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ON THE DISCRETE CONTROL PROBLEM BY THE BOUNDARY CONDITIONS

Abstract

One discrete control problem which is controlled by using boundary conditions is considered in this paper. The necessary optimality conditions of the first and second order have been obtained.

A number of necessary and sufficient conditions of optimality for the different discrete two parametric systems, in case, when the control function is on the right hand side of system have been found in papers [1-4] and etc.

In the present paper the case of boundary controls are considered and the necessary conditions of optimality of the first and second order are derived.

1. Assume that the controlled process is described by the following system of nonlinear difference equations

$$z(t+1,x+1)=f(t,x,z(t,x),z(t+1,x),z(t,x+1)) \qquad (1)$$

with the boundary conditions

$$\begin{gathered} z(t_0,x)=a(x),\ x=x_0,x_0+1,\dots,X \\ z(t,x_0)=b(t),\ t=t_0,\ t_0+1,\dots,T \\ a(x_0)=b(t_0) \end{gathered} \qquad (2)$$

where $f(t,x,z,l,m)$ is a given n-dimensional vector-function which is continuous on totality of variables together with partial derivatives up to second order inclusively with respect to (z,l,m), $b(t)$ is a given n-dimensional vector-function, t_0,x_0,X,T are given, $a(x)$ is an n-dimensional vector function which is a solution of the following system

$$a(x+1)=g(x,a(x),u(x)),\ x=x_0,\ x_0+1,\dots,X-1,\ a(x_0)=a_0. \qquad (3)$$

Here $g(x,a,u)$ is a given n-dimensional vector-function which is continuos on totality of variables together with partial derivatives with respect to (a,u) up to second order inclusively, a_0 is a given constant vector, $u(x)$ is a r-dimensional vector of control influences with values from the given nonempty, open and bounded set $U\subset R^r$ (admissible control)

$$u(x)\in U\subset R^r,\ x=x_0,x_0+1,\dots,X-1. \qquad (4)$$

Thus, the process is controlled by means of selection of boundary condition. In the solutions of system (1)-(3) generated by various admissible controls we we'll define the functional

$$S(u)=\varphi(z(T,X)). \qquad (5)$$

The problem is in minimization of the functional (5) at the constraints (1)-(4).

Here $\varphi(z)$ is a given twice continuous-differentiable scalar function.

The admissible control $u(x)$ which is the solution of the formulated problem we'll call as the optimal control and the corresponding process $(u(x),z(t,x))$ as optimal process.

2. Considering $(u(x),z(t,x))$ as fixed process we'll introduce the following notations

$$H(t,x,z,l,m,\psi)=\psi' f(t,x,z,l,m),$$
$$M(x,a,u,p)=p' g(x,a,u),$$
$$H_z[t,x]\equiv H_z(t,x,z(t,x),z(t+1,x),z(t,x+1),\psi(t,x)),$$
$$M_a[x]\equiv M_a(x,a(x),u(x),p(x)),$$
$$M_{aa}[x]\equiv M_{aa}(x,a(x),u(x),p(x)),$$
$$H_l[t,x]\equiv H_l(t,x,z(t,x),z(t+1,x),z(t,x+1),\psi(t,x))$$

Here $\psi(t,x)$ and $p(x)$ are n-dimensional vector functions which are the solutions of the following problem

$$\begin{gathered}\psi(t-1,x-1)=H_z[t,x]+H_1[t-1,x]+H_m[t,x-1],\\ \psi(T-1,x-1)=H_1[T-1,x],\\ \psi(t-1,X-1)=H_m[t,X-1],\\ \psi(T-1,X-1)=-\varphi_z(z(T,X)),\\ p(x-1)=\psi(t_0-1,x-1)+M_a[x]-H_1[t_0-1,x],\\ p(X-1)=\psi(t_0-1,X-1).\end{gathered} \tag{6}$$

Using Taylor formulae and allowing for (6) one can represent the first and second variation (in classical sense) of the functional $S(u)$ in the form

$$\delta^1 S(u;\delta u)=-\sum_{x=x_0}^{X-1} M_u'[x]\delta u(x), \tag{7}$$

$$\begin{aligned}\delta^2 S(u;\delta u)=\ &\delta z'(T,X)\varphi_{zz}(z(T,X))\delta z(T,X)-\sum_{t=t_0}^{T-1}\sum_{x=x_0}^{X-1}[\delta z'(t,x)H_{zz}[t,x]\delta z(t,z)+\\ &+\delta z'(t,x)H_{zl}[t,x]\delta z(t+1,x)+\delta z(t,x)H_{zm}[t,x]\delta z(t,x+1)+\delta z'(t+1,x)H_{lz}[t,x]\times\\ &\times\delta z(t,x)+\delta z'(t,x+1)H_{mz}[t,x]\delta z(t,x)+\delta z'(t+1,x)H_{ll}[t,x]\delta z(t+1,x)+\\ &+\delta z'(t,x+1)H_{mm}[t,x]\delta z(t,x+1)+\delta z'(t,x+1)H_{ml}[t,x]\delta z(t+1,x)+\\ &+\delta z'(t+1,x)H_{lm}[t,x]\delta z(t,x+1)]-\sum_{x=x_0}^{X-1}[\delta a'(x)M_{aa}[x]\delta a(x)+\\ &+2\delta u'(x)M_{ua}[x]\delta a(x)+\delta u'(x)M_{uu}(x)\delta u(x)].\end{aligned} \tag{8}$$

Here $(\delta z(t,x),\delta a(x))$ is a variation of trajectories which are the solutions of the following system

$$\delta z(t+1,x+1)=f_z[t,x]\delta z(t,x)+f_1[t,x]\delta z(t+1,x)+f_m[t,x]\delta z(t,x+1), \tag{9}$$

$$\begin{gathered}\delta z(t_0,x)=\delta a(x),\ x=x_0,x_0+1,...,X,\\ \delta z(t,x_0)=0,\ t=t_0,t_0+1,...,T,\end{gathered} \tag{10}$$

$$\delta a(x+1)=g_a[x]\delta a(x)+g_u[x]\delta u(x), \tag{11}$$

$$\delta a(x_0)=0 \tag{12}$$

and $\delta u(x)\in R^r$, $x=x_0$, $x_0+1,...,X-1$ is an arbitrary vector-function (variation of a control).

It is known (see for ex. [5]) that as the domain of control U is open along the optimal process $(u(x),z(t,x))$ for all

$$\delta u(x)\in R^r,\ \ x=x_0,\ \ x_0+1,...,X-1,$$

$$\delta^1 S(u;\delta u)=0, \tag{13}$$

$$\delta^2 S(u;\delta u) \geq 0. \tag{14}$$

From relation (13) it follows

Theorem 1. *For the optimality of admissible control $u(t,x)$ it is necessary that along the process $(u(x), z(t,x))$ the relation*

$$M_u[x] = 0, \quad x = x_0, \dots, X-1 \tag{15}$$

is fulfilled.

The optimality condition (15) is analogue of the Euler equation for the considered problem.

Let's begin to derive necessary optimality conditions of the second order.

At first let's give one notion following to [5].

Definition. *The admissible control $u(x)$ satisfying the condition (15) is called a classical extremal.*

It is obvious from this definition that the optimal control is among the classical extremals.

Assume that the right-hand side of the system (1) has the form

$$f(t,x,z,l,m) = A(t,x)m + F(t,x,z,l). \tag{16}$$

In this case the second variation (8) of the functional $S(u)$ will get the following form

$$\begin{aligned} \delta^2 S(u) = {} & \delta z'(T,X)\varphi_{zz}(z(T,X))\delta z(T,X) - \sum_{t=t_0}^{T-1}\sum_{x=x_0}^{X-1}\big[\delta z'(t,x)H_{zz}[t,x]\delta z(t,z) + \\ & + \delta z'(t,x)H_{zl}[t,x]\delta z(t+1,x) + \delta z'(t+1,x)H_{lz}[t,x]\delta z(t,x) + \\ & + \delta z'(t+1,x)H_{ll}[t,x]\delta z(t+1,x)\big] - \\ & - \sum_{x=x_0}^{X-1}\big[\delta a'(x)M_{aa}[x]\delta a(x) + 2\delta a'(x)M_{ua}[x]\delta a(x) + \delta u'(x)M_{uu}(x)\delta u(x)\big]. \end{aligned} \tag{17}$$

By analogy with [6,3,4] one can show that the solution of the system (9)-(10), (11)-(12) admits the following representations

$$\delta a(x) = \sum_{s=x_0}^{x-1} \Phi(x,s) g_u[s]\delta u(s), \tag{18}$$

$$\delta z(t,x) = R(t,x,t_0-1,x-1)\delta a(x) = \sum_{s=x_0}^{x-1} Q(t,x,s) g_u[s]\delta u(s). \tag{19}$$

Here by the definition

$$Q(t,x,s) = R(t,x,t_0-1,x-1)\Phi(x,s),$$

and $R(t,x;\tau,s)$ and $\Phi(x,s)(n\times n)$ are matrix functions which are the solutions of the following problem [4,6]

$$\begin{aligned} & R(t,x;\tau-1,s-1) = R(t,x;\tau,s)f_z[\tau,s] + R(t,x;\tau-1,s)f_l[\tau-1,s] + R(t,x;\tau,s-1)f_m[\tau,s-1], \\ & R(t,x;t-1,s-1) = R(t,x:t-1,s)f_l[t-1,s], \\ & R(t,x;\tau-1,x-1) = R(t,x:\tau,x-1)f_m[\tau,x-1], \\ & R(t,x:t-1,x-1) = E, \end{aligned} \tag{20}$$

$$\begin{aligned} & \Phi(x,s-1) = \Phi(x,s)g_a[s], \\ & \Phi(x,x-1) = E, \end{aligned} \tag{21}$$

where E is $(n\times n)$ unit matrix.

Let's introduce the notations

$$K(\tau,s)=-Q'(T,X,\tau)\varphi_{zz}(z(T,X))Q(T,X,s)+\sum_{x=\max(\tau,s)+1}^{X-1}\{\Phi'(x,\tau)M_{aa}[x]\Phi(x,s)+$$
$$+\sum_{t=t_1}^{T-1}Q'(t,x,\tau)H_{zz}[t,x]Q(t,x,s)+\sum_{t=t_1}^{T-1}Q'(t+1,x,\tau)H_{ll}[t,x]Q(t+1,x,s)+$$
$$+\sum_{t=t_1}^{T-1}Q'(t,x,\tau)H_{zl}[t,x]Q(t+1,x,s)+\sum_{t=t_1}^{T-1}Q'(t+1,x,\tau)H_{lz}[t,x]Q(t+1,x,s)\} \tag{22}$$

Subjected to the notation (22) using the methodology from [7] and the formulae (18) and (19) one can represent the second variation (17) of the functional $S(u)$ in the form

$$\delta^2 S(u)=-\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}\delta u'(\tau)g_u'[\tau]K(\tau,s)g_u[s]\delta u(s)-$$
$$-2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1}\delta u'(x)M_{ua}[x]\Phi(x,s)g_u[s]\delta u(s)\right]-\sum_{x=x_0}^{X-1}\delta u(x)M_{uu}(x)\delta u(x). \tag{23}$$

Theorem 2 follows from t he representation (23)

Theorem 2. *For the optimality of the classical extremal $u(x)$ in the problem (1)-(5), (16) it is necessary the fulfilling the inequality*

$$\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}\delta u'(\tau)g_u'[\tau]K(\tau,s)g_u[s]\delta u(s)+$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1}\delta u'(x)M_{ua}[x]\Phi(x,s)g_u[s]\delta u(s)\right]+\sum_{x=x_0}^{X-1}\delta u'(x)M_{uu}[x]\delta u(x)\le 0. \tag{24}$$

for all $\delta u(x)\in R^r$, $x\in x_0,\ x_0+1,\ X-1$.

The inequality (24) is a general necessary condition of the second order optimality. One can obtain the different necessary conditions of the second order optimality from this. Let's cite one of them

Theorem 3. *Along the optimal classical extreamal $u(x)$ in the problem (1)-(4), (16) the inequality*

$$v'g_u'[\xi]K(\xi,\xi)g_u[\xi]v+v'M_{uu}[\xi]v\le 0 \tag{25}$$

is fulfilled for all $\xi=x_0,\ x_0+1, X-1,\ v\in R^r$.

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