1/2}X_{N-5/2}-4X_{N-3/2}X_{N-5/2}+X_{N+1/2}^2+4X_{N-1/2}^2+X_{N-3/2}^2-$
$-4X_{N+1/2}X_{N-1/2}+2X_{N+1/2}X_{N-3/2}-4X_{N-1/2}X_{N-3/2}$.

That is why

$\delta_h = -X_{-1/2}^2 + X_{1/2}^2 - X_{3/2}X_{-1/2} + X_{1/2}X_{-3/2} + X_{N-1/2}^2 - X_{N+1/2}^2 - X_{N+1/2}X_{N-3/2} +$

$+ X_{N+3/2}X_{N-1/2}$.

By virtue of boundary conditions we get

$\delta_h = -X_{1/2}^2 + X_{1/2}^2 - X_{3/2}X_{1/2} + X_{1/2}X_{3/2} + X_{N-1/2}^2 - X_{N-1/2}^2 + X_{N-1/2}X_{N-3/2} -$

$- X_{N-3/2}X_{N-1/2} = 0$.

So, we get

$$\sum_{i=0}^{N-1} h\left(X_{i+3/2} - 2_{i+1/2} + X_{i-1/2}\right)^2 + h^4 \sum_{i=0}^{N-1} ha_{i+1/2}X_{i+1/2}^2 = h^4 \sum_{i=0}^{N-1} hf_{i+1/2}X_{i+1/2}.$$

Taking into account on $a(t)$, we arrive at the inequality

$$\sum_{i=0}^{N-1} h\left(X_{i+3/2} - 2x_{i+1/2} + X_{i-1/2}\right)^2 + a_{\min} h^4 \sum_{i=0}^{N-1} hX_{i+1/2}^2 \le$$

$$\le h^4/(4\varepsilon) \sum_{i=0}^{N-1} hf_{i+1/2}^2 + \varepsilon h^4 \sum_{i=0}^{N-1} hX_{i+1/2}^2.$$

Whence

$$\sum_{i=0}^{N-1} h\left(X_{i+3/2} - 2x_{i+1/2} + X_{i-1/2}\right)^2 + (a_{\min} - \varepsilon)h^4 \sum_{i=0}^{N-1} hX_{i+1/2}^2 \le$$

$$\le h^4/(4\varepsilon) \sum_{i=0}^{N-1} hf_{i+1/2}^2. \quad (9)$$

Here $0 < \varepsilon < a_{\min}$. Or

$$\sum_{i=0}^{N-1} h\left[\left(x_{i+3/2} - 2x_{i+1/2} + x_{i-1/2}\right)/h^2\right]^2 \le c_1, \quad \sum_{i=0}^{N-1} hx_{i+1/2}^2 \le c_2 \sum_{i=0}^{N-1} hf_{i+1/2}^2.$$

The received estimations prove the boundness of solution of difference problem. These estimations allow to prove the uniqueness of the solution of the difference problem and to get the estimation of speed of convergence. Let the problem (7) have two solutions $X_{i+1/2}$ and $\overline{X}_{i+1/2}$.

Let's consider the function $\eta_{i+1/2} = X_{i+1/2} - \overline{X}_{i+1/2}$ that satisfies the difference problem, analogously to (7)

$$\begin{aligned} &\eta_{i+5/2} - 4\eta_{i+3/2} + \left(6 + h^4 a_{i+1/2}\right)\eta_{i+1/2} - 4\eta_{i-1/2} + \eta_{i-3/2} = 0, \\ &\eta_{-1/2} = \eta_{1/2}, \ \eta_{-3/2} = \eta_{3/2}, \ \eta_{N+1/2} = -\eta_{N-1/2}, \ \eta_{N+3/2} = -\eta_{N-3/2} \end{aligned} \quad (10)$$

By virtue of received above estimation on boundedness of the solution of the problem (1), we'll get the estimation for the solution of the problem (10):

$$\sum_{i=0}^{N-1} h\eta_{i+1/2}^2 \le 0.$$

From which we get that $\eta_{i+1/2} \equiv 0$. Thus it is proved that the problem (7) has a unique solution.

3. The estimation of error.

Let's consider error the function $\varepsilon_{i+1/2} = X_{i+1/2} - x_{i+1/2}$.

Since $X_{i+1/2} = \varepsilon_{i+1/2} + x_{i+1/2}$ then

$$L_h^{(4)}\varepsilon_{i+1/2} + a_{i+1/2}\varepsilon_{i+1/2} = \left(\frac{d^4 x(t)}{dt^4}\right)_{i+1/2} - L_h^{(4)}x_{i+1/2},$$

$$\varepsilon_{-1/2} - \varepsilon_{1/2} = X_{-1/2} - X_{1/2} - (x_{-1/2} - x_{1/2}) = x_{1/2} - x_{-1/2},$$

$$\varepsilon_{-3/2} - \varepsilon_{3/2} = X_{-3/2} - X_{3/2} - (x_{-3/2} - x_{3/2}) = x_{3/2} - x_{-3/2},$$

$$\varepsilon_{N+1/2} + \varepsilon_{N-1/2} = X_{N+1/2} + X_{N-1/2} - (x_{N+1/2} - x_{N-1/2}) = -(x_{N+1/2} - x_{N-1/2}),$$

$$\varepsilon_{N+3/2} + \varepsilon_{N-3/2} = X_{N+3/2} + X_{N-3/2} - (x_{N+3/2} + x_{N-3/2}) = -(x_{N+3/2} - x_{N-3/2}).$$

By virtue of received above estimations we get:

$$L_4^{(4)}\varepsilon_{i+2} + a_{i+1/2}\varepsilon_{i+1/2} = 0(h^2),$$

$$\varepsilon_{-1/2} - \varepsilon_{1/2} = 0(h^{r_1})$$

$$\varepsilon_{-3/2} - \varepsilon_{3/2} = 0(h^{r_1}), \quad \varepsilon_{N+1/2} + \varepsilon_{N-1/2} = 0(h^{r_2}),$$

$$\varepsilon_{N+3/2} + \varepsilon_{N-3/2} = 0(h^{r_2}). \tag{11}$$

Applying the proof for the received problem we'll have analogous proof of boundedness of the solutions

$$\sum_{i=0}^{N-1} h(\varepsilon_{i+3/2} - 2\varepsilon_{i+1/2} + \varepsilon_{i-1/2})^2 + h^4 \sum_{i=0}^{N-1} h a_{i+1/2}\varepsilon_{i+1/2}^2 = h^4 \sum_{i=0}^{N-1} h R_{i+1/2} - h\delta_{h,\varepsilon}, \tag{12}$$

where

$$\delta_{h,\varepsilon} = -\varepsilon_{-1/2}^2 + \varepsilon_{1/2}^2 - \varepsilon_{3/2}\varepsilon_{-1/2} + \varepsilon_{1/2}\varepsilon_{-3/2} + \varepsilon_{N-1/2}^2 - \varepsilon_{N+1/2}^2 - \varepsilon_{N+1/2}\varepsilon_{N-3/2} + $$
$$+ \varepsilon_{N+3/2}\varepsilon_{N-1/2}. \tag{13}$$

Let's estimate the relation (12):

$$\sum_{i=0}^{N-1} h(\varepsilon_{i+3/2} - 2\varepsilon_{i+1/2} + \varepsilon_{i-1/2})^2 + (a_{\min} - \varepsilon)h^4 \sum_{i=0}^{N-1} h\varepsilon_{i+1/2}^2 \le$$

$$\le h^4/(4\varepsilon)\sum_{i=0}^{N-1} hR_{i+1/2}^2 + h|\delta_{h,\varepsilon}|.$$

Let's take $\varepsilon = a_{\min}/2$, then:

$$\sum_{i=0}^{N-1} h(\varepsilon_{i+3/2} - 2\varepsilon_{i+1/2} + \varepsilon_{i-1/2})^2 + (a_{\min}/2)h^4 \sum_{i=0}^{N-1} h\varepsilon_{i+1/2}^2 \le$$

$$\le h^4/(2a_{\min})\sum_{i=0}^{N-1} hR_{i+1/2}^2 + h|\delta_{h,\varepsilon}|. \tag{14}$$

Now let's estimate quantity $\delta_{h,\varepsilon}$:

$$\delta_{h,\varepsilon} = -(\varepsilon_{1/2} + 0(h^{r_1}))^2 + \varepsilon_{1/2}^2 - \varepsilon_{3/2}(\varepsilon_{1/2} + 0(h^{r_1})) + \varepsilon_{1/2}(\varepsilon_{3/2} + 0(h^{r_1})) + \varepsilon_{N-1/2}^2 -$$

$$-(0(h^{r_2}) - \varepsilon_{N-1/2})^2 - \varepsilon_{N-3/2}(0(h^{r_2}) - \varepsilon_{N-1/2}) + (0(h^{r_2}) - \varepsilon_{N-3/2})\varepsilon_{N-1/2} = -(0(h^{r_1}))\times$$

$$\times\left(2\varepsilon_{1/2}+0\left(h^{r_1}\right)\right)-\varepsilon_{3/2}0\left(h^{r_1}\right)+\varepsilon_{1/2}0\left(h^{r_1}\right)+0\left(h^{r_2}\right)\left(2\varepsilon_{N-1/2}-0\left(h^{r_2}\right)\right)-\varepsilon_{N-3/2}0\left(h^{r_2}\right)+$$
$$+\varepsilon_{N-3/2}\varepsilon_{N-1/2}+0\left(h^{r_2}\right)\varepsilon_{N-1/2}-\varepsilon_{N-3/2}\varepsilon_{N-1/2}=\left(\varepsilon_{1/2}+\varepsilon_{3/2}\right)0\left(h^{r_1}\right)+$$
$$+\left(\varepsilon_{N-1/2}-\varepsilon_{N-3/2}\right)0\left(h^{r_2}\right).$$

So,

$$h\left|\delta_{h,\varepsilon}\right|\le c\left(h^{r_1+1}+h^{r_2+1}\right).$$

Substituting the obtained estimate in (14) we get:

$$\sum_{i=0}^{N-1}h\left(\varepsilon_{i+3/2}-2\varepsilon_{i+1/2}+\varepsilon_{i-1/2}\right)^2+(a_{\min}/2)h^4\sum_{i=0}^{N-1}h\varepsilon_{i+1/2}^2\le$$
$$\le h^4/(2a_{\min})\sum_{i=0}^{N-1}hR_{i+1/2}^2+c\left(h^{r_1+1}+h^{r_2+1}\right).$$

Whence

$$\|\varepsilon_{\bar{t}t}\|^2+(a_{\min}/2)\|\varepsilon\|^2\le 1/(2a_{\min})\|R\|^2+c\left(h^{r_1-3}+h^{r_2-3}\right),$$
$$\|\varepsilon\|^2=\sum_{i=0}^{N-1}h\varepsilon_{i+1/2}^2,\quad \varepsilon_{\bar{t}t}=1/h^2\left[x(t+h)-2x(t)+x(t-h)\right]$$

or

$$\|\varepsilon_{\bar{t}t}\|^2+(a_{\min}/2)\|\varepsilon\|^2\le c_3h^4+c\left(h^{r_1-3}+h^{r_2-3}\right).$$

Whence if the condition 1) is fulfilled then

$$\|\varepsilon_{\bar{t}t}\|+\|\varepsilon\|\le c_0h^{1/2},$$

if the condition 2) is fulfilled, then

$$\|\varepsilon_{\bar{t}t}\|+\|\varepsilon\|\le c_0h$$

and finally if the condition 3) is fulfilled then

$$\|\varepsilon_{\bar{t}t}\|+\|\varepsilon\|\le c_0h^2.$$

References

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