SADYGOV M.A.

ON AN EXTREMAL PROBLEM FOR GOURSAT-DARBOUX TYPE DIFFERENTIAL INCLUSIONS, II

Abstract

In the paper non-convex extremal problem for differential inclusions with phase constraints is studied.

Extremal problems for multidimensional differential inclusions are considered in [1-6].

Let $a:[0,1]\times[0,1]\times R^{3n}\to 2^{R^n}$, where $a(\tau,\nu,z)$ be compact for all (τ,ν,z), $M_0\subset R^n$, $b_1:[0,1]\times R^n\to 2^{R^n}$, $b_2:[0,1]\times R^n\to 2^{R^n}$. The functions $u(\cdot)\in$ $\in A^n([0,1]\times[0,1])$ satisfying the inclusions

$$u_{ts}(t,s)\in a(t,s,u(t,s),u_t(t,s),u_s(t,s))$$
$$u_t(t,0)\in b_1(t,u(t,0)),u_s(0,s)\in b_2(s,u(0,s)),u(0,0)\in M_0 \tag{1}$$

for almost all $t,s\in[0,1]$ is called a solution of the problem (1).

Let $f:[0,1]\times[0,1]\times R^{4n}\to\overline{R}$, $\varphi_1:[0,1]\times R^{2n}\to\overline{R}$, $\varphi_2:[0,1]\times R^{2n}\to\overline{R}$ be measurable integrants, $q:R^{4n}\to\overline{R}$. The solution of the inclusion (1) minimizing the functional

$$J(u)=\int_0^1\int_0^1 f(t,s,u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))dtds+\int_0^1\varphi_1(t,u(t,0),u_t(t,0))dt+$$
$$+\int_0^1\varphi_2(s,u(0,s),u_s(0,s))ds+q(u(0,0),u(1,0),u(0,1),u(1,1)) \tag{2}$$

among all the solutions of the problem (1) is called, optimal. It's required to find the necessary conditions optimality of a solution of the problem (1)-(2).

Proposition 1. *If the set $a(t,s,x,y,z)$ is non-empty and compact, the map $(t,s)\to a(t,s,x,y,z)$ is measurable,*

$$\rho_X(a(t,s,x_1,y_1,z_1),a(t,s,x_2,y_2,z_2))\le K(|x_2-x_1|+|y_2-y_1|+|z_2-z_1|),$$

where $K>0$, and for $\overline{u}\in A^n([0,1]\times[0,1])$ the conditions

$$d(\overline{u}_{ts}(t,s),a(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s))\le\rho(t,s),$$
$$d(\overline{u}(0,s),\varphi(s))\le\xi_1(s),\quad d(\overline{u}(t,0),\psi(t))\le\eta_1(t),$$
$$d(\overline{u}_s(0,s),\dot\varphi(s))\le\xi_2(s),\quad d(\overline{u}_t(t,0),\dot\psi(t))\le\eta_2(t),$$

where $\rho(\cdot)\in L_1([0,1]\times[0,1])$, $\xi_1(\cdot)\in C[0,1]$, $\eta_1(\cdot)\in C[0,1]$, $\xi_2(\cdot)\in L_1[0,1]$, $\eta_2(\cdot)\in L_1[0,1]$, $\varphi(\cdot)\in W^n_{1,1}[0,1]$, $\varphi(\cdot)\in W^n_{1,1}[0,1]$, $\varphi(0)=\psi(0)$ *are satisfied, then exist such solutions of the problem*

$$u_{ts}(t,s)\in a(t,s,u(t,s),u_t(t,s),u_s(t,s)),$$
$$u(t,0)=\psi(t),\ u(0,s)=\varphi(s),$$

that

$$|u(t,s)-\overline{u}(t,s)|\le \xi_1(s)+\eta_1(t)+de^{3k(t+s)}+\xi_1(0),$$

$$|u_t(t,s)-\overline{u}_t(t,s)|\le 3dke^{3k(t+s)}+\eta(t)e^{ks}+\int_0^s e^{k(s-v)}\rho(t,v)dv,$$

$$|u_s(t,s)-\overline{u}_s(t,s)|\le 3dke^{3k(t+s)}+\xi(s)e^{kt}+\int_0^t e^{k(t-\tau)}\rho(\tau,s)d\tau,$$

$$|u_{ts}(t,s)-\overline{u}_{ts}(t,s)|\le 2dk^2+kd+\rho(t,s)+\eta(t)ke^{ks}+\xi(s)ke^{kt}+$$

$$+9k^2de^{3k(t+s)}+k\int_0^t e^{k(t-\tau)}\rho(\tau,s)d\tau+k\int_0^s e^{k(s-v)}\rho(t,v)dv,$$

where $\xi(s)=\xi_1(s)+\xi_2(s)+\xi_1(0)$, $\eta(t)=\eta_1(t)+\eta_2(t)$, $a_1=\int_0^1\xi(s)ds$, $a_2=\int_0^1\eta(t)dt$,

$b=\int_0^1\int_0^1\rho(\tau,v)d\tau dv$, $d=\max(a_1,a_2,b)$.

Proposition 2. *Let* $a(t,s,x,y,z)$ *be non-empty and compact for* $t,s\in[0,1]$, $|x-u_0(t,s)|\le\varepsilon$, $y,z\in R^n$, $gr\,a_{ts}=\{(x,y,z,v): v\in a(t,s,x,y,z)\}$ *be closed and convex almost for all* $(t,s)\in[0,1]\times[0,1]$, *there exists* $c>0$ *such that* $\|a(t,x,w)\|\le c(1+|w|)$, $w\in R^{3n}$, *where* $\|a(t,s,w)\|=\sup\{|v|: v\in a(t,s,w)\}$, $\|\varnothing\|=0$, *and let* $u_0(t,s)$ *be a solution of the problem (1),* $(t,s)\to a(t,s,w)$ *be measurable,* $b_1(t,x)$ *be non-empty and compact for* $t\in[0,1]$, $|x-u_0(t,0)|\le\varepsilon_1$; $grb_{1_t}=\{(x,y): y\in b_1(t,x)\}$ *be closed and convex almost for all* $t\in[0,1]$, $t\to b_1(t,x)$ *is measurable;* $b_2(s,y)$ *is non-empty and compact for* $s\in[0,1]$, $|y-u_0(0,s)|\le\varepsilon_2$, $grb_{2_s}=\{(y,z): z\in b_2(s,y)\}$ *be closed and convex almost for all* $s\in[0,1]$, $s\to b_2(s,y)$ *is measurable. Besides there exists the summable functions* $\lambda_1(t)$ *and* $\lambda_2(s)$ *such that*

$$\|b_1(t,x)\|\le\lambda_1(t)(1+|x|),\ \ \|b_2(s,y)\|\le\lambda_2(s)(1+|y|),\ \ x,y\in R^n.$$

Then for $\eta>0$ there exists $\delta>0$ such that for $z\in A^n([0,1]\times[0,1])$, $z(0,0)=0$, $\|z(\cdot)\|_{A^n}\le\delta$ there exists a solution $u_z(t,s)$ of the problem

$$u_{ts}(t,s)\in a(t,s,u(t,s),u_t(t,s)+z_t(t,s),u_s(t,s)+z_s(t,s))-z_{ts}(t,s),$$

$$u_t(t,0)\in b_1(t,u(t,0))-z_t(t,0),\ u_s(0,s)\in b_2(s,u(0,s))-z_s(0,s),\ u(0,0)=u_0(0,0) \qquad (1')$$

and $\|u_0-u_z\|_{A^n}\le\eta$.

Assume

$$\omega(t,s,x,y_1,y_2,z)=\begin{cases}0, & z\in a(t,s,x,y_1,y_2)\\ +\infty, & z\notin a(t,s,x,y_1,y_2),\end{cases}$$

$$\omega_1(t,x,y_1)=\begin{cases}0, & y_1\in b_1(t,x)\\ +\infty, & y_1\notin b_1(t,x),\end{cases}$$

$$\omega_2(s,x,y_2)=\begin{cases}0, & y_2\in b_2(s,x)\\ +\infty, & y_2\notin b_2(s,x),\end{cases}\qquad \omega_0(x)=\begin{cases}0, & x\in M_0\\ +\infty, & x\notin M_0.\end{cases}$$

The problem (1), (2) is equivalent to the following problem

$$\Phi_0(u)=J(u)+\int_0^1\int_0^1\omega(t,s,u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))dtds+\int_0^1\omega_1(t,u(t,0),u_t(t,0))dt+$$
$$+\int_0^1\omega_2(s,u(0,s),u_s(0,s))ds+\omega_0(u(0,0))\xrightarrow{u\in A^n([0,1]\times[0,1])}\inf \tag{3}$$

Consider the functional

$$\Phi(u,z)=\int_0^1\int_0^1 f(t,s,u(t,s),u_t(t,s)+z_t(t,s),u_s(t,s)+z_s(t,s),u_{ts}(t,s)+z_{ts}(t,s))dtds+$$
$$+\int_0^1\varphi_1(t,u(t,0),u_t(t,0)+z_t(t,0))dt+\int_0^1\varphi_2(s,u(0,s),u_s(0,s)+z_s(0,s))ds+q(u(0,0),u(1,0),$$
$$,u(0,1),u(1,1))+\int_0^1\int_0^1\omega(t,s,u(t,s),u_t(t,s)+z_t(t,s),u_s(t,s)+z_s(t,s),u_{ts}(t,s)+z_{ts}(t,s))dtds+$$
$$+\int_0^1\omega_1(t,u(t,0),u_t(t,0)+z_t(t,0))dt+\int_0^1\omega_2(s,u(0,s),u_s(0,s)+z_s(0,s))ds+\omega_0(u(0,0)),$$

where $z\in A_0^n=\{u\in A^n([0,1]\times[0,1]):u(0,0)=0\}$ and let f,φ_1 and φ_2 be convex normal integrants, q be convex, the mappings $w\to a(t,s,w)$, $x\to b_1(t,x)$, $y\to b_2(s,y)$ be convex and upper semi-continuous, the mappings $(t,s)\to a(t,s,w)$, $t\to b_1(t,x)$, $s\to b_2(s,y)$ be measurable, the set M_0 be convex.

It's clear that Φ is a convex function (u,z) and $\Phi(u,0)=\Phi_0(u)$. For any $z\in A_0^n$ consider the minimization problem

$$\inf_u \Phi(u,z). \tag{4}$$

The problem (4) is called perturbation of the problem (3). The problem

$$\sup\{-\Phi^*(0,z^*)\}, \tag{5}$$

where $z^*\in A_0^n([0,1]\times[0,1])^*$, is called dual to (3) with respects the given function Φ. It's case to check that

$$\sup\{-\Phi^*(0,z^*)\}\le\inf\Phi(u,0).$$

But the equality

$$\sup\{-\Phi^*(0,z^*)\}=\inf\Phi(u,0) \tag{6}$$

is of particular interest.

Assume $h(z)=\inf_u\Phi(u,z)$. By lemma 3.2.1 [7] h is a convex function. The problem (3) is called stable, if $h(0)$ is finite and is h subdifferentiable in zero. From proposition 3.2.2 and remark 3.2.3 [7] it follows that if the problem (3) is stable, then the relation (6) is satisfied and the problem (5) has at least one solution.

Lemma 1. *Let $u_0(t,s)$ be a solution of the problem (1), $a(t,s,x,y,z)$ be non-empty and compact for $t,s\in[0,1]$, $|x-u_0(t,s)|\le\varepsilon$, $y,z\in R^n$; $b_1(t,x)$ be non-empty and compact for $t\in[0,1]$, $|x-u_0(t,0)|\le\varepsilon_1$, $b_2(s,y)$ be non-empty and compact for $s\in[0,1]$, $|y-u_0(0,s)|\le\varepsilon_2$, the mappings $w\to a(t,s,w)$, $x\to b_1(t,x)$, $y\to b_2(s,y)$ be convex and upper semi-continuous, the mappings $(t,s)\to a(t,s,w)$, $t\to b_1(t,x)$, $s\to b_2(s,y)$ be measurable, there exist the number λ and the summable functions $\lambda_1(t)$ and $\lambda_2(s)$ such* *that* $\|a(t,s,z)\| \le \lambda \cdot (1+|z|)$, $\|b_1(t,x)\| \le \lambda_1(t)\cdot(1+|x|)$, $\|b_2(s,y)\| \le \lambda_2(s)\cdot(1+|y|)$, *the set* M_0 *be convex,* f, φ_1 *and* φ_2 *be convex normal integrants,* q *be convex, there exist the functions* $\alpha(\cdot)\in L_1[0,1]^2$, $\alpha_1(\cdot), \alpha_2(\cdot)\in L_1[0,1]$, *the number* $c \ge 0$, $r > 0$ *such that* $|f(t,s,u_0(t,s)+y,z)| \le \alpha(t,s)+c\cdot|z|$, $|\varphi_1(t,u_0(t,0)+y,z_1)| \le \alpha_1(t)+c\cdot|z_1|$, $|\varphi_2(s,u_0(0,s)+y,z_2)| \le \alpha_2(s)+c\cdot|z_2|$ *for* $y\in R^n$, $|y|\le r$, *function* $q(u_0(0,0),\cdot)$ *be continuous at the point* $(u_0(1,0),u_0(0,1),u_0(1,1))$ *and* $\inf \Phi_0(u)$ *be finite. Then the problem (3) is stable.*

Proof. By the condition the functional

$$J_1(u,z)=\int_0^1\int_0^1 f(t,s,u_0(t,s)+u(t,s),u_{0_t}(t,s)+u_t(t,s)+z_t(t,s),u_{0_s}(t,s)+u_s(t,s)+z_s(t,s),$$

$$,u_{0_{ts}}(t,s)+u_{ts}(t,s)+z_{ts}(t,s))dtds+\int_0^1\varphi_1(t,u_0(t,0)+u(t,0),u_{0_t}(t,0)+u_t(t,0)+z_t(t,0))dt+$$

$$+\int_0^1\varphi_2(s,u_0(0,s)+u(0,s),u_{0_s}(0,s)+u_s(0,s)+z_s(0,s))ds+q(u_0(0,0),u_0(1,0)+u(1,0),$$
$$u_0(0,1)+u(0,1),u_0(1,1)+u(1,1))$$

is continuous at the point $(0,0)$ with respect to the topology of the space $A_0^n\times A_0^n$. From convexity and continuity of $J_1(u,z)$ at zero it follows that there exist the numbers $\alpha>0$ and M such that $J_1(u,z)\le M$ for $(u,z)\in A_0^n\times A_0^n$, $\|(u,z)\|\le\alpha$. By proposition 2 for $\frac{\alpha}{2}$ there exist $\delta>0$ such that for $z\in A_0^n$, $\|z(\cdot)\|_{A_0^n}\le\delta$ the solution $u_z(t,s)$ of the problem (1`) exists and $\|u_0-u_z\|_{A^n}\le\frac{\alpha}{2}$, $u_z(0,0)=u_0(0,0)$. Therefore we obtain the following estimation

$$h(z)=\inf\{\Phi(u,z):u\in A^n\}\le\Phi(u_z,z)\le M$$

for $z\in A_0^n$, $\|z\|\le\frac{\alpha}{2}$. According to proposition 1.2.5 [7] from here it follows that the function h is continuous at zero. Then from proposition 1.2.5 [7] it follows that the function h is subdifferentiable at zero. The lemma is proved.

Lemma 2. *Let the mappings* a, b_1 *and* b_2 *satisfy the conditions of proposition 1. Besides let* $\omega^0(t,s,u_0(t,s),y(t,s))$ *be summable. Then for any functions* $y(\cdot)\in L_\infty^{3n}([0,1]\times[0,1])$, $y_1(\cdot), y_2(\cdot)\in L_\infty^n[0,1]$ *the functionals*

$$u\to\int_0^1\int_0^1\omega^0(t,s,u(t,s),y(t,s))dtds,\ x\to\int_0^1\omega_1^0(t,x(t),y_1(t))dt,\ y\to\int_0^1\omega_2^0(s,y(s),y_2(s))ds$$

are continuous with respect to uniform topology at the points $u_0(t,s)$, $u_0(t,0)$ *and* $u_0(0,s)$ *respectively.*

Proof. Let $|x|<\varepsilon$. Since

$\omega^0(t,s,u_0(t,s)+x,y(t,s))=\inf\{(z|y(t,s):z_1\in a(t,s,u_0(t,s)+x,z_2,z_3),\ z=(z_1,z_2,z_3)\}\le$

$\leq \min\{(u_{0_t}(t,s)|y_1(t,s))+(u_{0_s}(t,s)|y_2(t,s))+(z_1|y_3(t,s)): z_1 \in a(t,s,u_0(t,s)+$
$+x, u_{0_t}(t,s), u_{0_s}(t,s))\} = (u_{0_t}(t,s)|y_1(t,s)) + (u_{0_s}(t,s)|y_2(t,s)) + (\bar{z}(t,s)|y_3(t,s))$,

where $\bar{z}(t,s) \in a(t,s,u_0(t,s)+x, u_{0_t}(t,s), u_{0_s}(t,s))$. By the condition $|\bar{z}(t,s)| \leq \lambda \cdot (1+ |u_0(t,s)+x| + |u_{0_t}(t,s)| + |u_{0_s}(t,s)|)$, therefore $\bar{z}(t,s)$ is summable. It's clear that

$$\int_0^1\int_0^1 \omega^0(t,s,u_0(t,s)+x, y(t,s))dtds \leq \int_0^1\int_0^1 (u_{0_t}(t,s)|y_1(t,s))dtds + \int_0^1\int_0^1 (u_{0_s}(t,s)|y_2(t,s))dtds +$$

$$+\|y_3(\cdot)\| \cdot \int_0^1\int |\bar{z}(t,s)|dtds < +\infty .$$

Using proposition 1.2.5 [7], hence we obtain the continuous of the functional $u \to \int_0^1\int_0^1 \omega^0(t,s,u(t,s),y(t,s))dtds$ in the space $C^n([0,1]\times[0,1])$ at the point $u_0(t,s)$.

The continuity of the other functionals is proved analogously. The lemma is proved.

Using lemmas 1 and 2 following theorems 1 and 2 are proved analogously to theorem 1 and 2 [8].

Theorem 1. *Let f, φ_1 and φ_2 be convex normal integrants, q be convex, the mappings $w \to a(t,s,w)$, $x \to b_1(t,x)$, $y \to b_2(s,y)$ be convex and upper semi-continuous, the mappings $(t,s) \to a(t,s,w)$, $t \to b_1(t,x)$, $s \to b_2(s,y)$ be measurable, the set M_0 be convex. For $\bar{u}$ to be a minimum point of the functional $\Phi_0(u)$ on the space $A^n([0,1]\times[0,1])$ it is sufficient, and if the conditions of lemma 1 are satisfied and it's necessary that the functions $\overline{P}_1 \in L^n_\infty([0,1]\times[0,1])$, $\overline{P}_2 \in L^n_\infty([0,1]\times[0,1])$, the measures $\lambda \in frm([0,1]\times[0,1])^n$, $\mu \in frm[0,1]^n$, $\gamma \in frm[0,1]^n$, the functional $\bar{v}^* = (0, v_1(\cdot), v_2(\cdot), v(\cdot)) \in A^n([0,1]\times[0,1])^*$ and the vectors $\bar{c}, d_1, d_2, d \in R^n$ were found, such that for $T = S = 1$ and by substitution of the functions $f, \varphi_1, \varphi_2, q$ by the functions $f+\omega, \varphi_1+\omega_1, \varphi_2+\omega_2, q+\omega_0$ respectively, relations 1)-11) of theorem 1 [8] were satisfied.*

Assuming $f(t,s,x,y_1,y_2,z)+\delta_{M(t,s)}(x)$ we obtain that the problem (1), (2) covers phase boundaries in the form of $u(t,s) \in M(t,s)$, where $M: [0,1]\times[0,1] \to 2^{R^n}$.

Theorem 2. *Let f, φ_1 and φ_2 be convex normal integrants, q be convex, the mappings $w \to a(t,s,w)$, $x \to b_1(t,x)$, $y \to b_2(s,y)$ be convex and upper semi-continuous, the mappings $(t,s) \to a(t,s,w)$, $t \to b_1(t,x)$, $s \to b_2(s,y)$ be measurable, the set M_0 be convex. For $\bar{u}$ to be a minimum point of the functional $\Phi_0(u)$ on the space $A^n([0,1]\times[0,1])$ it is sufficient, and if for $u_0 = \bar{u}$ the conditions of lemma 1 are satisfied and it's necessary that the functions $\overline{P}_1 \in L^n_\infty([0,1]\times[0,1])$, $\overline{P}_2 \in L^n_\infty([0,1]\times[0,1])$, $v(\cdot) \in A^n([0,1]\times[0,1])$, where $v(1,s) = v(t,1)$ for $t,s \in [0,1]$, $v_1(\cdot) \in W^n_{1,1}[0,1]$, $v_2(\cdot) \in W^n_{1,1}[0,1]$, were found, such that for $T = S = 1$ and by substitution of the functions*

$f, \varphi_1, \varphi_2, q$ by the functions $f+\omega, \varphi_1+\omega_1, \varphi_2+\omega_2, q+\omega_0$, respectively, relations 1)-7) of theorem 2 [8] were satisfied.

Assume $\psi(t,s,z,\omega)=\inf\{|u-\omega|:u\in a(t,s,z)\}$, $q_1(t,x,y)=\inf\{|z-y|:z\in b_1(t,x)\}$, $q_2(s,x,y)=\inf\{|u-y|:u\in b_2(s,x)\}$, $q_0(x)=\inf\{|u-x|:u\in M_0\}$, $F(u)=\int_0^1\int_0^1\psi(t,s,u(t,s),$

$$u_t(t,s),u_s(t,s),u_{ts}(t,s))dtds+\int_0^1 q_1(t,u(t,0),u_t(t,0))dt+\int_0^1 q_2(s,u(0,s),u_s(0,s))ds+q_0(u(0,0)),$$

$I_m(u)J(u)+l\cdot F(u)$.

Theorem 3. *Let $(t,s)\to a(t,s,z)$, $t\to b_1(t,x)$, $s\to b_2(s,x)$ be measurable, M_0, $b_1(t,x)$, $b_2(s,x)$ and $a(t,s,z)$ be non-empty, compact, there exist the functions $k_1(\cdot)\in L_1[0,1]$, $k_2(\cdot)\in L_1[0,1]$ and the number k such that*

$$\rho_X(a(t,s,x_1,y_1,z_1),a(t,s,x_2,y_2,z_2))\le k\cdot(|x_2-x_1|+|y_2-y_1|+|z_2-z_1|),$$

$$\rho_X(b_1(t,x_1),b_1(t,x_2))\le k_1(t)\cdot|x_2-x_1|,\ \ \rho_X(b_2(s,y_1),b_2(s,y_2))\le k_2(s)\cdot|y_2-y_1|.$$

Besides let there exist the functions $k_3(\cdot)\in L_\infty([0,1]\times[0,1])$, $k_4(\cdot)\in L_\infty[0,1]$, $k_5(\cdot)\in L_\infty[0,1]$ and the number k_0 such that

1) $|f(t,s,z)-f(t,s,z_1)|\le k_3(t,s)\cdot|z-z_1|$ *for* $z,z_1\in R^{4n}$,

2) $|\varphi_1(t,x)-\varphi_1(t,x_1)|\le k_4(t)\cdot|x-x_1|$ *for* $x,x_1\in R^{2n}$

3) $|\varphi_2(s,y)-\varphi_2(s,y_1)|\le k_5(s)\cdot|y-y_1|$ *for* $y,y_1\in R^{2n}$,

4) $|q(\mathrm{v})-q(\mathrm{v}_1)|\le k_0\cdot|\mathrm{v}-\mathrm{v}_1|$ *for* $\mathrm{v},\mathrm{v}_1\in R^{4n}$

and let the mappings $(t,s)\to f(t,s,z)$, $t\to\varphi_1(t,x)$, $s\to\varphi_2(s,y)$ be measurable. Then if $\bar u\in A^n([0,1]\times[0,1])$ is a solution of the problem (1), (2), then there exists the number m_0 such that for $u\in A^n([0,1]\times[0,1])$ and $m\ge m_0$

$$I_m^{\{\beta\}-}(\bar u;u)=\varliminf_{\lambda\downarrow 0}\frac{1}{\lambda^\beta}\cdot(I_m(\bar u+\lambda\cdot u)-I_m(\bar u))\ge 0.$$

Proof. Let $u\in A^n([0,1]\times[0,1])$. Assume $P_1(t)=q_1(t,u(t,0),u_t(t,0))$, $P_2(s)=q_2(s,u(0,s),u_s(0,s))$, $\varphi(0)=\psi(0)\in M_0$, where $q_0(u(0,0))=|u(0,0)-\varphi(0)|$, $\delta=q_0(u(0,0))$. By lemma 2.1 [2] there exist the solutions of $\bar x(t)$ and $\bar y(s)$, respectively, of the problems $\dot x(t)\in b_1(t,x(t))$, $\dot y(s)\in b_2(s,y(s))$, $x(0)=y(0)=\varphi(0)$ that

$$|\bar x(t)-u(t,0)|\le\eta_1(t),\ |\dot{\bar x}(t)-u_t(t,0)|\le k_1(t)\cdot\eta_1(t)+P_1(t),$$

$$|\bar y(s)-u(0,s)|\le\xi_1(s),\ |\dot{\bar y}(s)-u_s(0,s)|\le k_2(s)\cdot\xi_1(s)+P_2(s),$$

where $\eta_1(t)=\delta\cdot e^{m_1(t)}+\int_0^t e^{m_1(t)-m_1(\tau)}\cdot P_1(\tau)d\tau$, $\xi_1(s)=\delta\cdot e^{m_2(s)}+\int_0^s e^{m_2(s)-m_2(v)}\times$ $\times P_2(v)dv$, $m_1(t)=\int_0^t k_1(\tau)d\tau$, $m_2(s)=\int_0^s k_2(v)dv$. Assume in proposition 1 $\psi(t)=\bar x(t)$, $\varphi(s)=\bar y(s)$, $\eta_2(t)=k_1(t)\eta_1(t)+P_1(t)$, $\xi_2(s)=k_2(s)\xi_1(s)+P_2(s)$, $\rho(t,s)=\psi(t,s,u(t,s),$

$u_t(t,s),u_s(t,s),u_{ts}(t,s))$ it's obtained that there exists such solution $u_0(t,s)$ of the problem (1) for which the statement of proposition 1 is satisfied.

It's clear that

$$|J(u)-J(u_0)|\le \int_0^1\int_0^1 k_3(\tau,\nu)|(u(\tau,\nu),u_t(\tau,\nu),u_s(\tau,\nu),u_{ts}(\tau,\nu))-(u_0(\tau,\nu),u_{0_t}(\tau,\nu),$$

$$,u_{0_s}(\tau,\nu),u_{0_{ts}}(\tau,\nu))|d\tau d\nu+\int_0^1 k_4(\tau)|(u(\tau,0),u_t(\tau,0))-(u_0(\tau,0),u_{0_t}(\tau,0))|d\tau+$$

$$+\int_0^1 k_5(\nu)|(u(0,\nu),u_s(0,\nu))-(u_0(0,\nu),u_{0_s}(0,\nu))|d\nu+|q(u(0,0),u(1,0),u(0,1),$$

$$,u(1,1))-q(u_0(0,0),u_0(1,0),u_0(0,1),u_0(1,1))|\le\|k_3(\cdot)\|\cdot(\int_0^1\int_0^1|u(\tau,\nu)-u_0(\tau,\nu)|d\tau d\nu+$$

$$+\int_0^1\int_0^1|u_t(\tau,\nu)-u_{0_t}(\tau,\nu)|d\tau d\nu+\int_0^1\int_0^1|u_s(\tau,\nu)-u_{0_s}(\tau,\nu)|d\tau d\nu+\int_0^1\int_0^1|u_{ts}(\tau,\nu)-$$

$$-u_{0_{ts}}(\tau,\nu)|d\tau d\nu+\|k_4(\cdot)\|(\int_0^1|u(\tau,0)-u_0(\tau,0)|d\tau+\int_0^1|u_t(\tau,0)-u_{0_t}(\tau,0)|d\tau)+$$

$$+\|k_5(\cdot)\|(\int_0^1|u(0,\nu)-u_0(0,\nu)|d\nu+\int_0^1|u_s(0,\nu)-u_{0_s}(0,\nu)|d\nu)+k_0(|u(0,0)-u_0(0,0)|+$$

$$+|u(1,0)-u_0(1,0)|+|u(0,1)-u_0(0,1)|+|u(1,1)-u_0(1,1)|).$$

Using proposition 1 from here we obtain that

$$|J(u)-J(u_0)|\le c_0(\|k_3(\cdot)\|,\|k_4(\cdot)\|,\|k_5(\cdot)\|,k_0)\cdot F(u).$$

We show that for $m\ge c_0$ the function $\overline{u}$ minimized the functions I_m in the space $A^n([0,1]\times[0,1])$ too. We assume contrary. Let there exists $\tilde{u}\in A^n([0,1]\times[0,1])$ such that $I_m(\tilde{u})<I_m(\overline{u})$. Since for $\tilde{u}$ there exists such solution $\tilde{u}_0$ of the problem (1) that

$$|J(\tilde{u})-J(\tilde{u}_0)|\le m\cdot F(\tilde{u}),$$

then

$$J(\tilde{u}_0)\le J(\tilde{u})+m\cdot F(\tilde{u})=I_m(\tilde{u})<J(\overline{u}).$$

The obtained contradiction means that $\overline{u}$ minimizes $I_m(u)$ in $A^n([0,1]\times[0,1])$. Therefore we obtain that $I_m^{\{\beta\}-}(\overline{u};u)\ge 0$. The theorem is proved.

From the conditions $\rho_X(a(t,s,x_1,y_1,z_1),a(t,s,x_2,y_2,z_2))\le$ $\le k(|x_2-x_1|+|y_2-y_1|+|z_2-z_1|)$, $\rho_X(b_1(t,x_1),b_1(t,x_2))\le k_1(t)\cdot|x_2-x_1|$, $\rho_X(b_2(s,y_1),$ $b_2(s,y_2))\le k_2(s)\cdot|y_2-y_1|$, it follows that $|\psi(t,s,z,\omega)|\le k\,|z-(\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s))|+$ $+|\omega-\overline{u}_{ts}(t,s)|$, $|q_1(t,x,y)|\le k_1(t)\cdot|x-\overline{u}(t,0)|+|y-\overline{u}_t(t,0)|$, $|q_2(s,x,y)|\le k_2(s)\times$ $\times|x-\overline{u}(0,s)|+|y-\overline{u}_s(0,s)|$. Therefore, if $(t,s)\to a(t,s,z)$, $t\to b_1(t,x)$, $s\to b_2(s,y)$ are measurable, then for any $u\in A^n([0,1]\times[0,1])$ of the function $\psi(t,s,u(t,s),u_t(t,s),u_s(t,s),$ $u_{ts}(t,s))$, $q_1(t,u(t,0),u_t(t,0))$, $q_2(s,u(0,s),u_s(0,s))$ are summable.

Let X be a Banach space, $f:X\to\overline{R}$, $|f(x)|<+\infty$, $\nu\in X$. Assume [9]

$$f^0(x;v)=\limsup_{\substack{y\to x\\ t\downarrow 0}}\frac{f(y+t\cdot v)-f(y)}{t},\ f^+(x;v)=\limsup_{\substack{(y,\alpha)\downarrow f^x,\omega\to v\\ \lambda\downarrow 0}}\frac{f(y+\lambda\cdot\omega)-\alpha}{\lambda},$$

$$f^{\uparrow}(x;v)=\lim_{\varepsilon\downarrow 0}\limsup_{\substack{(y,\alpha)\downarrow f^x\\ \lambda\downarrow 0}}\inf_{\omega\in v+\varepsilon\cdot B}\frac{f(y+\lambda\cdot\omega)-\alpha}{\lambda},$$

where the symbol $(y,\alpha)\downarrow f^x$ means that $(y,\alpha)\in epi\, f$, $y\to x$, $\alpha\to f(x)$.

Theorem 4. *Let $\overline{u}\in A^n([0,1]\times[0,1])$ be a solution of the problem (1), (2) and the conditions of theorem 3 are satisfied. Then there exist the number $m>0$ and the functions $\overline{P}_1\in L^n_\infty([0,1]\times[0,1])$, $\overline{P}_2\in L^n_\infty([0,1]\times[0,1])$, $v(\cdot)\in A^n([0,1]\times[0,1])$, where $v(1,s)=v(t,1)$ for $t,s\in[0,1]$, $v_1(\cdot)\in W^n_{1,1}[0,1]$, $v_2(\cdot)\in W^n_{1,1}[0,1]$, such that*

1) $(v_{ts}(t,s),-\overline{P}_1(t,s),-\overline{P}_2(t,s),\int_0^1\overline{P}_1(t,v)dv-\int_0^s\overline{P}_1(t,v)dv+\int_0^1\overline{P}_2(\tau,s)d\tau-\int_0^t\overline{P}_2(\tau,s)d\tau-$
$-v(t,s))\in\partial(f(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s))+m\psi(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s))$,

2) $(v_t(t,0)-\dot{v}_1(t),\int_0^1\overline{P}_1(t,v)dv-v_1(t))\in\partial(\varphi_1(t,\overline{u}(t,0),\overline{u}_t(t,0))+m\cdot q_1(t,\overline{u}(t,0),\overline{u}_t(t,0))$,

3) $(v_s(0,s)-\dot{v}_2(s),\int_0^1\overline{P}_2(\tau,s)d\tau-v_2(s))\in\partial(\varphi_2(s,\overline{u}(0,s),\overline{u}_s(0,s))+mq_2(s,\overline{u}(0,s),\overline{u}_s(0,s))$,

4) $(v(0,0)-v_1(0)-v_2(0),v_1(1)-v(1,0),v_2(1)-v(0,1),v(1,1))\in\partial(q(\overline{u}(0,0),\overline{u}(1,0),$
$\overline{u}(0,1),\overline{u}(1,1))+mq_0(\overline{u}(0,0)))$.

Proof. In theorem 3 it's proved that there exists such number $m>0$ that $\overline{u}$ minimizes $I_m(u)=J(u)+m\cdot F(u)$ in $A^n([0,1]\times[0,1])$. Therefore $I^0_m(\overline{u};u)\geq 0$. Using the Fatou lemma ([10]) we obtain that

$$0\leq I^0_m(\overline{u};u)\leq\Phi(u)=\int_0^1\int_0^1 f^0(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s);u(t,s),u_t(t,s),u_s(t,s),$$

$$,u_{ts}(t,s))dtds+\int_0^1\varphi_1^0(t,\overline{u}(t,0),\overline{u}_t(t,0);u(t,0),u_t(t,0))dt+\int_0^1\varphi_2^0(s,\overline{u}(0,s),\overline{u}_s(0,s);u(0,s),$$

$$,u_s(0,s))ds+q^0(\overline{u}(0,0),\overline{u}(1,0),\overline{u}(0,1),\overline{u}(1,1);u(0,0),u(1,0),u(0,1),u(1,1))+$$

$$+m\cdot(\int_0^1\int_0^1\psi^0(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s);u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))dtds+$$

$$+\int_0^1 q_1^0(t,\overline{u}(t,0),\overline{u}_t(t,0);u(t,0),u_t(t,0))dt+\int_0^1 q_2^0(s,\overline{u}(0,s),\overline{u}_s(0,s);u(0,s),u_s(0,s))ds+$$

$$+q^0(\overline{u}(0,0);u(0,0)).$$

It's clear that $u=0$ minimizes $\Phi(u)$ on $A^n([0,1]\times[0,1])$ and the conditions of theorem 2 [8] are satisfied. Therefore there exists the function $\overline{P}_1\in L^n_\infty([0,1]\times[0,1])$, $\overline{P}_2\in L^n_\infty([0,1]\times[0,1])$, $v(\cdot)\in A^n([0,1]\times[0,1])$, where $v(1,s)=v(t,1)$ for $t,s\in[0,1]$,

Theorem 5. *Let $\overline{u}(t,s)$ be a solution of the problem (2), (7), the condition of theorem 3 be satisfied, the mapping $(t,s)\to M(t,s)$ be continuous and $\operatorname{int}T_{M(t,s)}(\overline{u}(t,s))$ be non-empty. Besides $\lim\limits_{l\to+\infty}\inf\limits_{u\in A^n([0,1]\times[0,1])}E_l(u)=J(\overline{u})$. Then there exist the functions $\overline{p}_1\in L_\infty^n([0,1]\times[0,1])$, $\overline{p}_2\in L_\infty^n([0,1]\times[0,1])$, $\omega_1(\cdot)\in L_1^n[0,1]$, $\omega_2(\cdot)\in L_1^n[0,1]$, the measure $\lambda\in frm([0,1]\times[0,1])^n$, the vectors $\overline{c},d_1,d_2,d\in R^n$ and the functional $\overline{\mathrm{v}}^*=(0,\mathrm{v}_1,\mathrm{v}_2,\mathrm{v})\in A^n([0,1]\times[0,1])^*$ such that*

1) $$\mathrm{v}^*(u)=\int_0^1\int_0^1(u(t,s)|d\lambda)+\int_0^1(u(t,0)|\omega_1(t))dt+\int_0^1(u(0,s)|\omega_2(s))ds+(u(0,0)|\overline{c}-d_1-d_2+d)+$$
$$+(u(1,0)|d_1-d)+(u(0,1)|d_2-d)+(u(1,1)|d);$$

2) $$\left(\overline{\omega}(t,s),-\overline{p}_1(t,s),-\overline{p}_2(t,s),-\mathrm{v}(t,s)+\int_0^1\overline{p}_1(t,v)dv-\int_0^s\overline{p}_1(t,v)dv+\int_0^1\overline{p}_2(\tau,s)d\tau-\right.$$
$$\left.-\int_0^t\overline{p}_2(\tau,s)d\tau\right)\in\partial f(t,s,\overline{u}(t,s),\overline{u}_t(t,s)\overline{u}_s(t,s)\overline{u}_{ts}(t,s))+\partial\omega(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),$$
$$\overline{u}_{ts}(t,s))+(N_{M(t,s)}(\overline{u}(t,s)),0);$$

3) $$\left(\omega_1(t),\int_0^1\overline{p}_1(t,v)dv-\mathrm{v}_1(t)\right)\in\partial\varphi_1(t,\overline{u}(t,0),\overline{u}_t(t,0))+N_{grb_{1_t}}(\overline{u}(t,0),\overline{u}_t(t,0));$$

3) $$\left(\omega_2(s),\int_0^1\overline{p}_2(\tau,s)d\tau-\mathrm{v}_2(s)\right)\in\partial\varphi_2(s,\overline{u}(0,s),\overline{u}_s(0,s))+N_{grb_{2_s}}(\overline{u}(0,s),\overline{u}_s(0,s));$$

4) $$(\overline{c}-d_1-d_2+d,d_1-d,d_2-d,d)\in\partial q(\overline{u}(0,0),\overline{u}(1,0),\overline{u}(0,1),\overline{u}(1,1))+(N_{M_0}(\overline{u}(0,0)),0);$$

5) $$\max\left\{\int_0^1\int_0^1(z(t,s)|d\lambda_s):z(t,s)\in T_{M(t,s)}(\overline{u}(t,s)),z(\cdot)\in A^n([0,1]\times[0,1])\right\}=0,$$

where $\lambda(E)=\int_E\overline{\omega}(t,s)ds+\lambda_s(E)$ is a Lebesque expansion of λ.

Proof. From the equality $\lim\limits_{l\to+\infty}\inf\limits_{u\in A^n([0,1]\times[0,1])}E_l(u)=J(\overline{u})$ it follows that for $\frac{1}{k}$ there exists such l_k that for $l\ge l_k$

$$\left|\inf_{u\in A^n([0,1]\times[0,1])}E_l(u)-J(\overline{u})\right|<\frac{1}{k}.$$

Without losing generality we can assume that $l_{k_1}<l_{k_2}$ for $k_1<k_2$. Assume $v(k)=l_k$ and consider the function

$$\overline{E}_k(u)=J(u)+v(k)\left(F(u)+\int_0^1\int_0^1W(t,s,u(t,s))dtds\right)+\int_0^1\int_0^1\delta_{\overline{u}(t,s)+\varepsilon\overline{B}}(u(t,s))dtds+$$
$$+\int_0^1\int_0^1\delta_{\lambda_1(t,s)\overline{B}}(u_{ts}(t,s))dtds+\int_0^1\delta_{\lambda_2(t)\overline{B}}(u_t(t,0))dt+\int_0^1\delta_{\lambda_3(s)\overline{B}}(u_s(0,s))ds.$$

By conditions the functional $\overline{E}_k(u)$ is lower semi-continuous in the space $A^n([0,1]\times[0,1])$. Since $\inf \overline{E}_k(u)+\frac{1}{k}\geq J(\overline{u})=\overline{E}_k(\overline{u})$, then by Ekeland theorem there exist $u_k(\cdot)\in A^n([0,1]\times[0,1])$, that $\overline{E}_k(u_k)\leq J(\overline{u})$, $\|u_k-\overline{u}\|\leq\frac{1}{\sqrt{k}}$ and minimizes the functional

$$\widetilde{E}_k(u)=\overline{E}_k(u)+\frac{1}{\sqrt{k}}\|u-u_k\|.$$

Using the Fatou theorem we obtain

$$\widetilde{E}_0(u_k;u)\leq S_k(u)=\int_0^1\int_0^1\Big(f^0(t,s,u_k(t,s),u_{k_t}(t,s),u_{k_s}(t,s),u_{k_{ts}}(t,s);u(t,s),u_t(t,s),$$

$$u_s(t,s),u_{ts}(t,s))+\nu(k)\psi^0(t,s,u_k(t,s),u_{k_t}(t,s),u_{k_s}(t,s),u_{k_{ts}}(t,s);u(t,s),u_t(t,s),u_s(t,s),$$

$$u_{ts}(t,s)\Big)dtds+\int_0^1\Big(\varphi_1^0(t,u_k(t,0),u_{k_t}(t,0);u(t,0),u_t(t,0))+\nu(k)q_1^0(t,u_k(t,0),u_{k_t}(t,0);u(t,0),$$

$$u_t(t,0))\Big)dt+\int_0^1\Big(\varphi_2^0(s,u_k(0,s),u_{k_s}(0,s);u(0,s),u_s(0,s))+\nu(k)q_2^0(s,u_k(0,s),u_{k_s}(0,s);$$

$$u(0,s),u_s(0,s))\Big)ds+q^0(u_k(0,0),u_k(1,0),u_k(0,1),u_k(1,1);u(0,0),u(1,0),u(0,1),u(1,1))+$$

$$+\nu(k)q_0^0(u_k(0,0);u(0,0))+\nu(k)\int_0^1\int_0^1 W^0(t,s,u_k(t,s);u(t,s))dtds+\frac{1}{\sqrt{k}}\|u\|.$$

Subject to $0=S_k(0)\leq S_k(u)$ and the derivative in the direction of Klark is upper semi-continuous [9], then passing to the limit in $S_k(u)$ where $k\to+\infty$ and using lemma 3.2.4 [5] we obtain

$$0\leq\int_0^1\int_0^1\Big(f^0(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s);u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))+$$

$$+\omega^{\uparrow}(t,s,\overline{u}(t,s),\overline{u}_t(t,s),\overline{u}_s(t,s),\overline{u}_{ts}(t,s);u(t,s),u_t(t,s),u_s(t,s),u_{ts}(t,s))+$$

$$\delta_{T_{M(t,s)}(\overline{u}(t,s))}(u(t,s))\Big)dtds+\int_0^1\Big(\varphi_1^0(t,\overline{u}(t,0),\overline{u}_t(t,0);u(t,0),u_t(t,0))+$$

$$+\delta_{T_{grb_{1_t}}(\overline{u}(t,0),\overline{u}_t(t,0))}(u(t,0),u_t(t,0))\Big)dt+\int_0^1\Big(\varphi_2^0(s,\overline{u}(0,s),\overline{u}_s(0,s);u(0,s),u_s(0,s))+$$

$$+\delta_{T_{grb_{2_s}}(\overline{u}(0,s),\overline{u}_s(0,s))}(u(0,s),u_s(0,s))\Big)ds+q^0(\overline{u}(0,0),\overline{u}(1,0),\overline{u}(0,1),\overline{u}(1,1);u(0,0),u(1,0),$$

$$u(0,1),u(1,1))+\delta_{T_{M_0}(\overline{u}(0,0))}(u(0,0)).$$

Using theorem 1[8] we obtain the validity of theorem 5.

References

[1]. Sadygov M.A. *On minimization of integral functionals in Sobolev spaces.* Preprint №165, Baku, 1986, 48p. (in Russian).

[2]. Sadygov M.A. *Necessary condition of extremum for differential inclusions.* Baku-1991, Preprint №426, 42p. (in Russian).

[3]. Sadygov M.A. *On necessary conditions of minimum for multi-dimensional differential inclusions.* The questions of applied non-linear analysis, Baku, "Elm", 1994, p.3-28 (in Russian).
[4]. Sadygov M.A. *On extermal problem for two-dimensional differential inclusions.* Izv. AN Azerb., 1995, №1-3, p.71-81 (in Russian).
[5]. Sadygov M.A. *Extermal problems for non-smooth systems.* Baku, 1996, 148p. (in Russian).
[6]. Sadygov M.A. *On extermal problem for two-dimensional Goursat-Darboux type differential inclusions.* Trudy IMM AN Azerb., 1998, v.IX(XVI), p.91-101 (in Russian).
[7]. Ekeland I., Temam P. *Convex analysis and variation problems.* M., Mir, 1979, 399.
[8]. Sadygov M.A. *On extermal problem for Goursat-Darboux type differential inclusions, I.*
[9]. Clark F. *Optimization and non-smooth analysis.* M., Nauka, 1988, 280p.
[10]. Federer G. *Geometrical theory of measure.* M., Nauka, 1987.

Sadygov M.A.
Institute of Mathematics & Mechanics of NAS of Azerbaijan.
9, F.Agayeva str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20(off.).

Received November 16, 2000; Revised March 12, 2001.
Translated by Mirzoyeva K.S.