2001 VOL. XV(XXIII)

ASLANOVA N.M.

THE STABILITY OF THE INVERSE PROBLEM OF SCATTERING THEORY FOR NONSELF-ADJOINT OPERATOR ON ALL AXIS

Abstract

The stability for non-self-adjoint operator on all axis is studied. The estimations for the solutions and potentials of two non-self-adjoint operators have been obtained.

The questions about which information about the function $q(x)$ or generally about the operator L one can extract if the scattering data are known only on an interval of variation of spectral parameter, has an important value. Therefore from physical point of view the natural statement of the question about stability of inverse problems is such: how much strongly can differ two problems whose scattering data coincide in the given interval of variation ρ^2. Stability of the inverse problem was studied in the series of works ([1], [4], [5]). For the self-adjoint operator on semi-axis it has got the solution in the V.A.Marchenko work [1].

In the present paper the stability for non-self-adjoint operator on all axis is studied.

The results of [1], [2], [3] are used.

1. Operator L.

We'll denote by L an operator given in Hilbert space $L^2(-\infty,\infty)$ by the differential expression

$$ly = -y'' + q(x)y . \tag{1.1}$$

We'll assume that $q(x)$ satisfies

$$|q(x)|e^{\varepsilon|x|} \in L^1(-\infty,\infty) \tag{1.2}$$

for $\varepsilon > 0$.

We'll denote by $V\{C_\pm(x)\}$ the set of boundary-value problems, whose

$$\int_x^\infty e^{\varepsilon t}|q(t)|dt \le C_+(x)\ , \quad \int_{-\infty}^x e^{\varepsilon|t|}|q(t)|dt \le C_-(x)\ ,$$

where $C_\pm(x)$ are continuous monotone functions, correspondingly.

2. Special solutions of the equation $ly = \rho^2 y$.

We'll denote by $e_\pm(x,\rho)$ the solution of the differential equation $ly = S^2 y$, which have the following asymptotics on infinity for $\operatorname{Im}\rho > -\dfrac{\varepsilon}{2}$

$$e_+(x,\rho) \sim e^{ix\rho} \text{ for } x \to \infty, \quad e_-(x,\rho) \sim e^{-ix\rho} \text{ for } x \to -\infty .$$

These solutions are holomorphic by ρ in half-plane $\operatorname{Im}\rho > -\dfrac{\varepsilon}{2}$ (ε is the number that in the condition (1.2)) and admit the following representations with the help of transformation operators

$$e_+(x,\rho)=e^{ix\rho}+\int_x^{\infty}K_+(x,t)e^{it\rho}dt\,, \qquad (2.1)$$

$$e_-(x,\rho)=e^{-ix\rho}+\int_{-\infty}^{x}K_-(x,t)e^{-it\rho}dt\,. \qquad (2.2)$$

The kernel $K_\pm(x,t)$ have continuous derivatives which satisfy the inequalities

$$|K_+(x,t)|\le C_a^+ e^{-\varepsilon\frac{x+t}{2}} \quad \text{for } t\ge x\ge a\,, \qquad (2.3)$$

$$|K_-(x,t)|\le C_a^- e^{\varepsilon\frac{x+t}{2}} \quad \text{for } t\le x\le a\,. \qquad (2.4)$$

Moreover, the kernels $K_\pm(x,t)$ are connected with the potential $q(x)$ by the following way:

$$q(x)=2\frac{d}{dx}K_-(x,x)=-2\frac{d}{dx}K_+(x,x).$$

3. Scattering data of the operator L.

According to [3] the Wronskian $w(\rho)$

$$w(\rho)=e_-(x,\rho)e'+(x,\rho)-e'_-(x,\rho)e_+(x,\rho)$$

is called denominator (of kernel of the resolving of the operator L). The denominator $w(\rho)$ has a finite number of zeros in the half-plane $\operatorname{Im}\rho\ge 0$. These zeros are called singular numbers. Let's denote by $\rho_1,...,\rho_2$ the non-real singular numbers, and singular numbers laying on real axis by $\rho_{\alpha+1},...,\rho_\beta$. Multiplicity of the root ρ_k of the equation $w(\rho)=0$ is called multiplicity of the singular number ρ_k and is denoted by m_k $\left(k=\overline{1,\beta}\right)$.

The function $S(\rho)=\dfrac{v(\rho)}{w(\rho)}$, where

$$w(\rho)=e_+(x,_-\rho)e'_-(x,\rho)_-e'_+(x,_-\rho)e_-(x,\rho)\,,$$

is called a scattering function. Parallel with $S(\rho)$ the function $S_1(\rho)$:

$$S_1(\rho)=\frac{S(_-\rho)w(_-\rho)}{w(\rho)}$$

is used. The Fourier transformation of the operator L

$$f_S^+(x)=\frac{1}{2\pi}\int_+ S(\rho)e^{ix\rho}d\rho\,, \qquad (3.1)$$

$$f_S^-(x)=\frac{1}{2\pi}\int_+ S_1(\rho)e^{-ix\rho}d\rho \qquad (3.2)$$

is its important spectral characteristic together with scattering function.

Since $S(\rho)$, $S_1(\rho)$ are the functions of (S) type [3], the contour of integration is any straight line in the strip $0<\operatorname{Im}\rho<\varepsilon_0$ $(\varepsilon_0=\min\{\varepsilon/2,\varepsilon_1\}$, ε_1 is a distance from the axis $\operatorname{Im}\rho=0$ to nonreal zeros of the $w(\rho)$).

The functions

$$f_k^\pm(x)=p_k^\pm(x)e^{\pm ix\rho_k} \qquad (3.3)$$

characterize the operator L on the pointwise spectrum. Here $p_k^{\pm}(x)$ are the polynomials of the degree m_k-1 (see [3]). In the case of real potential and the multiplicity $m_k=1$ the polynomials $p_k^{\pm}(x)$ coincide with the normalized factors considered in the work [2].

The scattering functions $S(\rho)$, the non-real singular numbers $\rho_1,...,\rho_\alpha$ and the corresponding to them normalized factors are called the scattering data of the operator L.

4. Estimation of the special solutions.

Let's write the integral equations which the kernels $K_{\pm}(x,t)$

$$K_+(x,t)+\int_x^{\infty}K_+(x,u)f_+(u+t)du+f_+(x+t)=0,\quad -\infty<x\le t<\infty, \tag{4.1}$$

$$K_-(x,t)+\int_{-\infty}^{x}K_-(x,u)f_-(u+t)du+f_-(x+t)=0,\quad -\infty<t\le x<\infty \tag{4.2}$$

satisfy, where

$$f_{\pm}(x)=f_S^{\pm}(x)+\sum_{k=1}^{\alpha}f_k^{\pm}(x). \tag{4.2`}$$

Let's consider two problems with the potentials $q_1(x), q_2(x)$. Let's compose the main integral equations for them, and subtract one from another

$$K_{1,2+}(x,t)+\int_x^{\infty}f_{1+}(u+t)K_{1,2+}(x,u)du+\int_x^{\infty}f_{1,2+}(u+t)K_{2+}(x,u)du+f_{1,2+}(x,t)=0, \tag{4.3}$$

$$K_{1,2-}(x,t)+\int_{-\infty}^{x}f_{1-}(u+t)K_{1,2-}(x,u)du+\int_{-\infty}^{x}f_{1,2-}(u+t)K_{2-}(x,u)du+f_{1,2-}(x+t)=0, \tag{4.4}$$

where

$$\begin{aligned}K_{1,2\pm}(x,t)&=K_{1\pm}(x,t)-K_{2\pm}(x,t),\\ f_{1,2\pm}(x)&=f_{1\pm}(x)-f_{2\pm}(x).\end{aligned} \tag{4.5}$$

$K_{1\pm}$, $f_{1\pm}$ and $K_{2\pm}$, $f_{2\pm}$ are functions from the equations (4.1), (4.2) for the problems with the potentials $q_1(x)$, $q_2(x)$ correspondingly. Since the estimation of the difference $e_{1+}(x,\rho)-e_{2+}(x,\rho)$ is obtained analogously to the problem considered on the positive semi-axis, we'll calculate the difference of the solutions $e_{1-}(x,\rho)-e_{2-}(x,\rho)$.

For every fixed x solving (4.4) with respect to $K_{1,2-}(x,t)$ we obtain

$$K_{1,2-}(x,t)=-\left(\mathbf{I}+\mathbf{F}_{1-,x}\right)^{-1}\left\{f_{1,2-}(x+t)+\int_{-\infty}^{x}f_{1,2-}(u+t)K_{2-}(x,u)du\right\}, \tag{4.6}$$

where

$$\mathbf{F}_{1-,x}u=\int_{-\infty}^{x}f_{1-}(y+t)u(t)dt.$$

Note that the validity of the identity

$$\left(\mathbf{I}+\mathbf{F}_{1-,x}\right)^{-1}=\left(\mathbf{I}+\overline{\mathbf{K}}_{1-,x}\right)\left(\mathbf{I}+\mathbf{K}_{1-,x}\right) \tag{4.7}$$

follows from the main equation. The operators $\mathbf{K}_{1-,x}$, $\overline{\mathbf{K}}_{1-,x}$ are defined by the formula

$$\mathbf{K}_{1-,x}f=\int_{-\infty}^{y}K_{1-}(y,t)f(t)dt\,,\quad \overline{\mathbf{K}}_{1-,x}f=\int_{y}^{x}K_{1-}(t,y)f(t)dt$$

Granting (3.2), (3.3), (4.2) we find

$$f_{1,2-}(x)=\frac{1}{2\pi}\int_{+}\left(S_1^1(\rho)-S_1^2(\rho)\right)e^{-ix\rho}d\rho+\sum_{k=1}^{\alpha}\left[P_{k_1}^-(x)-P_{k_2}^-(x)\right]e^{-ix\rho_k}\,, \tag{4.8}$$

where $S_1^1(\rho)$, $S_1^2(\rho)$ are the same functions that in the formula (3.2) for the equations with the potentials $q_1(x)$, $q_2(x)$ correspondingly. Using these equalities and also the formula (2.1), (2.2) the following lemma is proved.

Lemma 4.1. *For all values of* μ *from the domain* $\operatorname{Im}\mu>-\frac{\varepsilon}{2}$, $\operatorname{Im}\mu\neq\eta$, $\mu\neq\rho_k$

$$-\{e_{1-}(x,\mu)-e_{2-}(x,\mu)\}^2=\int_{-\infty}^{x}\{q_1(t)-q_2(t)\}\{A_{1,2-}(\mu,x,t)-A_{1,2-}(\mu,t,x)\}dt\,, \tag{4.9}$$

$$-\{e_{1+}(x,\mu)-e_{2+}(x,\mu)\}^2=\int_{x}^{\infty}\{q_1(t)-q_2(t)\}\{A_{1,2+}(\mu,x,t)-A_{1,2+}(\mu,t,x)\}dt\,,$$

where

$$A_{1,2-}(\mu,x,t)=\frac{e_{1-}(x,\mu)e_{2-}(x,\mu)}{2\pi}\int_{+}\frac{\left[S_1^1(\rho)-S_1^2(\rho)\right]}{\rho^2-\mu^2}e_{1-}(t,\rho)e_{2-}(t,\rho)d\rho+$$
$$+e_{1-}(x,\mu)e_{2-}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^-\left(\frac{d}{id\rho}\right)-P_{k_2}^-\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1-}(t,\rho)e_{2-}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k},$$

$$A_{1,2+}(\mu,x,t)=\frac{e_{1+}(x,\mu)e_{2+}(x,\mu)}{2\pi}\int_{+}\frac{\left[S^1(\rho)-S^2(\rho)\right]}{\rho^2-\mu^2}e_{1+}(t,\rho)e_{2+}(t,\rho)d\rho+$$
$$+e_{1+}(x,\mu)e_{2+}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^+\left(\frac{d}{id\rho}\right)-P_{k_2}^+\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1+}(t,\rho)e_{2+}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k}.$$

$S^1(\rho),S^2(\rho)$ *are the functions from (3.1) for the equations with the potentials* $q_1(x), q_2(x)$.

Let the data $\left\{S^j(\rho),S_1^j(\rho),\rho_k(j),P_{kj}\right\}$ of the problems with the potentials $q_j(x)\,(j=1,2)$ coincide for $\operatorname{Re}\rho^2\in(-\infty,N)$:

$S^1(\rho)=S^2(\rho)$,

$S_1^1(\rho)=S_1^2(\rho)$, $\quad for\,|\operatorname{Re}\rho|<\sqrt{N+\eta^2}$, $\operatorname{Im}\rho=\eta$, $\eta<\varepsilon_0$,

$\rho_k(1)=\rho_k(2)$, $\quad P_{k_1}=P_{k_2}$.

Let's prove the following theorem.

Theorem 4.1. *If the scattering data of two problems* $q_j(x)\in V\{C_\pm(x)\}$ $(j=1,2)$ *coincide for all* $\operatorname{Re}\rho^2\in(-\infty,N)$, *then for* $\operatorname{Re}\mu^2\in(-\infty,N)$ *the following inequalities*

$$\mathbf{K}_{1-,x}f=\int_{-\infty}^{y}K_{1-}(y,t)f(t)dt\,,\quad \overline{\mathbf{K}}_{1-,x}f=\int_{y}^{x}K_{1-}(t,y)f(t)dt$$

Granting (3.2), (3.3), (4.2) we find

$$f_{1,2-}(x)=\frac{1}{2\pi}\int_{+}\left(S_1^1(\rho)-S_1^2(\rho)\right)e^{-ix\rho}d\rho+\sum_{k=1}^{\alpha}\left[P_{k_1}^{-}(x)-P_{k_2}^{-}(x)\right]e^{-ix\rho_k}\,, \tag{4.8}$$

where $S_1^1(\rho)$, $S_1^2(\rho)$ are the same functions that in the formula (3.2) for the equations with the potentials $q_1(x)$, $q_2(x)$ correspondingly. Using these equalities and also the formula (2.1), (2.2) the following lemma is proved.

Lemma 4.1. *For all values of* μ *from the domain* $\operatorname{Im}\mu>-\frac{\varepsilon}{2}$, $\operatorname{Im}\mu\neq\eta$, $\mu\neq\rho_k$

$$-\{e_{1-}(x,\mu)-e_{2-}(x,\mu)\}^2=\int_{-\infty}^{x}\{q_1(t)-q_2(t)\}\{A_{1,2-}(\mu,x,t)-A_{1,2-}(\mu,t,x)\}dt\,, \tag{4.9}$$

$$-\{e_{1+}(x,\mu)-e_{2+}(x,\mu)\}^2=\int_{x}^{\infty}\{q_1(t)-q_2(t)\}\{A_{1,2+}(\mu,x,t)-A_{1,2+}(\mu,t,x)\}dt\,,$$

where

$$A_{1,2-}(\mu,x,t)=\frac{e_{1-}(x,\mu)e_{2-}(x,\mu)}{2\pi}\int_{+}\frac{\left[S_1^1(\rho)-S_1^2(\rho)\right]}{\rho^2-\mu^2}e_{1-}(t,\rho)e_{2-}(t,\rho)d\rho+$$

$$+e_{1-}(x,\mu)e_{2-}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^{-}\left(\frac{d}{id\rho}\right)-P_{k_2}^{-}\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1-}(t,\rho)e_{2-}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k},$$

$$A_{1,2+}(\mu,x,t)=\frac{e_{1+}(x,\mu)e_{2+}(x,\mu)}{2\pi}\int_{+}\frac{\left[S^1(\rho)-S^2(\rho)\right]}{\rho^2-\mu^2}e_{1+}(t,\rho)e_{2+}(t,\rho)d\rho+$$

$$+e_{1+}(x,\mu)e_{2+}(x,\mu)\sum_{k=1}^{\alpha}\left(P_{k_1}^{+}\left(\frac{d}{id\rho}\right)-P_{k_2}^{+}\left(\frac{d}{id\rho}\right)\right)\left(\frac{e_{1+}(t,\rho)e_{2+}(t,\rho)}{\rho^2-\mu^2}\right)_{\rho=\rho_k}.$$

$S^1(\rho), S^2(\rho)$ *are the functions from (3.1) for the equations with the potentials* $q_1(x)$, $q_2(x)$.

Let the data $\left\{S^j(\rho), S_1^j(\rho), \rho_k(j), P_{kj}\right\}$ of the problems with the potentials $q_j(x)\,(j=1,2)$ coincide for $\operatorname{Re}\rho^2\in(-\infty,N)$:

$S^1(\rho)=S^2(\rho)$,

$S_1^1(\rho)=S_1^2(\rho)$, $\quad for\,|\operatorname{Re}\rho|<\sqrt{N+\eta^2}$, $\operatorname{Im}\rho=\eta$, $\eta<\varepsilon_0$,

$\rho_k(1)=\rho_k(2)$, $\quad P_{k_1}=P_{k_2}$.

Let's prove the following theorem.

Theorem 4.1. *If the scattering data of two problems* $q_j(x)\in V\{C_\pm(x)\}$ $(j=1,2)$ *coincide for all* $\operatorname{Re}\rho^2\in(-\infty,N)$, *then for* $\operatorname{Re}\mu^2\in(-\infty,N)$ *the following inequalities*

$$\left|e_{1-}(x,\mu)-e_{2-}(x,\mu)\right|^2<\frac{4e^{2x\operatorname{Im}\mu}\left(1+C_a^-e^{\varepsilon x}\right)^4}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}C_-(x), \tag{4.10}$$

$$\left|e_{1+}(x,\mu)-e_{2+}(x,\mu)\right|^2<\frac{4e^{-2x\operatorname{Im}\mu}\left(1+C_a^+e^{-\varepsilon x}\right)^4}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}C_+(x) \tag{4.11}$$

are correct.

Proof. Let's bring proof for $\left|e_{1-}(x,\mu)-e_{2-}(x,\mu)\right|$.

For the conditions of the theorem and from lemma 4.1 for $\operatorname{Re}\mu^2\in(-\infty,N)$

$$A_{1,2-}(\mu,x,t)=\frac{e_{1-}(x,\mu)e_{2-}(x,\mu)}{2\pi}\int\limits_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\frac{S_1^1(\rho)-S_1^2(\rho)}{\rho^2-\mu^2}e_{1-}(t,\rho)e_{2-}(t,\rho)d\rho$$

follows.

Granting the formula (2.2), (2.4) we obtain the estimation

$$\left|e_{j-}(x,\rho)\right|\le e^{x\operatorname{Im}\rho}\left(1+\int\limits_{-\infty}^{x}\left|K_{j-}(x,t)\right|dt\right)\le e^{x\operatorname{Im}\rho}\left(1+C_a^-e^{\varepsilon x}\right)$$

for the solutions $e_{j-}(x,\rho)\ (j=1,2)$.

Using this estimation, by the correlation ([3]

$$S_1^j(\rho)=O\left(\frac{1}{\rho}\right),\ |\rho|\to\infty$$

(uniformly in the strip $\left|\operatorname{Im}\rho\right|\le\eta$ for every $\eta<\varepsilon_0$) we obtain

$$\left|A_{1,2-}(\mu,x,t)\right|\le\frac{e^{2x\operatorname{Im}\mu}\left(1+C_a^-e^{\varepsilon x}\right)^2 e^{2t\operatorname{Im}\rho}\left(1+C_a^-e^{\varepsilon t}\right)^2}{2\pi}\times$$

$$\times\int\limits_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\frac{\left|S_1^1(\rho)-S_1^2(\rho)\right|}{\left|\rho^2-\mu^2\right|}d\rho<\frac{e^{2x\operatorname{Im}\mu}e^{2t\eta}\left(1+C_a^-e^{\varepsilon x}\right)^2\left(1+C_a^-e^{\varepsilon t}\right)^2}{\pi\left(N+\eta^2\right)\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)\left(1-\frac{\eta^2}{N+\eta^2}\right)}=$$

$$=\frac{e^{2x\operatorname{Im}\mu}e^{2t\eta}\left(1+C_a^-e^{\varepsilon x}\right)^2\left(1+C_a^-e^{\varepsilon t}\right)^2}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}.$$

Now the inequality (4.9) follows from (4.7)

$$|e_{1-}(\mu,x)-e_{2-}(\mu,x)|^2 \le \frac{2e^{2x\operatorname{Im}\mu}\left(1+C_a^- e^{\varepsilon x}\right)^4}{\pi N\left(1-\dfrac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}\int_{-\infty}^{x}|q_1(t)-q_2(t)|e^{2t\eta}dt <$$

$$< \frac{4e^{2x\operatorname{Im}\mu}\left(1+C_a^- e^{\varepsilon x}\right)^4}{\pi N\left(1-\dfrac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}C_-(x).$$

Theorem is proved.

Remark.

$$|e_{1-}(\mu,x)-e_{2-}(\mu,x)|^2 \le 4e^{2x\operatorname{Im}\mu}\left(1+C_a^- e^{\varepsilon x}\right)^2$$

is obtained from the representation (2.2), therefore (4.8) is nontrivial in the domain, where $C_-(x)<N$.

5. Estimation of potentials.

Now let's estimate difference of the potentials $q_1(x)-q_2(x)$ of the considered problem. For definiteness in (2.3), (2.4) we'll take $a=0$ and we'll lead the further calculations for $x\in(-\infty,0)$, since the case $x\in(0,\infty)$ is analogous to the problem considered on the positive semi-axis.

By fulfilling of the conditions of Theorem 4.1 we obtain from the formula (4.6), (4.7), (4.8)

$$\frac{1}{2}\int_{-\infty}^{x}(q_1(t)-q_2(t))dt = \frac{1}{2\pi}\int_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\left(S_1^2(\rho)-S_1^1(\rho)\right)e_{1-}(x,\rho)e_{2-}(x,\rho)d\rho. \tag{5.1}$$

Chosen by the same method that in [1] the sufficiently smooth function $g(x)$ which is equal to zero outside of the interval (x_0-h,x_0), is multiplied by (5.1) and is integrated. After the integration by parts we find

$$\frac{1}{2}\int_{x_0-h}^{x_0}(q_1(t)-q_2(t))g(t)dt = \frac{1}{2\pi}\int_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\left[S_1^1(\rho)-S_1^2(\rho)\right]\times$$
$$\times\int_{x_0-h}^{x_0}e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dt\,d\rho. \tag{5.2}$$

The following lemma is proved.

Lemma 5.1. *Let the problems* $\{q_j(x)\}\in V\{C_-(x)\}$, *potentials* $q_j(x)$ *be bounded in the interval* (x_0-h,x_0) *and*

$$Q_-(x)=\int_{-\infty}^{x}(q_1(t)+q_2(t))dt.$$

Then for any continuous differentiable function $g(x)$, which is equal to zero outside of the interval (x_0-h,x_0), the identity

$$\int_{x_0-h}^{x_0} e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dt = \int_{x_0-h}^{x_0}\{g'(t)-g(t)Q_-(t)\}e^{-2i\rho t}dt + r(\rho,x_0,h)$$

is correct, where

$$|r(\rho,x_0,h)| \le \frac{C_-^2(x_0)m^2-(\rho,x_0)}{4\rho^2}\left(3|\tilde{g}'(2\rho)|+|\tilde{g}'(-2\rho)|\right)+$$

$$+\frac{4hC_-(x_0)m_-^2(\rho,x_0)\beta_-(h,x_0)}{\rho^2}\int_{x_0-h}^{x_0}|g'(t)|dt,$$

$$m_-(\rho,x_0)=\max_{j=1,2}\left\{\sup_{-\infty<t\le x_0}\left|e_{j-}(t,\rho)\right|\right\},$$

$$\beta_-(h,x_0)=\max_{j=1,2}\left\{\sup_{x_0-h<t<x_0}\left|q_j(t)e^{\varepsilon/2|t|}\right|\right\},$$

$$\tilde{g}'(2\rho)=\int_{x_0-h}^{x_0} e^{-2i\rho t}g'(t)dt.$$

$g(x)$ is chosen by the following way ([1]). Let

$$\delta_0^-(t)=\frac{n}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^n e^{-2in\lambda t}d\lambda,\quad n>3,$$

$$\delta_-(t)=\frac{1}{h}\delta_0^-\left(-\frac{1}{2}+\frac{x_0-t}{h}\right).$$

$\delta_-(t)$ do not decrease on the interval $\left(x_0-h,x_0-\frac{h}{2}\right)$ do not increase on $\left(x_0-\frac{h}{2},x_0\right)$, have $n-2$ continuous derivatives, be negative, even, be equal to zero outside of the interval (x_0-h,x_0) and integral from it on all straight-line be equal to unit.

As the function $g(x)$ we'll take the solution of the differential equation

$$g'(x)+g(x)Q_-(x)=\delta_-'(x)+C\delta_-(x),$$

$(C-const)$ vanishing for $x\ge x_0$. For $x_0-h\le x\le x_0$ we obtain the inequality

$$|g(x)-\delta_-(x)|\le\omega(h,x_0)\delta_-(x)\frac{h}{2}e^{C_-(x_0)h},$$

$$|g'(x)-\delta_-'(x)|\le\omega(h,x_0)\delta_-(x)\left(1+hC_-(x_0)e^{hC_-(x_0)}\right),$$

$$\omega(h,x_0)=\max_{x_0-h\le x,y\le x_0}|Q_-(x)-Q_-(y)|.$$

These inequalities together with Lemma 5.1 lead to the estimation

$$\left|\int_{x_0-h}^{x_0} e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dt\right|\le\left|\int_{x_0-h}^{x_0}\{g'(t)-g(t)Q_-(t)\}e^{-2i\rho t}dt\right|+$$

$$+r(\rho,x_0,h)\le 2\left(\frac{n}{h}\right)^n|\rho|^{-n+1}\left\{1+\frac{C_-(x_0)}{|\rho|}+\frac{C_-^2(x_0)m_-^2(\rho,x_0)}{|\rho|^2}+\right.$$

$$+\frac{C_-(x_0)\beta_-(h,x_0)m_-^2(\rho,x_0)}{|\rho|^2}\left[8n+18C_-(x_0)h\left(1+C_-(x_0)he^{hC_-(x_0)}\right)\right]\Bigg\}.$$

Whence according to the identity (5.2) it follows, that

$$\frac{1}{2}\left|\int_{x_0-h}^{x_0} g(t)\{q_1(t)-q_2(t)\}dt\right|=\frac{1}{2\pi}\left|\int_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}}\left[S_1^1(\rho)-S_1^2(\rho)\right]\times\right.$$

$$\left.\times\int_{x_0-h}^{x_0} e_{1-}(t,\rho)e_{2-}(t,\rho)g'(t)dtd\rho\right|\le\frac{2}{\pi}\left(\frac{n}{h}\right)^n\frac{\left(N+\eta^2\right)^{-\frac{n-2}{2}}}{n-2}\left\{1+\frac{C_-(x_0)}{\sqrt{N+\eta^2}}+\frac{C_-^2(x_0)m_N^2(x_0)}{N+\eta^2}+\right.$$

$$\left.+\frac{2C_-(x_0)\beta(h,x_0)m_N^2(x_0)}{\pi\sqrt{N+\eta^2}}\left(4n+9C_-(x_0)h\left(1+C_-(x_0)he^{hC_-(x_0)}\right)\right)\right\},$$

$$m_N(x_0)=\sup_{\substack{|\operatorname{Re}\rho|>\sqrt{N+\eta^2}\\ \operatorname{Im}\rho=\eta}} m_-(\rho,x_0).$$

Using the last inequality the following theorem is proved.

Theorem 5.1. *If scattering data of two problems* $q_j(x)\in V\{C_\pm(x)\}$ *coincide for all* $\operatorname{Re}\rho^2\in(-\infty,N)$ *and* $N+\eta^2\ge 1$, *then on domain where*

$$\frac{5\left\{\left|\ln\left(N+\eta^2\right)\right|+1\right\}}{\sqrt{N+\eta^2}}C_\pm(x)<1$$

the inequalities

$$|q_1(x)-q_2(x)|\le\frac{2\left\{\left|\ln\left(N+\eta^2\right)\right|+3\right\}}{\sqrt{N+\eta^2}}\{38C_\pm(x)\beta_\pm(h,x)+5\gamma_\pm(h,x)\}+$$

$$+\frac{1}{\sqrt{N+\eta^2}\left\{3\left[\ln\left(N+\eta^2\right)\right]+1\right\}}.$$

are correct. Here

$$h=5\left(N+\eta^2\right)^{-\frac{1}{2}}\left\{\left|\ln\left(N+\eta^2\right)\right|+1\right\},$$

$$\gamma_+(h,x)=\max_{j=1,2}\left\{\sup_{x<t<x+h}\left|q_j'(t)\right|\right\},$$

$$\beta_+(h,x)=\max_{j=1,2}\left\{\sup_{x<t<x+h}\left|q_j(t)e^{\varepsilon/2t}\right|\right\},\quad x>0,$$

$$\gamma_-(h,x)=\max_{j=1,2}\left\{\sup_{x-h<t<x}\left|q_j'(t)\right|\right\},\quad x<0.$$

References

[1]. Marchenko V.A. *Spectral theory of the Stourm-Luiville operators.* Izd. "Naukova dumka", Kiev, 1972, p.187-207.

[2]. Fadeev L.D. Tr.MIAN SSSR, 2, 1964, p.314-336.

[3]. Blashak V.A. *The analogue the inverse problem of scattering theory for non-self-adjoint operator.* Dif.urav., v.IV, №8, p.1518-1533.

[4]. Sundin D.S., Marken S.A. *The stability of the inverse problem of the scattering theory in class of smooth potentials.* Theory of functions, funct. Analyses and their applications, M., 1971, p.14.

[5]. Marchemko V.A., Maslov K.V. *The stability of restoration problem of the Stourm-Luiville operator on spectral functions.* Mat.sbor., 1970, 81(125), 525-551.

Aslanova N.M.
Institute of Mathematics & Mechanics of NAS of Azerbaijan.
9, F.Agayeva str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20(off.).

Received September 20, 2000; Revised March 12, 2001.
Translated by Nazirova S.H.