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ON ONE BOUNDARY PROBLEM FOR THE 6[th] ORDER POLYHARMONIC EQUATIONS

Abstract

The article is dedicated to the solution of one of the main boundary value problems for a polyharmonic equation of the 6-th order, taken in two dimensional space. The solution of the given problem is found as the sum of three potentials. The bound formulas are proved for these potentials and their derivatives. Using these formulas the solution of the boundary problem is reduced to the solution of series of Fredholm type integral equations.

As it is known one of steps M.L. Rasulov is contour integral method [4], [2] is the investigation of a boundary value problem with complex parameter called a spectral problem we have to note that transference of this theory to more than second order equations is of special interest. There exist a lot of articles dedicated to the solution of such problems (for ex. [3]-[7]).

Article [3] in which biharmonic potentials for space and plane cases, is found very fruitful in this domain. The question on limit values of potentials themselves, their normal derivatives, Laplacian operators and normal derivatives of Laplacian operator is also investigated. Using the results of paper [3] in papers [5]-[7] the special potentials depending on a complex parameter, good decreasing for sufficiently large values of this parameter from some infinite part of a complex plane and having weak pointwise singularity by a space variable are constructed. With the help of these potentials the existence of solutions of boundary value problems in domain of three dimensions for the 4^{th} order [5] and 6^{th} order [6] polyharmonic equations by means of reducing them to the system of regular integral equations is proved.

The present paper is dedicated to the investigation of a boundary value problem for the 6^{th} order polyharmonic equations of 2 dimensional domain and namely the problem of determination of solution of the equation

$$L\left(x,\frac{\partial}{\partial x};\Delta;\lambda^2\right)U(x,\lambda)=\sum_{k=0}^{3}A_k(x)\lambda^{2k}\Delta^{3-k}U(s,\lambda)+\sum_{i=1}^{2}\sum_{j=0}^{2}B_{ij}(x)\lambda^{2j}\Delta^{2-j}\times$$
$$\times\frac{\partial U(x,\lambda)}{\partial x_i}+C(x)U(x,\lambda)=F(x,\lambda) \qquad (1)$$

satisfying the boundary conditions

$$\lim_{x\to z\in\Gamma}U(x,\lambda)=\varphi_0(z,\lambda);\qquad \lim_{x\to z\in\Gamma}\frac{dU(x,\lambda)}{dn_z}=\varphi_1(z,\lambda),$$
$$\lim_{x\to z\in\Gamma}\frac{d}{dn_z}\Delta^2U(x,\lambda)=\varphi_2(z,\lambda), \qquad (2)$$

where $x=(x_1,x_2)$ is a point of some bounded domain D of two dimensional space E_2, Δ is a Laplacian operator, Γ is a boundary of the domain D, n_z is the direction of the interior normal to the boundary Γ at the point $z\in\Gamma$, is considered. Without losing generality assume that $A_3(x)\equiv 1$. We assume the fulfillment of the conditions:

1^0. The real parts of roots of the equation

$$\nu^3 - A_2(x)\nu^2 + A_1(x)\nu - A_0(x) = 0 \tag{3}$$

are negative $(\operatorname{Re}\nu < 0)$, the arguments of these roots and their differences are independent of x.

2^0. The coefficients of the equation (1) $A_k(x)$ $(k = 0,1)$ have continuous bounded derivatives to the $(3-2k)^{\text{th}}$ order for all $x \in D + \Gamma$, $A_2(x)$ is continuous differentiable, $B_{ij}(x)$ $\left(i = 1,2;\ j = \overline{0,2}\right)$ and $C(x)$ are continuous functions for all $x \in D$.

3^0. The boundary Γ of the domain D is a Lyapunov line with the number $0 < \alpha < 1$ and $\Gamma \in \Lambda_2$.

4^0. The right hand side $F(x,\lambda)$ of the equation (1) is continuous and has first order continuous partial derivatives with respect to $x_i (i = 1,2)$ for all $x \in D$ is analytic by λ in the domain R_δ of the complex λ plane, where R_δ is determined by the following form:

$$R_\delta = \left\{ \lambda;\ |\lambda| > R,\ |\arg\lambda| \le \frac{\pi}{4} + \delta \right\}, \tag{4}$$

where R is sufficiently large, and $\delta > 0$, small numbers.

5^0. The boundary functions $\varphi_s(x,\lambda)$ $\left(s = \overline{0,2}\right)$ analytic by λ functions in R_δ tending to zero when $|\lambda| \to \infty$, uniformly with respect to $\arg\lambda$, besides $\varphi_s(z,\lambda)$ $s = (0,1)$ have continuous partial derivatives by $z \in \Gamma$ to the 2^{nd} order, and $\varphi_2(z,\lambda)$ is continuous on Γ.

In connection with that solution of non-homogeneous equation for homogeneous boundary conditions is constructed with the help of Green functions. We'll find a solution of the problem for a homogeneous equation corresponding to the equation (1), i.e. for the equation

$$L\left(x, \frac{\partial}{\partial x}, \Delta, \lambda^2\right) U(x,\lambda) = 0 \tag{1`}$$

under the boundary conditions (2).

By the scheme of potentials method the solution of the problem (1`), (2) is sought in the form of the sum of the potentials

$$U(x,\lambda) = W_1(x,\lambda) + W_2(x,\lambda) + W_3(x,\lambda), \tag{5}$$

where $W_k(x,\lambda)$ $\left(k = \overline{1,3}\right)$ are potentials determined by the following form

$$W_k(x,\lambda) = \int_\Gamma P_k(x,y,\lambda)\mu_k(y,\lambda)d\Gamma_y, \tag{6}$$

$$(k = 1,2)$$

$$W_3(x,\lambda) = \int_\Gamma \frac{dP_3(x,y,\lambda)}{dn_y}\mu_3(y,\lambda)d\Gamma_y, \tag{7}$$

where $P_1(x,y,\lambda)$ is fundamental and $P_k(x,y,\lambda)$ $(k = 2,3)$ are partial solutions of the equation (1`), and $\mu_k(y,\lambda)$ $\left(k = \overline{1,3}\right)$ are the unknown densities to be defined.

The solutions $P_k(x,\xi,\lambda)$ are determined by the formulas

$$P_k(x,\xi,\lambda) = P_k(x-\xi,\xi,\lambda) + \iint_D P_1(x-\eta,\eta,\lambda)h_k(\eta,\xi,\lambda)d_\eta D,\ \left(k = \overline{1,3}\right)$$

where $P_k(x-\xi,\xi,\lambda)$ is fundamental, and $P_k(x-\xi,\xi,\lambda)$ $(k = 2,3)$ are partial solutions of the main part of the equation (1`) with frozen at the point $x = \xi \in D$ coefficients, i.e. the next equations are valid

$$\sum_{k=0}^{3} A_k(\xi)\lambda^{2k}\Delta^{3-k}U(x,\lambda)=0. \qquad (1``)$$

By the scheme of the mentioned in article [4]-[6] the partial solutions $P_k(x-\xi,\xi,\lambda)$ $(k=2,3)$ of the equation (1``) are determined by the following form

$$P_k(x-\xi,\xi,\lambda)=\frac{-1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{v_k^2(\xi)}{\prod\limits_{k\neq s=1}^{3}(v_k(\xi)-v_s(\xi))}K_0\left(\frac{\lambda|x-\xi|}{\sqrt{-v_k(\xi)}}\right),$$

$$P_3(x-\xi,\xi,\lambda)=P_{31}(x-\xi,\xi,\lambda)-\frac{2A_0(\xi)}{3\lambda^2}\frac{d^2}{d\eta^2}P_{32}(x-\xi,\xi,\lambda),$$

where

$$P_{31}(x-\xi,\xi,\lambda)=\frac{-1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{\left(v_k^2(\xi)-A_2(\xi)v_k(\xi)\right)}{\prod\limits_{k\neq s=1}^{3}(v_k(\xi)-v_s(\xi))}K_0\left(\frac{\lambda|x-\xi|}{\sqrt{-v_k(\xi)}}\right),$$

$$P_{32}(x-\xi,\xi,\lambda)=\frac{-1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{A_1(\xi)v_k(\xi)-A_0(\xi)}{\prod\limits_{k\neq s=1}^{3}(v_k(\xi)-v_s(\xi))}K_0\left(\frac{\lambda|x-\xi|}{\sqrt{-v_k(\xi)}}\right).$$

Finally, the fundamental solution $P_1(x-\xi,\xi,\lambda)$ of the equations (1``) is determined by the formula

$$P_1(x-\xi,\xi,\lambda)=\frac{-1}{4\pi\lambda^4}\sum_{k=1}^{3}\frac{1}{\prod\limits_{k\neq s=1}^{3}(v_k(\xi)-v_s(\xi))}K_0\left(\frac{\lambda|x-\xi|}{\sqrt{-v_k(\xi)}}\right),$$

where $v_k(\xi)$ $(k=\overline{1,3})$ are the different roots of the equation (3) $K_0(z)$ is the zero order Bessel functions of the second kind (see [8]).

With the help of asymptotic representations for $K_0(z)$ and its derivatives ([8]) the correctness of the following estimates is proved for fundamental and partial solutions and their derivatives

$$\left|\frac{\partial^s P_k(x,\xi,\lambda)}{\partial x_i^s}\right|\leq\frac{C\exp[-\varepsilon|\lambda||x-\xi|]}{|\lambda|^{4-s}|x-\xi|^{\frac{1}{2}(1+s)}},\quad (s=0,1;\ k=\overline{1,3},\ i=1,2),$$

$$\left|\frac{\partial^s \Delta P_k(x,\xi,\lambda)}{\partial x_i^s}\right|\leq\frac{C\exp[-\varepsilon|\lambda||x-\xi|]}{|\lambda|^{2-s}|x-\xi|^{\frac{1}{2}(1+s)}},\quad (s=0,1;\ k=\overline{1,3},\ i=1,2) \qquad (8)$$

valid for $x\neq\xi\in D$, $\lambda\in R_\delta$ and

$$\left|\frac{d}{dn_y}\Delta^s P_k(z,y,\lambda)\right|\leq\frac{C\exp[-\varepsilon|\lambda||x-\xi|]}{|\lambda|^{2(2-s)}|z-y|^{1-\alpha}},\quad (s=\overline{0,2};\ k=\overline{1,3}) \qquad (9)$$

valid for $z,y\in\Gamma$, $\lambda\in R_\delta$.

Finally, with the help of above mentioned estimates the jump formula is proved for the potentials $W_k(x,\lambda)$ $(k=\overline{1,3})$ and their necessary derivatives.

For the potential $W_1(x,\lambda)$ the jump formula

$$W_{1i,e}(z,\lambda)=W_1(z,\lambda)=\int_\Gamma P_1(z,y,\lambda)\mu_1(y,\lambda)d\Gamma_y,$$

$$\left[\frac{dW_1(z,\lambda)}{dn_z}\right]_{i,e}=\frac{dW_1(z,\lambda)}{dn_z}=\int_\Gamma\frac{dP_1(z,y,\lambda)}{dn_z}\mu_1(y,\lambda)d\Gamma_y\,, \tag{10}$$

$$\left[\frac{d}{dn_z}\Delta^2W_1(z,\lambda)\right]_{i,e}=\mp\frac{\mu_1(z,\lambda)}{A_0(z)}+\frac{d}{dn_z}\Delta^2W_1(z,\lambda)=$$

$$=\mp\frac{\mu_1(z,\lambda)}{A_0(z)}+\int_\Gamma\frac{d}{dn_z}\Delta^2P_1(z,y,\lambda)\mu_1(y,\lambda)d\Gamma_y\,,$$

valid for $\lambda\in R_\delta$, $\Gamma\in\Lambda_1$, $\mu_1(z,\lambda)\in C$, holds.

For the potential $W_2(x,\lambda)$ we have

$$W_{2i,e}(z,\lambda)=W_2(z,\lambda)=\int_\Gamma P_2(z,y,\lambda)\mu_2(y,\lambda)d\Gamma_y\,, \tag{11}$$

$$\left[\frac{dW_2(z,\lambda)}{dn_z}\right]_{i,e}=\mp\frac{\mu_2(z,\lambda)}{\lambda^4}+\frac{dW_2(z,\lambda,\lambda)}{dn_z}=\mp\frac{\mu_2(z,\lambda)}{\lambda^4}+$$

$$+\int_\Gamma\frac{d}{dn_z}P_2(z,y,\lambda)\mu_2(y,\lambda)d\Gamma_y\,, \tag{11`}$$

$$\left[\frac{d}{dn_z}\Delta^2W_2(z,\lambda)\right]_{i,e}=\frac{d\Delta^2W_2(z,\lambda)}{dn_z}=\int_\Gamma\frac{d}{dn_z}\Delta^2P_2(z,y,\lambda)\mu_2(y,\lambda)d\Gamma_y \tag{11``}$$

valid for all $\lambda\in R_\delta$, $\Gamma\in\Lambda_1$ and $\mu_2(z,\lambda)\in C$.

Finally for the potential $W_3(x,\lambda)$ the validity of the following jump formula is proved

$$W_{3i,e}(z,\lambda)=\pm\frac{\mu_3(z,\lambda)}{3\lambda^4}+W_3(z,\lambda)=\pm\frac{\mu_3(z,\lambda)}{\lambda^4}+$$

$$+\int_\Gamma\frac{d}{dn_y}P_3(z,y,\lambda)\mu_3(y,\lambda)d\Gamma_y \tag{12}$$

$$\left[\frac{dW_3(z,\lambda)}{dn_z}\right]_{i,e}=\pm\frac{2\overline{\chi}(z)\mu_3(z,\lambda)}{3A_0(z)\lambda^4}+\frac{dW_3(z,\lambda)}{dn_z}=$$

$$=\pm\frac{2\overline{\chi}(z)\mu_3(z,\lambda)}{3A_0(z)\lambda^4}+\int_\Gamma\frac{d^2}{dn_zdn_y}P_3(z,y,\lambda)\mu_3(y,\lambda)d\Gamma_y\,, \tag{12`}$$

$$\left[\frac{d}{dn_z}\Delta^2W_3(z,\lambda)\right]_{i,e}=\mp\frac{2A^2(z)\mu_3(z,\lambda)}{3A_0(z)}+\frac{d}{dn_z}\Delta^2W_3(z,\lambda)=$$

$$=\mp\frac{2A_2(z)_3(z,\lambda)}{3A_0(z)}+\int_\Gamma\frac{d^2}{dn_zdn_y}\Delta^2P_3(z,y,\lambda)\mu_3(y,\lambda)d\Gamma_y\,,$$

valid for $\lambda\in R_\delta$, $\Gamma\in\Lambda_2$, $\mu_3(z,\lambda)\in C_1$, where $\overline{\chi}(z)$ is a mean curvature of the curve Γ at the point $z\in\Gamma$ determined by the formula

$$\overline{\chi}(z)=\int_0^{2\pi}\chi(\varphi)d\varphi\,.$$

Substituting (5) to the left hand side of the boundary conditions (2) allowing for the jump formula (10)-(12) for the unknown densities $\left(k=\overline{1,3}\right)$ we obtain the following system of integral equations

$$\mu(z,\lambda)=\varphi(z,\lambda)+\int_{\Gamma}K(z,y,\lambda)\mu(y,\lambda)d\Gamma_y, \qquad (13)$$

where

$$\mu(z,\lambda)=\begin{pmatrix}\mu_1(z,\lambda)\\ \mu_2(z,\lambda)\\ \mu_3(z,\lambda)\end{pmatrix}; \quad \psi(z,\lambda)=\begin{pmatrix}\psi_1(z,\lambda)\\ \psi_2(z,\lambda)\\ \psi_3(z,\lambda)\end{pmatrix}$$

$$K(z,y,\lambda)=\left(K_{ij}(z,y,\lambda)\right)_{i,j=1}^{3},$$

where

$\psi_1(z,\lambda)=-2\lambda^4 A_2(z)\overline{\chi}(z)\varphi_0(z,\lambda)-A_0(z)\varphi_2(z,\lambda)$,

$\psi_2(z,\lambda)=-2\lambda^4 A_0(z)\overline{\chi}(z)\varphi_0(z,\lambda)-\lambda^4\varphi_1(z,\lambda)$,

$\psi_3(z,\lambda)=3\lambda^4\varphi_0(z,\lambda)$.

The elements of the kernel $K(z,y,\lambda)$ are determined by the formulas

$$K_{11}(z,y,\lambda)=2\lambda^4\left\{A_2(z)\overline{\chi}(z)+\frac{A_0(z)}{\lambda^4}\frac{d}{dn_z}\Delta^2\right\}P_1(z,y,\lambda)$$

$$K_{12}(z,y,\lambda)=2\lambda^4\left\{A_2(z)\overline{\chi}(z)+\frac{A_0(z)}{\lambda^4}\frac{d}{dn_z}\Delta^2\right\}P_2(z,y,\lambda),$$

$$K_{13}(z,y,\lambda)=2\lambda^4\left\{A_2(z)\overline{\chi}(z)+\frac{A_0(z)}{\lambda^4}\frac{d}{dn_z}\Delta^2\right\}\frac{dP_3(z,y,\lambda)}{dn_y},$$

$$K_{21}(z,y,\lambda)=2\lambda^4\left\{A_0(z)\overline{\chi}(z)+\frac{1}{2}\frac{d}{dn_z}\right\}P_1(z,y,\lambda),$$

$$K_{22}(z,y,\lambda)=2\lambda^4\left\{A_0(z)\overline{\chi}(z)+\frac{1}{2}\frac{d}{dn_z}\right\}P_2(z,y,\lambda),$$

$$K_{23}(z,y,\lambda)=2\lambda^4\left\{A_0(z)\overline{\chi}(z)+\frac{1}{2}\frac{d}{dn_z}\right\}\frac{dP_3(z,y,\lambda)}{dn_y},$$

$$K_{31}(z,y,\lambda)=-3\lambda^4 P_1(z,y,\lambda),$$

$$K_{32}(z,y,\lambda)=-3\lambda^4 P_2(z,y,\lambda),$$

$$K_{33}(z,y,\lambda)=-3\lambda^4\frac{dP_3(z,y,\lambda)}{dn_y}.$$

Taking into account the estimates (8) and (9) for the kernel $K(z,y,\lambda)$ we can show the correctness of the estimate

$$\left|K(z,y,\lambda)\right|\le\frac{C\exp\left[-\varepsilon|\lambda||z-y|\right]}{|z-y|^{1-\alpha}} \qquad (14)$$

valid for all $\lambda\in R_\delta$ and $z,y\in\Gamma$.

By virtue of the inequality (14) a system of the integral equations (13) is regular, consequently we can solve it by the method of successive approximations, where the solution $\mu(z,\lambda)$ of the system (13) is analytic, bounded by λ function in the domain R_δ and continuous by z on Γ.

Thus the following theorem is proved.

Theorem. *Under conditions 1^0, 3^0, 5^0 the problem (1`), (2) has a unique analytical by* λ *in* R_δ *solution* $U(x,\lambda)$ *representable in the form of the sum of the positials (5), whose densities are the solutions of a system of regular integral equations (13). And to the solution* $U(x,\lambda)$ *the estimates*

$$\left|\frac{\partial^k U(x,\lambda)}{\partial x_i^k}\right| \leq \frac{C\exp[-\varepsilon|\lambda|d(x)]}{|\lambda|^{4-k} d^{k+\frac{1}{2}}(x)}, \quad \left(k=\overline{0,4}\right),$$

$$\left|\frac{\partial^k U(x,\lambda)}{\partial x_i^k}\right| \leq \frac{C\exp[-\varepsilon|\lambda|d(x)]}{d^{k+\frac{1}{2}}(x)}, \quad (k=5,6)$$

which are valid all $\lambda \in R_\delta$, $x \in D_1 \subset D$, *and* $d(x)$ *is the shortest distance from the point* $x \in D_1$ *to the boundary* Γ *of the domain* D, *holds.*

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THE STABILITY OF THE INVERSE PROBLEM OF SCATTERING THEORY FOR NONSELF-ADJOINT OPERATOR ON ALL AXIS

Abstract

The stability for non-self-adjoint operator on all axis is studied. The estimations for the solutions and potentials of two non-self-adjoint operators have been obtained.

The questions about which information about the function $q(x)$ or generally about the operator L one can extract if the scattering data are known only on an interval of variation of spectral parameter, has an important value. Therefore from physical point of view the natural statement of the question about stability of inverse problems is such: how much strongly can differ two problems whose scattering data coincide in the given interval of variation ρ^2. Stability of the inverse problem was studied in the series of works ([1], [4], [5]). For the self-adjoint operator on semi-axis it has got the solution in the V.A.Marchenko work [1].

In the present paper the stability for non-self-adjoint operator on all axis is studied.

The results of [1], [2], [3] are used.

1. Operator L.

We'll denote by L an operator given in Hilbert space $L^2(-\infty,\infty)$ by the differential expression

$$ly = -y'' + q(x)y. \tag{1.1}$$

We'll assume that $q(x)$ satisfies

$$|q(x)|e^{\varepsilon|x|} \in L^1(-\infty,\infty) \tag{1.2}$$

for $\varepsilon > 0$.

We'll denote by $V\{C_{\pm}(x)\}$ the set of boundary-value problems, whose

$$\int_x^{\infty} e^{\varepsilon t}|q(t)|dt \le C_+(x), \quad \int_{-\infty}^{x} e^{\varepsilon|t|}|q(t)|dt \le C_-(x),$$

where $C_{\pm}(x)$ are continuous monotone functions, correspondingly.

2. Special solutions of the equation $ly = \rho^2 y$.

We'll denote by $e_{\pm}(x,\rho)$ the solution of the differential equation $ly = S^2 y$, which have the following asymptotics on infinity for $\operatorname{Im}\rho > -\frac{\varepsilon}{2}$

$$e_+(x,\rho) \sim e^{ix\rho} \text{ for } x \to \infty, \quad e_-(x,\rho) \sim e^{-ix\rho} \text{ for } x \to -\infty.$$

These solutions are holomorphic by ρ in half-plane $\operatorname{Im}\rho > -\frac{\varepsilon}{2}$ (ε is the number that in the condition (1.2)) and admit the following representations with the help of transformation operators

$$e_+(x,\rho)=e^{ix\rho}+\int_x^{\infty}K_+(x,t)e^{it\rho}dt\,, \qquad (2.1)$$

$$e_-(x,\rho)=e^{-ix\rho}+\int_{-\infty}^{x}K_-(x,t)e^{-it\rho}dt\,. \qquad (2.2)$$

The kernel $K_\pm(x,t)$ have continuous derivatives which satisfy the inequalities

$$|K_+(x,t)|\le C_a^+e^{-\varepsilon\frac{x+t}{2}} \text{ for } t\ge x\ge a\,, \qquad (2.3)$$

$$|K_-(x,t)|\le C_a^-e^{\varepsilon\frac{x+t}{2}} \text{ for } t\le x\le a\,. \qquad (2.4)$$

Moreover, the kernels $K_\pm(x,t)$ are connected with the potential $q(x)$ by the following way:

$$q(x)=2\frac{d}{dx}K_-(x,x)=-2\frac{d}{dx}K_+(x,x).$$

3. Scattering data of the operator L.

According to [3] the Wronskian $w(\rho)$

$$w(\rho)=e_-(x,\rho)e'+(x,\rho)-e'_-(x,\rho)e_+(x,\rho)$$

is called denominator (of kernel of the resolving of the operator L). The denominator $w(\rho)$ has a finite number of zeros in the half-plane $\operatorname{Im}\rho\ge 0$. These zeros are called singular numbers. Let's denote by $\rho_1,...,\rho_2$ the non-real singular numbers, and singular numbers laying on real axis by $\rho_{\alpha+1},...,\rho_\beta$. Multiplicity of the root ρ_k of the equation $w(\rho)=0$ is called multiplicity of the singular number ρ_k and is denoted by m_k $\left(k=\overline{1,\beta}\right)$.

The function $S(\rho)=\dfrac{v(\rho)}{w(\rho)}$, where

$$w(\rho)=e_+(x,_-\rho)e'_-(x,\rho)_-e'_+(x,_-\rho)e_-(x,\rho)\,,$$

is called a scattering function. Parallel with $S(\rho)$ the function $S_1(\rho)$:

$$S_1(\rho)=\frac{S(_-\rho)w(_-\rho)}{w(\rho)}$$

is used. The Fourier transformation of the operator L

$$f_S^+(x)=\frac{1}{2\pi}\int_+ S(\rho)e^{ix\rho}d\rho\,, \qquad (3.1)$$

$$f_S^-(x)=\frac{1}{2\pi}\int_+ S_1(\rho)e^{-ix\rho}d\rho \qquad (3.2)$$

is its important spectral characteristic together with scattering function.

Since $S(\rho)$, $S_1(\rho)$ are the functions of (S) type [3], the contour of integration is any straight line in the strip $0<\operatorname{Im}\rho<\varepsilon_0$ $(\varepsilon_0=\min\{\varepsilon/2,\varepsilon_1\}$, ε_1 is a distance from the axis $\operatorname{Im}\rho=0$ to nonreal zeros of the $w(\rho))$.

The functions

$$f_k^\pm(x)=p_k^\pm(x)e^{\pm ix\rho_k} \qquad (3.3)$$

characterize the operator L on the pointwise spectrum. Here $p_k^{\pm}(x)$ are the polynomials of the degree m_k-1 (see [3]). In the case of real potential and the multiplicity $m_k=1$ the polynomials $p_k^{\pm}(x)$ coincide with the normalized factors considered in the work [2].

The scattering functions $S(\rho)$, the non-real singular numbers $\rho_1,...,\rho_\alpha$ and the corresponding to them normalized factors are called the scattering data of the operator L.

4. Estimation of the special solutions.

Let's write the integral equations which the kernels $K_{\pm}(x,t)$

$$K_+(x,t)+\int_x^{\infty}K_+(x,u)f_+(u+t)du+f_+(x+t)=0,\quad -\infty<x\le t<\infty, \tag{4.1}$$

$$K_-(x,t)+\int_{-\infty}^{x}K_-(x,u)f_-(u+t)du+f_-(x+t)=0,\quad -\infty<t\le x<\infty \tag{4.2}$$

satisfy, where

$$f_{\pm}(x)=f_S^{\pm}(x)+\sum_{k=1}^{\alpha}f_k^{\pm}(x). \tag{4.2`}$$

Let's consider two problems with the potentials $q_1(x)$, $q_2(x)$. Let's compose the main integral equations for them, and subtract one from another

$$K_{1,2+}(x,t)+\int_x^{\infty}f_{1+}(u+t)K_{1,2+}(x,u)du+\int_x^{\infty}f_{1,2+}(u+t)K_{2+}(x,u)du+f_{1,2+}(x,t)=0, \tag{4.3}$$

$$K_{1,2-}(x,t)+\int_{-\infty}^{x}f_{1-}(u+t)K_{1,2-}(x,u)du+\int_{-\infty}^{x}f_{1,2-}(u+t)K_{2-}(x,u)du+f_{1,2-}(x+t)=0, \tag{4.4}$$

where

$$\begin{aligned}K_{1,2\pm}(x,t)&=K_{1\pm}(x,t)-K_{2\pm}(x,t),\\ f_{1,2\pm}(x)&=f_{1\pm}(x)-f_{2\pm}(x).\end{aligned} \tag{4.5}$$

$K_{1\pm}$, $f_{1\pm}$ and $K_{2\pm}$, $f_{2\pm}$ are functions from the equations (4.1), (4.2) for the problems with the potentials $q_1(x)$, $q_2(x)$ correspondingly. Since the estimation of the difference $e_{1+}(x,\rho)-e_{2+}(x,\rho)$ is obtained analogously to the problem considered on the positive semi-axis, we'll calculate the difference of the solutions $e_{1-}(x,\rho)-e_{2-}(x,\rho)$.

For every fixed x solving (4.4) with respect to $K_{1,2-}(x,t)$ we obtain

$$K_{1,2-}(x,t)=-\left(\mathbf{I}+\mathbf{F}_{1-,x}\right)^{-1}\left\{f_{1,2-}(x+t)+\int_{-\infty}^{x}f_{1,2-}(u+t)K_{2-}(x,u)du\right\}, \tag{4.6}$$

where

$$\mathbf{F}_{1\text{-},x}u=\int_{-\infty}^{x}f_{1-}(y+t)u(t)dt.$$

Note that the validity of the identity

$$\left(\mathbf{I}+\mathbf{F}_{1-,x}\right)^{-1}=\left(\mathbf{I}+\overline{\mathbf{K}}_{1-,x}\right)\left(\mathbf{I}+\mathbf{K}_{1-,x}\right) \tag{4.7}$$

follows from the main equation. The operators $\mathbf{K}_{1\text{-},x}$, $\overline{\mathbf{K}}_{1\text{-},x}$ are defined by the formula

$$\mathbf{K}_{1-,x}f=\int_{-\infty}^{y}K_{1-}(y,t)f(t)dt\,,\quad \overline{\mathbf{K}}_{1-,x}f=\int_{y}^{x}K_{1-}(t,y)f(t)dt$$

Granting (3.2), (3.3), (4.2) we find

$$f_{1,2-}(x)=\frac{1}{2\pi}\int_{+}\left(S_1^1(\rho)-S_1^2(\rho)\right)e^{-ix\rho}d\rho+\sum_{k=1}^{\alpha}\left[P_{k_1}^-(x)-P_{k_2}^-(x)\right]e^{-ix\rho_k}\,, \qquad (4.8)$$

where $S_1^1(\rho)$, $S_1^2(\rho)$ are the same functions that in the formula (3.2) for the equations with the potentials $q_1(x)$, $q_2(x)$ correspondingly. Using these equalities and also the formula (2.1), (2.2) the following lemma is proved.

Lemma 4.1. *For all values of μ from the domain* $\operatorname{Im}\mu>-\frac{\varepsilon}{2}$, $\operatorname{Im}\mu\neq\eta$, $\mu\neq\rho_k$

$$-\{e_{1-}(x,\mu)-e_{2-}(x,\mu)\}^2=\int_{-\infty}^{x}\{q_1(t)-q_2(t)\}\{A_{1,2-}(\mu,x,t)-A_{1,2-}(\mu,t,x)\}dt\,, \qquad (4.9)$$

$$-\{e_{1+}(x,\mu)-e_{2+}(x,\mu)\}^2=\int_{x}^{\infty}\{q_1(t)-q_2(t)\}\{A_{1,2+}(\mu,x,t)-A_{1,2+}(\mu,t,x)\}dt\,,$$

where

$$A_{1,2-}(\mu,x,t)=\frac{e_{1-}(x,\mu)e_{2-}(x,\mu)}{2\pi}\int_{+}\frac{\left[S_1^1(\rho)-S_1^2(\rho)\right]}{\rho^2-\mu^2}e_{1-}(t,\rho)e_{2-}(t,\rho)d\rho+$$
$$+e_{1-}(x,\mu)e_{2-}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^-\left(\frac{d}{id\rho}\right)-P_{k_2}^-\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1-}(t,\rho)e_{2-}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k},$$

$$A_{1,2+}(\mu,x,t)=\frac{e_{1+}(x,\mu)e_{2+}(x,\mu)}{2\pi}\int_{+}\frac{\left[S^1(\rho)-S^2(\rho)\right]}{\rho^2-\mu^2}e_{1+}(t,\rho)e_{2+}(t,\rho)d\rho+$$
$$+e_{1+}(x,\mu)e_{2+}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^+\left(\frac{d}{id\rho}\right)-P_{k_2}^+\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1+}(t,\rho)e_{2+}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k}.$$

$S^1(\rho), S^2(\rho)$ *are the functions from (3.1) for the equations with the potentials* $q_1(x), q_2(x)$.

Let the data $\left\{S^j(\rho), S_1^j(\rho), \rho_k(j), P_{kj}\right\}$ of the problems with the potentials $q_j(x)\,(j=1,2)$ coincide for $\operatorname{Re}\rho^2\in(-\infty,N)$:

$S^1(\rho)=S^2(\rho)$,

$S_1^1(\rho)=S_1^2(\rho)$, $\quad$ for $|\operatorname{Re}\rho|<\sqrt{N+\eta^2}$, $\operatorname{Im}\rho=\eta$, $\eta<\varepsilon_0$,

$\rho_k(1)=\rho_k(2)$, $\quad P_{k_1}=P_{k_2}$.

Let's prove the following theorem.

Theorem 4.1. *If the scattering data of two problems* $q_j(x)\in V\{C_\pm(x)\}$ $(j=1,2)$ *coincide for all* $\operatorname{Re}\rho^2\in(-\infty,N)$, *then for* $\operatorname{Re}\mu^2\in(-\infty,N)$ *the following inequalities*

$$\mathbf{K}_{1-,x}f=\int_{-\infty}^{y}K_{1-}(y,t)f(t)dt\,,\quad \overline{\mathbf{K}}_{1-,x}f=\int_{y}^{x}K_{1-}(t,y)f(t)dt$$

Granting (3.2), (3.3), (4.2) we find

$$f_{1,2-}(x)=\frac{1}{2\pi}\int_{+}\left(S_1^1(\rho)-S_1^2(\rho)\right)e^{-ix\rho}d\rho+\sum_{k=1}^{\alpha}\left[P_{k_1}^{-}(x)-P_{k_2}^{-}(x)\right]e^{-ix\rho_k}\,, \tag{4.8}$$

where $S_1^1(\rho)$, $S_1^2(\rho)$ are the same functions that in the formula (3.2) for the equations with the potentials $q_1(x)$, $q_2(x)$ correspondingly. Using these equalities and also the formula (2.1), (2.2) the following lemma is proved.

Lemma 4.1. *For all values of* μ *from the domain* $\operatorname{Im}\mu>-\frac{\varepsilon}{2}$, $\operatorname{Im}\mu\neq\eta$, $\mu\neq\rho_k$

$$-\{e_{1-}(x,\mu)-e_{2-}(x,\mu)\}^2=\int_{-\infty}^{x}\{q_1(t)-q_2(t)\}\{A_{1,2-}(\mu,x,t)-A_{1,2-}(\mu,t,x)\}dt\,, \tag{4.9}$$

$$-\{e_{1+}(x,\mu)-e_{2+}(x,\mu)\}^2=\int_{x}^{\infty}\{q_1(t)-q_2(t)\}\{A_{1,2+}(\mu,x,t)-A_{1,2+}(\mu,t,x)\}dt\,,$$

where

$$A_{1,2-}(\mu,x,t)=\frac{e_{1-}(x,\mu)e_{2-}(x,\mu)}{2\pi}\int_{+}\frac{\left[S_1^1(\rho)-S_1^2(\rho)\right]}{\rho^2-\mu^2}e_{1-}(t,\rho)e_{2-}(t,\rho)d\rho+$$

$$+e_{1-}(x,\mu)e_{2-}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^{-}\left(\frac{d}{id\rho}\right)-P_{k_2}^{-}\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1-}(t,\rho)e_{2-}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k},$$

$$A_{1,2+}(\mu,x,t)=\frac{e_{1+}(x,\mu)e_{2+}(x,\mu)}{2\pi}\int_{+}\frac{\left[S^1(\rho)-S^2(\rho)\right]}{\rho^2-\mu^2}e_{1+}(t,\rho)e_{2+}(t,\rho)d\rho+$$

$$+e_{1+}(x,\mu)e_{2+}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^{+}\left(\frac{d}{id\rho}\right)-P_{k_2}^{+}\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1+}(t,\rho)e_{2+}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k}.$$

$S^1(\rho),S^2(\rho)$ *are the functions from (3.1) for the equations with the potentials* $q_1(x)$, $q_2(x)$.

Let the data $\left\{S^j(\rho),S_1^j(\rho),\rho_k(j),P_{kj}\right\}$ of the problems with the potentials $q_j(x)\,(j=1,2)$ coincide for $\operatorname{Re}\rho^2\in(-\infty,N)$:

$S^1(\rho)=S^2(\rho)$,

$S_1^1(\rho)=S_1^2(\rho)$, $\quad for\,|\operatorname{Re}\rho|<\sqrt{N+\eta^2}$, $\operatorname{Im}\rho=\eta$, $\eta<\varepsilon_0$,

$\rho_k(1)=\rho_k(2)$, $\quad P_{k_1}=P_{k_2}$.

Let's prove the following theorem.

Theorem 4.1. *If the scattering data of two problems* $q_j(x)\in V\{C_\pm(x)\}$ $(j=1,2)$ *coincide for all* $\operatorname{Re}\rho^2\in(-\infty,N)$, *then for* $\operatorname{Re}\mu^2\in(-\infty,N)$ *the following inequalities*

$$\left|e_{1-}(x,\mu)-e_{2-}(x,\mu)\right|^2<\frac{4e^{2x\operatorname{Im}\mu}\left(1+C_a^-e^{\varepsilon x}\right)^4}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}C_-(x), \tag{4.10}$$

$$\left|e_{1+}(x,\mu)-e_{2+}(x,\mu)\right|^2<\frac{4e^{-2x\operatorname{Im}\mu}\left(1+C_a^+e^{-\varepsilon x}\right)^4}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}C_+(x) \tag{4.11}$$

are correct.

Proof. Let's bring proof for $\left|e_{1-}(x,\mu)-e_{2-}(x,\mu)\right|$.

For the conditions of the theorem and from lemma 4.1 for $\operatorname{Re}\mu^2\in(-\infty,N)$

$$A_{1,2-}(\mu,x,t)=\frac{e_{1-}(x,\mu)e_{2-}(x,\mu)}{2\pi}\int\limits_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\frac{S_1^1(\rho)-S_1^2(\rho)}{\rho^2-\mu^2}e_{1-}(t,\rho)e_{2-}(t,\rho)d\rho$$

follows.

Granting the formula (2.2), (2.4) we obtain the estimation

$$\left|e_{j-}(x,\rho)\right|\le e^{x\operatorname{Im}\rho}\left(1+\int\limits_{-\infty}^{x}\left|K_{j-}(x,t)\right|dt\right)\le e^{x\operatorname{Im}\rho}\left(1+C_a^-e^{\varepsilon x}\right)$$

for the solutions $e_{j-}(x,\rho)\,(j=1,2)$.

Using this estimation, by the correlation ([3]

$$S_1^j(\rho)=O\left(\frac{1}{\rho}\right),\ |\rho|\to\infty$$

(uniformly in the strip $|\operatorname{Im}\rho|\le\eta$ for every $\eta<\varepsilon_0$) we obtain

$$\left|A_{1,2-}(\mu,x,t)\right|\le\frac{e^{2x\operatorname{Im}\mu}\left(1+C_a^-e^{\varepsilon x}\right)e^{2t\operatorname{Im}\rho}\left(1+C_a^-e^{\varepsilon t}\right)^2}{2\pi}\times$$

$$\times\int\limits_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\frac{\left|S_1^1(\rho)-S_1^2(\rho)\right|}{\left|\rho^2-\mu^2\right|}d\rho<\frac{e^{2x\operatorname{Im}\mu}e^{2t\eta}\left(1+C_a^-e^{\varepsilon x}\right)^2\left(1+C_a^-e^{\varepsilon t}\right)^2}{\pi\left(N+\eta^2\right)\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)\left(1-\frac{\eta^2}{N+\eta^2}\right)}=$$

$$=\frac{e^{2x\operatorname{Im}\mu}e^{2t\eta}\left(1+C_a^-e^{\varepsilon x}\right)^2\left(1+C_a^-e^{\varepsilon t}\right)^2}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}.$$

Now the inequality (4.9) follows from (4.7)

$$\left|e_{1-}(\mu,x)-e_{2-}(\mu,x)\right|^2 \le \frac{2e^{2x\operatorname{Im}\mu}\left(1+C_a^- e^{\varepsilon x}\right)^4}{\pi N\left(1-\dfrac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}\int_{-\infty}^{x}\left|q_1(t)-q_2(t)\right|e^{2t\eta}dt <$$

$$< \frac{4e^{2x\operatorname{Im}\mu}\left(1+C_a^- e^{\varepsilon x}\right)^4}{\pi N\left(1-\dfrac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}C_-(x).$$

Theorem is proved.

Remark.

$$\left|e_{1-}(\mu,x)-e_{2-}(\mu,x)\right|^2 \le 4e^{2x\operatorname{Im}\mu}\left(1+C_a^- e^{\varepsilon x}\right)^2$$

is obtained from the representation (2.2), therefore (4.8) is nontrivial in the domain, where $C_-(x)<N$.

5. Estimation of potentials.

Now let's estimate difference of the potentials $q_1(x)-q_2(x)$ of the considered problem. For definiteness in (2.3), (2.4) we'll take $a=0$ and we'll lead the further calculations for $x\in(-\infty,0)$, since the case $x\in(0,\infty)$ is analogous to the problem considered on the positive semi-axis.

By fulfilling of the conditions of Theorem 4.1 we obtain from the formula (4.6), (4.7), (4.8)

$$\frac{1}{2}\int_{-\infty}^{x}(q_1(t)-q_2(t))dt = \frac{1}{2\pi}\int\limits_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\left(S_1^2(\rho)-S_1^1(\rho)\right)e_{1-}(x,\rho)e_{2-}(x,\rho)d\rho. \qquad (5.1)$$

Chosen by the same method that in [1] the sufficiently smooth function $g(x)$ which is equal to zero outside of the interval (x_0-h,x_0), is multiplied by (5.1) and is integrated. After the integration by parts we find

$$\frac{1}{2}\int_{x_0-h}^{x_0}(q_1(t)-q_2(t))g(t)dt = \frac{1}{2\pi}\int\limits_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\left[S_1^1(\rho)-S_1^2(\rho)\right]\times$$

$$\times\int_{x_0-h}^{x_0}e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dt\,d\rho. \qquad (5.2)$$

The following lemma is proved.

Lemma 5.1. *Let the problems* $\{q_j(x)\}\in V\{C_-(x)\}$, *potentials* $q_j(x)$ *be bounded in the interval* (x_0-h,x_0) *and*

$$Q_-(x)=\int_{-\infty}^{x}(q_1(t)+q_2(t))dt.$$

Then for any continuous differentiable function $g(x)$, which is equal to zero outside of the interval (x_0-h,x_0), the identity

$$\int_{x_0-h}^{x_0} e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dt=\int_{x_0-h}^{x_0}\{g'(t)-g(t)Q_-(t)\}e^{-2i\rho t}dt+r(\rho,x_0,h)$$

is correct, where

$$|r(\rho,x_0,h)|\le\frac{C_-^2(x_0)m^2-(\rho,x_0)}{4\rho^2}\left(3|\tilde g'(2\rho)|+|\tilde g'(-2\rho)|\right)+$$

$$+\frac{4hC_-(x_0)m_-^2(\rho,x_0)\beta_-(h,x_0)}{\rho^2}\int_{x_0-h}^{x_0}|g'(t)|dt,$$

$$m_-(\rho,x_0)=\max_{j=1,2}\left\{\sup_{-\infty<t\le x_0}|e_{j-}(t,\rho)|\right\},$$

$$\beta_-(h,x_0)=\max_{j=1,2}\left\{\sup_{x_0-h<t<x_0}\left|q_j(t)e^{\varepsilon/2|t|}\right|\right\},$$

$$\tilde g'(2\rho)=\int_{x_0-h}^{x_0} e^{-2i\rho t}g'(t)dt.$$

$g(x)$ is chosen by the following way ([1]). Let

$$\delta_0^-(t)=\frac{n}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^n e^{-2in\lambda t}d\lambda,\quad n>3,$$

$$\delta_-(t)=\frac{1}{h}\delta_0^-\left(-\frac{1}{2}+\frac{x_0-t}{h}\right).$$

$\delta_-(t)$ do not decrease on the interval $\left(x_0-h,x_0-\frac{h}{2}\right)$ do not increase on $\left(x_0-\frac{h}{2},x_0\right)$, have $n-2$ continuous derivatives, be negative, even, be equal to zero outside of the interval (x_0-h,x_0) and integral from it on all straight-line be equal to unit.

As the function $g(x)$ we'll take the solution of the differential equation

$$g'(x)+g(x)Q_-(x)=\delta_-'(x)+C\delta_-(x),$$

$(C-const)$ vanishing for $x\ge x_0$. For $x_0-h\le x\le x_0$ we obtain the inequality

$$|g(x)-\delta_-(x)|\le\omega(h,x_0)\delta_-(x)\frac{h}{2}e^{C_-(x_0)h},$$

$$|g'(x)-\delta_-'(x)|\le\omega(h,x_0)\delta_-(x)\left(1+hC_-(x_0)e^{hC_-(x_0)}\right),$$

$$\omega(h,x_0)=\max_{x_0-h\le x,y\le x_0}|Q_-(x)-Q_-(y)|.$$

These inequalities together with Lemma 5.1 lead to the estimation

$$\left|\int_{x_0-h}^{x_0} e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dt\right|\le\left|\int_{x_0-h}^{x_0}\{g'(t)-g(t)Q_-(t)\}e^{-2i\rho t}dt\right|+$$

$$+r(\rho,x_0,h)\le 2\left(\frac{n}{h}\right)^n|\rho|^{-n+1}\left\{1+\frac{C_-(x_0)}{|\rho|}+\frac{C_-^2(x_0)m_-^2(\rho,x_0)}{|\rho|^2}+\right.$$

$$+\frac{C_-(x_0)\beta_-(h,x_0)m_-^2(\rho,x_0)}{|\rho|^2}\left[8n+18C_-(x_0)h\left(1+C_-(x_0)he^{hC_-(x_0)}\right)\right]\Bigg\}.$$

Whence according to the identity (5.2) it follows, that

$$\frac{1}{2}\left|\int_{x_0-h}^{x_0} g(t)\{q_1(t)-q_2(t)\}dt\right|=\frac{1}{2\pi}\left|\int_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\left[S_1^1(\rho)-S_1^2(\rho)\right]\times\right.$$

$$\left.\times\int_{x_0-h}^{x_0} e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dtd\rho\right|\le\frac{2}{\pi}\left(\frac{n}{h}\right)^n\frac{\left(N+\eta^2\right)^{-\frac{n-2}{2}}}{n-2}\left\{1+\frac{C_-(x_0)}{\sqrt{N+\eta^2}}+\frac{C_-^2(x_0)m_N^2(x_0)}{N+\eta^2}\right\}+$$

$$+\frac{2C_-(x_0)\beta(h,x_0)m_N^2(x_0)}{\pi\sqrt{N+\eta^2}}\left(4n+9C_-(x_0)h\left(1+C_-(x_0)he^{hC_-(x_0)}\right)\right),$$

$$m_N(x_0)=\sup_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}} m_-(\rho,x_0).$$

Using the last inequality the following theorem is proved.

Theorem 5.1. *If scattering data of two problems* $q_j(x)\in V\{C_\pm(x)\}$ *coincide for all* $\operatorname{Re}\rho^2\in(-\infty,N)$ *and* $N+\eta^2\ge 1$, *then on domain where*

$$\frac{5\left\{\left|\ln\left(N+\eta^2\right)\right|+1\right\}}{\sqrt{N+\eta^2}}C_\pm(x)<1$$

the inequalities

$$|q_1(x)-q_2(x)|\le\frac{2\left\{\left|\ln\left(N+\eta^2\right)\right|+3\right\}}{\sqrt{N+\eta^2}}\{38C_\pm(x)\beta_\pm(h,x)+5\gamma_\pm(h,x)\}+$$

$$+\frac{1}{\sqrt{N+\eta^2}\left\{3\left[\ln\left(N+\eta^2\right)\right]+1\right\}}.$$

are correct. Here

$$h=5\left(N+\eta^2\right)^{-\frac{1}{2}}\left\{\left|\ln\left(N+\eta^2\right)\right|+1\right\},$$

$$\gamma_+(h,x)=\max_{j=1,2}\left\{\sup_{x<t<x+h}\left|q_j'(t)\right|\right\},$$

$$\beta_+(h,x)=\max_{j=1,2}\left\{\sup_{x<t<x+h}\left|q_j(t)e^{\varepsilon/2t}\right|\right\},\quad x>0,$$

$$\gamma_-(h,x)=\max_{j=1,2}\left\{\sup_{x-h<t<x}\left|q_j'(t)\right|\right\},\quad x<0.$$

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NECESSARY CONDITION OF BASISITY OF POWER SYSTEM IN L_p

Abstract

At the paper the system of power is considered $\{A(t)\varphi^n(t);\, B(t)\overline{\varphi}^k(t)\},\ n,k \ge 0$, *where* $A(t)$, $B(t)$ *and* $\varphi(t)$ *are complex valued functions on* $[a,b]$; $\overline{\varphi}$ *is a complex conjugation. It is proved that at definite conditions on the functions* $A(t)$, $B(t)$ *and* $\varphi(t)$ *necessary conditions of basisity of this system in* L_p, $1 < p < +\infty$ *is* $|\varphi(t)| \equiv const$.

Let's consider the following "double of powers"

$$\{A(t)\varphi^n(t); B(t)\overline{\varphi}^k(t)\},\quad n,k \ge 0\,, \tag{1}$$

$A(t)$, $B(t)$, $\varphi(t)$ are complex-valued functions on $[a,b]$, $\overline{\varphi}$ is a complex conjugation. The necessary and sufficient condition of completeness and minimality of the system (1) in L_p was obtained in paper [1]. The basic properties of the system (1) in L_p was completely investigated in paper [2], in case when $\varphi(t) \equiv e^{it}$. It appears, this case for basisity is exclusive.

So, let in some Banach space B "double" system

$$\{x_n^+, x_k^-\},\ n,k \ge 0 \tag{2}$$

be given.

Definition. *The system (2) makes up a basis in* B, *if for* $\forall x \in B$ *there exists a unique sequence of numbers* $\{a_n^+, a_k^-\}$, $n,k \ge 0$, *such that*

$$\lim_{N^+,N^- \to \infty} \left\| \sum_0^{N^+} a_n^+ x_n^+ + \sum_0^{N^-} a_n^- x_n^- - x \right\| = 0\,,$$

where $\|\cdot\|$ *is a norm in* B.

So, let the functions $A(t)$, $B(t)$ and $\varphi(t)$ satisfy the following conditions:

1) $A(t)$, $B(t)$ and $\varphi'(t)$ are measurable on (a,b), moreover
$$\sup_{[a,b]} \mathrm{rvai}\{|A(t)|^{\pm 1}; |B(t)|^{\pm 1}; |\varphi'(t)|^{\pm}\} \le M < +\infty\,;$$
2) $\Gamma = \varphi\{[a,b]\}$ is a closed $(\varphi(a) = \varphi(b))$ piecewise smooth simple $[0 \in \operatorname{int}\Gamma]$ Radon's curve or simple Lyapunov contour;
3) $\alpha(t) \equiv \arg A(t)$, $\beta(t) \equiv \arg B(t)$ are continuous functions with bounded variation on $[a,b]$.

For definiteness we'll assume that when the point $\varphi = \varphi(t)$ by increasing t runs along the curve Γ, the internal domain $\operatorname{int}\Gamma$ remains at the left.

Theorem. *Let the functions* $A(t)$, $B(t)$ *and* $\varphi(t)$ *satisfy the conditions 1)-3). If the system (1) makes up a basis in* L_p, $1 < p < \infty$, *then* $|\varphi(t)| \equiv const$.

Proof. Let's suppose the contrary. Let

$$R = \max_{[a,b]} |\varphi(t)| > \min_{[a,b]} |\varphi(t)| = r\,.$$

Since the system (1) make up a basis in $L_p(a,b)$, then $\forall f \in L_p$ it holds the following biorthogonal expansion:

$$f(t)=A(t)\sum_{n=0}^{\infty}a_n\varphi^n(t)+B(t)\sum_{n=0}^{\infty}b_n\overline{\varphi}^n(t).$$

Let's denote

$$f^+(t)=\sum_{n=0}^{\infty}a_n\varphi^n(t). \quad (3)$$

From the basisity and from condition 1) it follows that $f^+ \in L_p(a,b)$. Let's consider the following power series:

$$F(z)\equiv\sum_{n=0}^{\infty}a_n z^n.$$

The radius of convergence of this series we'll denote by R_0. Let's show that $R_0 \geq R$. Let $R_0 < R$. Since, $|\varphi(t)|$ is continuous on $[a,b]$, then it is evident that $\exists t_0 \in [a,b]$, for which $R=|\varphi(t_0)|$. Consequently, $\exists\delta$ is the neighborhood $\overline{G}_\delta(t_0)= =[t_0-\delta\ t_0+\delta]$ (at $t_0=a$ or $t_0=b$ one-sided neighborhood) of point t_0, such that for $\forall t \in \overline{G}_\delta(t_0)$ we have: $R_0<|\varphi(t)|\leq R$, i.e. $\min\limits_{G_\delta(t_0)}|\varphi(t)|=r_\delta>R_0$. From the convergence of the series (3) in L_p it follows that

$$\|a_n\varphi^n(t)\|L_p\to 0,\ \text{for } n\to\infty\,,\ \text{where } \|f\|_{L_p}^p=\int_a^b|f(t)|^p dt.$$

As a result $\|a_n\varphi^n(t)\|_{L_p(G_\delta(t_0))}\to 0,\ n\to\infty$.

As known, $r_0=\dfrac{1}{\overline{\lim\limits_n}\sqrt[n]{|a_n|}}$. So, there exists the sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, $n_k\to\infty$ for which $R_0=\left[\lim\limits_{k\to\infty}\sqrt[n_k]{|a_{n_k}|}\right]^{-1}$. Since $r_\delta>R_0$ then for sufficiently large k: $r_\delta>\dfrac{1}{\sqrt[n_k]{|a_{n_k}|}}$ as a result $\left|\varphi(t)\sqrt[n_k]{|a_{n_k}|}\right|>1$, $\forall t\in G_\delta(t_0)$, i.e. $\left|a_{n_k}\cdot\varphi^{n_k}(t)\right|>1$, $\forall t\in G_\delta(t_0)$.

Thus $\left\|a_{n_k}\cdot\varphi^{n_k}(t)\right\|_{L_p(G_\delta(t_0))}\geq 2\delta>0$, for sufficiently large k. We get the contradiction. Then $R_0\geq R$.

Whence it follows that for $\forall f\in L_p(a,b)$ the function $F(z)$ is analytic in the circle $C_R(0)=\{z\in C\,|\,|z|<R\}$. Since $r<R$, then $\exists\delta_0$ is a neighbourhood of some point $\tau\in(a,b)$, such that $|\varphi(t)|<R$ for $\forall t\in\overline{D}_{\delta_0}(\tau)\equiv[\tau-\delta_0,\tau+\delta_0]$. It is evident that $F^+[\varphi(t)]=f^+(t)$ almost everywhere on $D_{\delta_0}(\tau)$.

From the conditions 1), 2) it follows that the series $\sum_{n=0}^{\infty}a_n\xi^n$ converges in $L_p(\Gamma)$.

Consequently

$$\int_\Gamma\sum_{n=0}^{\infty}a_n\xi^n\cdot\xi^k d\xi=\sum_{n=0}^{\infty}a_n\int_\Gamma\xi^{n+k}d\xi=0,\quad\forall k\geq 0.$$

From this equation and Smirnov's [3, s.424] theorem it follows that there exists the function $F_1(z)$ from the class of Smirnov $E_1(\operatorname{int}\Gamma)$, for which $F_1^+(\xi)=\sum_{n=0}^{\infty}a_n\xi^n$, almost everywhere on Γ. By the condition 2) $\operatorname{int}\Gamma$ belongs to the class C of Smirnov's [4, s.90] domain. Since $F_1^+(\xi)\in L_p(\Gamma)$ then again by the Smirnov's theorem [4, 92] the function $F_1(z)$ belongs to the class $E_p(\operatorname{int}\Gamma)$. From the previous consideration it follows that $F_1^+[\varphi(t)]=F^+[\varphi(t)]$ almost everywhere on $D_{\delta_0}(\tau)$. Then from the uniqueness theorem of Privalov [3, s.413] it follows that $F_1(z)\equiv F(z)$ in $\operatorname{int}\Gamma$. Consequently $F(z)\in E_p(\operatorname{int}\Gamma)$ and $F^+[\varphi(t)]=f^+(t)$ almost everywhere on $[a,b]$. It is proved analogously that the function $\Phi(z)\equiv\sum_{n=0}^{\infty}\overline{b}_n z^n$ belongs to the class of Smirnov $E_p(\operatorname{int}\Gamma)$ and $\overline{\Phi}[\varphi(t)]=\sum_{n=0}^{\infty}b_n\overline{\varphi}^n(t)$ almost everywhere on $[a,b]$.

As a result we get that the function $F(z)$ and $\Phi(z)$ are the solutions of the following conjugate problem in Smirnov class $E_p(\operatorname{int}\Gamma)$:

$$A(t)F^+[\varphi(t)]+B(t)\overline{\Phi}^+[\varphi(t)]=f(t) \text{ almost everywhere on } [a,b]. \tag{4}$$

Let's consider the system:

$$\left\{\widetilde{A}(t)\varphi^n(t);\ \widetilde{B}(t)\overline{\varphi}^n(t)\right\}_{n\geq 0}. \tag{5}$$

where $\widetilde{A}(t)\equiv A(t)\cdot[\overline{\varphi}'(t)]^{-1}$; $\widetilde{B}(t)\equiv B(t)\cdot[\varphi'(t)]^{-1}$, so,

$$\widetilde{\alpha}(t)\equiv\arg\widetilde{A}(t)=\alpha(t)+\arg\varphi'(t) \text{ and } \widetilde{\beta}(t)\equiv\arg\widetilde{B}(t)=\beta(t)-\arg\varphi'(t).$$

It is easy to show that the system (5) is complete in $L_q(a,b)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ iff the conjugate problem

$$\widetilde{B}(t)\cdot\varphi'(t)\cdot\Phi_1^+[\varphi(t)]+\widetilde{A}(t)\cdot\overline{\varphi}'(t)\overline{\Phi}_2^+[\varphi(t)]=0 \text{ almost everywhere on } [a,b]$$

has only a trivial solution in $E_p(\operatorname{int}\Gamma)$. So we get that if the system (5) is complete in $L_q(a,b)$ then homogeneous problem of conjugation (4) has only a trivial solution in the class $E_p(\operatorname{int}\Gamma)$.

According to paper [1], let's find the value $\widetilde{\omega}$ for the system (5):

$$\widetilde{\omega}=\frac{1}{2\pi}\left[\widetilde{\beta}(a)-\widetilde{\beta}(b)+\widetilde{\alpha}(b)-\widetilde{\alpha}(a)+\frac{2}{p}(\arg\varphi'(a)-\arg\varphi'(b))\right]+\frac{2}{p}-1.$$

Since $\arg\varphi'(b)-\arg\varphi'(a)=2\pi$ we get:

$$\widetilde{\omega}=-\gamma+1,$$

where $\gamma=\frac{1}{2\pi}[\alpha(b)-\alpha(a)-\beta(b)+\beta(a)]$.

By the results of [1] the system (5) is complete in L_q iff $\widetilde{\omega}\leq\frac{1}{q}$. Again by the results of paper [1] the system (1) is minimal in $L_q(a,b)$ if $\omega=\gamma-1>-\frac{1}{q}$ i.e. $\widetilde{\omega}<\frac{1}{q}$. As a result the system (5) is complete in $L_q(a,b)$, so, the homogeneous conjugation problem

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By the results of [1] the system (5) is complete in L_q iff $\widetilde{\omega}\leq\frac{1}{q}$. Again by the results of paper [1] the system (1) is minimal in $L_q(a,b)$ if $\omega=\gamma-1>-\frac{1}{q}$ i.e. $\widetilde{\omega}<\frac{1}{q}$. As a result the system (5) is complete in $L_q(a,b)$, so, the homogeneous conjugation problem

$$A(t)F^{+}[\varphi(t)]+B(t)\overline{\Phi}^{+}[\varphi(t)]=0\text{, almost everywhere on } [a,b]$$

has only a trivial solution in $E_p(\mathrm{int}\Gamma)$. Thus, the problem (4) is uniquely solvable in $E_p(\mathrm{int}\Gamma)$.

Since $r<R$ then it's evident that $\exists z_0 \in ext\Gamma$ such that $|z_0|<R$. Let's consider the function

$$f(t)=\frac{A(t)}{\varphi(t)-z_0}.$$

It is evident that the function $f_0(z)\equiv\frac{1}{z-z_0}$ belongs to the class $E_p(\mathrm{int}\Gamma)$ and moreover $A(t)f_0^{+}[\varphi(t)]=f(t)$ on $[a,b]$.

Comparing this problem with (4) from the uniqueness we receive that $F(z)\equiv f_0(z)$, $\Phi(z)\equiv 0$ in $\mathrm{int}\Gamma$. So $F(z)$ is an analytic continuation of the function $f_0(z)$ from the domain $\mathrm{int}\Gamma$ in $C_R(0)\setminus\mathrm{int}\Gamma$. But from the uniqueness of analytic continuation it follows that it's impossible, because $z_0\in C_R$ is a pole of the function $f_0(z)$ in $C_R(0)$. The theorem is proved.

Let's formulate the following easily provable lemma.

Lemma: *The system*

$$\left\{\mathrm{Re}\left[A(t)\varphi^{n}(t)\right];\ \mathrm{Im}\left[A(t)\varphi^{n}(t)\right]\right\}_{n\geq 0}. \qquad (6)$$

makes up a basis in $L_p^R(a,b)$, $p\geq 1$ *iff the system*

$$\left\{A(t)\varphi^{n}(t);\ \overline{A}(t)\overline{\varphi}^{n}(t)\right\}_{n\geq 0}$$

makes up a basis in $L_p(a,b)$.

The corollary follows from this lemma.

Corollary: *Let the function* $A(t)$ *and* $\varphi(t)$ *satisfy the conditions 1)-3). If the system (6) makes up a basis in* $L_p^R(a,b)$, $1<p<+\infty$ *then* $|\varphi(t)|\equiv const$.

Note. The theorem can be proved in common assumptions with respect to the function $A(t)$, $B(t)$ and $\varphi(t)$.

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THE PROPERTIES OF TRACES OF FUNCTIONS ON THE BOUNDARY OF A SURFACE

Abstract

At the paper the properties of L_p - traces of functions of many variables defined on domain $G \subset E_n$ satisfying "the condition of σ-semi-horn" are considered. The estimations of norms of traces of functions by norm of known spaces $B^{<r>}_{p,\theta}(G;s)$ of the generalizing spaces S.M.Nikolsky-O.V.Besov $(s=1)$ and the B-spaces functions with dominate mixed derivatives are proved.

Introduction. At the paper the definitions of the known spaces

$$B^{<r>}_{p,\theta}(G;s), \tag{1}$$

of the functions $f = f(x)$ of the points $x = (x_1;...;x_s) \in E_n = E_{n_1} \times ... E_{n_s}$ of many groups of variables $x_k = (x_{k,1};...;x_{k,n_k}) \in E_{n_k}$ $(k = 1,2,...,s)$, $(n_1 + n_2 + . + n_s = n)$ defined on domain $G \subset E_n$, which satisfies "the condition σ-semi-horn" ([1]) are cited.

A class of surfaces

$$\Gamma_m \in \Pi^1 \tag{2}$$

of dimension $m = m_1 + m_2 + ... + m_\alpha < n = n_1 + n_2 + ... + n_s$ $(1 \le \alpha \le s \le n)$, where $1 \le m_k \le n_k$ $(k = 1,2,...,\alpha)$ is introduced the notation of L_p traces on the surface Γ_m is given, in case when this surface Γ_m is on the boundary ∂G of domain

$$G \in C(\sigma; H) \tag{3}$$

and on the base of new integral representations given in monography [1], the estimations of L_p-traces of functions and their corresponding derivatives on the surface Γ_m by norms of given spaces (1) is proved. Thus it is necessary to note that these spaces in case $s = 1$ are the generalizations of corresponding spaces

$$B^{<r>}_{p,\theta}(G;s),$$

of S.M.Nikolsky-O,V.Besov $(1 < p \le \theta < \infty,\ \theta = \infty)$ and in case $s = n$ generalizations of the known spaces $S^r_{p,\theta}B(G)$ - S.M.Nikolsky $(\theta = \infty)$ - A.D.Dzabrailov $(p = \theta)$ - I.T.Amanov $(1 < p \le \theta < \infty)$.

1.Main definitions and notations.

1.1. Spaces. Let

$$r = (r_1;...;r_s) \tag{1.1}$$

be a "positive vector" with coordinate vectors $r_k = (r_{k,1};...;r_{k,n_k})$ $(k = 1,2,..,s)$.

Let

$$\bar{r} = (\bar{r}_1;...;\bar{r}_s) \tag{1.2}$$

be an "integer non-negative vector" such that $\bar{r}_{k,j}$ is the greatest integer number smaller than $r_{k,j}$ at $j = 1,2,...,n_k$ for all $k = 1,2,...,s$.