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THE NECESSARY CONDITIONS OF OPTIMALITY FOR IMPULSE EXCITATIONS

Abstract

In the paper the recurrent necessary conditions of optimality singular in classical sense controls for systems with impulse excitations have been obtained.

For today the questions related with obtaining the necessary conditions of optimality of Pontryagin's maximum principle type for disconnected problems have been sufficiently complete studied [1-5], and in some works [5,6] singular by Pontryagin controls have been considered.

It is known [7] that any control which is singular in sense of Pontryagin's maximum principle under additional conditions of smoothness, is singular also in classical sense, but the inverse doesn't hold. Therefore investigating controls which are singular in classical sense, one can obtain additional information also about controls which aren't singular in Pontryagin's maximum principle sense.

In the present paper continuing the investigations [6] new necessary conditions of optimality for singular in classical sense controls in systems with impulse excitations, have been obtained.

1. Let motion of the object happen in the time interval $[t_0,T]$. Moments $t_0=\theta_0<\theta_1<\theta_2<...<\theta_{n-1}<\theta_n=T$ have been chosen and fixed in this interval. On each of the intervals $[\theta_{i-1},\theta_i]$ trajectory of the motion $x_i(t)\in R^{n_i}$ is described by the system of differential equations:

$$\dot{x}_i(t)=f_i(t,x_i(t),u_i(t)),\ t\in[\theta_{i-1},\theta_i],\ \ i=\overline{1,n}\,, \qquad (1)$$

where in moments θ_i, $i=\overline{0,n-1}$ the estimations

$$x_{i+1}(\theta_i)=G_i(x_i(\theta_i)),\ \ i=\overline{1,n-1},$$
$$x_1(\theta_0)=G_0(x_0(\theta_0))=x_0 \qquad (2)$$

are fulfilled.

Here $f_i(t,x_i,u_i)$ are the given n_i-dimensional vector functions, which are continuous in $[\theta_{i-1},\theta_i]\times R^{n_i}\times R^{r_i}$, $G_i(z)$ are the given n_{i+1}-dimensional twice continuously differentiable vector-functions, $x_0\in R^{n_1}$ is a given point, $u_i(t)$ are the r_i-dimensional piecewise continuous (continuous from the right at discontinuity points) vector-functions of control excitations with values from given bounded open sets $V_i\subset R^{r_i}$ (admissible control):

$$u_i(t)\in V_i\subset R^{r_i},\ \ t\in[\theta_{i-1},\theta_i],\ i=\overline{1,n}\,. \qquad (3)$$

The problem is in the minimization of the functional

$$S(u_1,...,u_n)=G_n^0(x_n(\theta_n)), \qquad (4)$$

defined on the solutions of the system (1), (2) generated by the all possible admissible controls.

Here $G_n^0(z)$ is the given twice continuously differentiable scalar function.

It is assumed that to each given admissible control $u_1(t),...,u_n(t)$ corresponds unique absolutely continuous solutions $x_1(t),...,x_n(t)$ of the system (1), (2).

The functions $f_i(t,x_i,u_i)$ are sufficiently smooth on totality of their arguments, at the points of continuity, and at the point of discontinuity is continuous from the right with its derivatives.

Further we'll call the problem of minimum of the problem (4) under the constraints (1)-(3) as the problem (1)-(4), the solution of this problem an optimal control, and corresponding process an optimal process.

2. Let $(u_1(t),...,u_n(t),x_1(t),...,x_n(t))$ be a fixed admissible process in t he problem (1)-(4). We'll introduce the designations:

$$H_i(t,x_i,u_i,\psi_i)=\psi_i' f_i(t,x_i,u_i),\ H_{ix_i}(t)=H_{ix_i}(t,x_i(t),u_i(t),\psi_i(t)).$$

Here ψ_i are n-dimensional vector-functions of adjoint variables determined by the correlations (see [6]):

$$\psi_n(t)=-F_n'(\theta_n,t)G_{nz}^0(x_n(\theta_n)),$$
$$\psi_i(t)=F_i'(\theta_i,t)G_{iz}^{n-i}(x_i(\theta_i)), \qquad (5)$$
$$G_i^{n-i}(x_i(\theta_i))=\psi_{i+1}'(\theta_i)G_i(x_i(\theta_i)).$$

and $F_i(t,\tau)$ is a matrix which is the solution of the following integral equation of Volterra type

$$F_i(t,\tau)=E_i+\int_\tau^t F_i(t,\alpha)f_{ix_i}(\alpha)d\alpha$$

E_i is a unit $n_i\times n_i$ matrix.

It is shown in [6] that the first and the second variations (in classical sense) of the functional (4) have the following form:

$$\delta^1 S(u_1,...,u_n)=-\sum_{i=1}^n\int_{\theta_{i-1}}^{\theta_i} H_{iu_i}'(t)\delta u_i(t)dt, \qquad (6)$$

$$\delta^2 S(u_1,...,u_n)=-\sum_{i=1}^n\Bigg[(-1)^{e(i,n)}\delta x_i'(\theta_i)\frac{\partial^2 G_i^{n-i}(x_i(\theta_i))}{\partial z^2}\delta x_i(\theta_i)+$$
$$+\int_{\theta_{i-1}}^{\theta_i}\left(\delta x_i'(t)H_{ix_ix_i}(t)\delta x_i(t)+2\delta u_i'(t)H_{iu_ix_i}(t)\delta x_i(t)+\delta u_i'(t)H_{iu_iu_i}(t)\delta u_i(t)\right)dt\Bigg], \qquad (7)$$

where $\delta u_i(t)$ are the variations of controls $u_i(t)$, and $\delta x_i(t)$ are the solutions of equations in the variations

$$\delta\dot{x}_i(t)=f_{ix_i}(t)\delta x_i(t)+f_{iu_i}(t)\delta u_i(t),$$
$$\delta x_i(\theta_{i-1})=\frac{\partial G_{i-1}(x_{i-1}(\theta_{i-1}))}{\partial z}\delta x_{i-1}(\theta_{i-1}),\ i=\overline{2,n},$$
$$\delta x_1(\theta_0)=0, \qquad (8)$$

and

$$e(i,n)=\begin{cases}1, & i=n,\\ 0, & i\neq n.\end{cases}$$

3. Decomposition of high order for the second variation of the functional. The second variation of the functional (7) for $\delta u_1(t)=0,\dots,\delta u_k(t)=\delta u_k(t),\dots,\delta u_n(t)=0$ one can represent in the form (see [6]):

$$\delta^2 S(u_1,\dots,u_n)=-\int\limits_{\theta_{k-1}}^{\theta_k}\Big[\delta u_k'(t)H_{ku_ku_k}(t)\delta u_k(t)+2\delta x_k'(t)H_{kx_ku_k}(t)\delta u_k(t)+ \\ +\delta x_k'(t)H_{kx_kx_k}(t)\delta x_k(t)\Big]dt-\delta x_k'(\theta_k)L_k^n\delta x_k(\theta_k), \quad (9)$$

where

$$L_k^n=(-1)^{e(k,n)}\frac{\partial^2 G_k^{n-k}(x_k(\theta_k))}{\partial z^2}+\frac{\partial G_k'(x_k(\theta_k))}{\partial z^2}\times \\ \times\sum_{i=k+1}^{n}\left[\prod_{j=2}^{i-k}\left(F_{i+1-j}'(\theta_{i+1-j},\theta_{i-j})\frac{\partial G_{i+1-j}'(x_{j+1-j}(\theta_{i+1-j}))}{\partial z}\right)\Psi_i^n\times\right. \\ \left.\times\prod_{j=2}^{i-k}\left(\frac{\partial G_{i+1-j}(x_{j+1-j}(\theta_{i+1-j}))}{\partial z}F_{i+1-j}(\theta_{i+1-j},\theta_{i-j})\right)\frac{\partial G_k(x_k(\theta_k))}{\partial z}\right],\quad \prod_{j=2}^{1}A=1,\quad \sum_{i=n+1}^{n}A=0,$$

$$\Psi_i^n=\int\limits_{\theta_{i-1}}^{\theta_i}F_i'(t,\theta_{i-1})H_{ix_ix_i}(t)F_i(t,\theta_{i-1})dt+(-1)^{e(i,n)}F_i'(\theta_i,\theta_{i-1})\frac{\partial^2 G_i^{n-i}(x_i(\theta_i))}{\partial z^2}F_i(\theta_i,\theta_{i-1}).$$

Introducing the function $\Psi_k(t)$ which is the solution of the problem

$$\dot\Psi_k(t)=-f_{kx_k}'(t)\Psi_k(t)-\Psi_k(t)f_{kx_k}(t)-H_{kx_kx_k}(t),\quad \Psi_k(\theta_k)=L_k^n,$$

analogously [7] the second variation of the functional (9) we represent in the form

$$\delta^2 S(u_1,\dots,u_n)=-\int\limits_{\theta_{k-1}}^{\theta_k}\delta u_k'(t)H_{ku_ku_k}(t)\delta u_k(t)dt- \\ -2\int\limits_{\theta_{k-1}}^{\theta_k}\delta u_k'(t)\big(H_{kx_ku_k}(t)+\Psi_k(t)f_{ku_k}(t)\big)'\delta x_k(t)dt. \quad (10)$$

Assume

$$\delta u_k(t)=\begin{cases}v_k, & t\in[\tau,\tau+\varepsilon)\subset[\theta_{k-1},\theta_k),\\ 0, & t\in[\theta_{k-1},\theta_k)\setminus[\tau,\tau+\varepsilon),\end{cases}$$

where $v_k\in V_k$, $\varepsilon>0$ is a parameter, τ is a point of continuity of $u_k(t)$.

By virtue of (10), applying Taylor formula at the point τ and Leibniz formula on differentiation of derivative similarly to [8] we'll obtain the following representation for $\delta^2 S(u_1,\dots,u_n)$

$$\delta^2 S(u_1,\dots,u_n)=-\int\limits_{\tau}^{\tau+\varepsilon}v_k'H_{ku_ku_k}(t)v_kdt-2v_k'\sum_{m=1}^{l+1}\sum_{i=1}^{m}C_m^i\frac{d^{m-i}}{dt^{m-i}}q_k'(t)\Big/_{t=\tau+0}\times \\ \times P_{i-1}^k(\tau)v_k\frac{\varepsilon^{m+1}}{(m+1)!}+o\big(\varepsilon^{l+2}\big). \quad (11)$$

Here

$$C_m^i = \frac{m!}{i!(m-i)!};\ q_k(t) = H_{kx_k u_k}(t) + \Psi_k(t) f_{ku_k}(t),$$

$$\frac{d^m}{dt^m}\delta x_k(t) = \omega_{m-1}^k(t)\delta x_k(t) + P_{m-1}^k(t)v_k,\ t\in[\tau,\tau+\varepsilon),\ m=1,2,...$$

$$\frac{d^m}{dt^m}\delta x_k(t)\Big/_{t=\tau+0} = P_{m-1}^k(\tau)v_k,\quad m=1,2,...$$

$$\omega_j^k(t) = \frac{d}{dt}\omega_{j-1}^k(t) + \omega_{j-1}^k(t)\omega_0^k(t),\ \omega_0^k(t) = f_{kx_k}(t),$$

$$P_j^k(t) = \frac{d}{dt}P_{j-1}^k(t) + \omega_{j-1}^k(t)P_0^k(t),\quad P_0^k(t) = f_{ku_k}(t),$$

$$j=1,2,...\quad k=1,2,...,n.$$

Using designations

$$Q_m^k(\tau)[v_k,v_k] = v_k'\sum_{i=1}^m C_m^i \frac{d^{m-i}}{dt^{m-i}}q_k'(t)\Big/_{t=\tau+0}P_{i-1}^k(\tau)v_k, \tag{12}$$

$$Q_0^k(\tau)[v_k,v_k] = v_k' H_{ku_k u_k}(\tau)v_k,\ k=\overline{1,n},\ m=1,2,...,$$

we'll write (11) in the form

$$\delta^2 S(u_1,...,u_n) = -\int_\tau^{\tau+\varepsilon} Q_0^k(t)[v_k,v_k]dt - 2\sum_{m=1}^{l+1} Q_m^k(\tau)[v_k,v_k]\frac{\varepsilon^{m+1}}{(m+1)!} + o(\varepsilon^{l+2}). \tag{13}$$

4. Optimality condition.

Let $u_1(t),...,u_n(t)$ be an optimal control for the problem (1)-(4). Then the following conditions (see [6]):

$$H_{ku_k}(t) = 0,\ \forall\tau\in[\theta_{k-1},\theta]\qquad \text{(Euler's equation)} \tag{14}$$

$$v_k' H_{ku_k u_k}(\tau)v_k \le 0,\ k=\overline{1,n},\quad \forall\tau\in[\theta_{k-1},\theta_k),\ \forall v_k\in R^{r_k}\qquad \text{(the Legendre-Klebsch condition)} \tag{15}$$

are satisfied.

Definition 1. *The admissible control $u_1(t),...,u_n(t)$ satisfying conditions (14). (!5) is called singular of zero-order, in classical sense, at the point $\tau\in[\theta_{k-1},\theta_k]$, if there exist $\alpha>0$ such that*

$$Ker Q_0^k(\tau)\ne\{0\},\ Ker Q_0^k(\tau)\subset Ker Q_0^k(t),\ \forall t\in[\tau,\tau+\alpha),\ k=\overline{1,n},$$

where $Ker Q_0^k(\tau)$ is the kernel of the quadratic form $Q_0^k(\tau)[v_k,v_k]$.

Definition 2. *The control $u_1(t),...,u_n(t)$ which is zero-order singular at the point τ is called l-th $(l>0)$ order singular at the point τ, if*

$$\bigcap_{i=0}^{l} Ker Q_i^k(\tau)\ne\{0\},\ \bigcap_{i=0}^{l+1} Ker Q_i^k(\tau)=\{0\},\ k=\overline{1,n}.$$

Then from (13) we'll obtain the following theorem.

Theorem. *Let the admissible control $u_1(t),...,u_n(t)$ be l-th $(l>0)$ order singular at the point τ. Then for the optimality $u_1(t),...,u_n(t)$ it is necessary satisfaction of the inequalities*

$$Q_{m+1}^{k}(\tau)[v_k, v_k] \le 0, \quad \forall v_k \in \bigcap_{i=0}^{m} Ker Q_i^k(\tau), \quad m = 0,1,...,l. \tag{16}$$

Note that the condition (16) for $l = 0$ has been found in [6].

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