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MATHEMATICS

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ON THE IMBEDDING AND COMPACTNESS THEOREMS FOR ANISOTROPIC WEIGHT SOBOLEV SPACES

Abstract

The imbedding and compactness theorems of anisotrop weighted Sobolev spaces to weighted Lebesque spaces are proved. In particular, to prove the comnpactness theorem the local compactness theorem and the theorem for obtaining the global compactness from local compactness are given.

The article is dedicated to the questions of continuous and compact imbedding anisotropic weight Sobolev spaces to weighted Lebesque spaces for weights of general form. P.I. Lizorkin and M. Otelbaev's articles [1] are one of the basic steps in development of the theory of imbedding theorems for Sobolev space with weights of general form. Later on these results were developed by a lot of authors and one of the last articles in this direction is a series of P. Gurka and B. Opic's articles [2]. In the mentioned articles the obtained results are based on classical non-weight imbedding theorems. The development of theory of weight estimates not based on non-weight results brought to ([3,4] and others) natural idea on corresponding development of the theory of imbedding theorems. Particularly, we can note for example Besov's articles [3,4].

In this report the imbedding and compactness theorems of anisotropic weight Sobolev space to weight Lebesque space for weights of general form are led in the case when none additional conditions are imposed upon bounded domain. The conditions on weight are formulated in terms of integrals, free from the notion of capacity [8] and pointwise relations [1,2]. Besides in the articles the relations connecting the exponents of corresponding spaces aren't given in an explicit form but are contained implicitly in conditions on weight functions.

We are to note that the results on imbedding and compactness given in the present article generalize the result of the known papers dedicated to the present themes in particular, our previous results [5, 11].

Let $\Omega \in \mathbf{R}^n$ $(n \geq 1)$ be in general an arbitrary bounded set. We'll call the weight an arbitrary measurable on Ω function $\nu(x)$ such that $0<\nu(x)<+\infty$ a.e. on Ω, and a set of all weights on Ω denote by $W(\Omega)$. By $|E|_\nu = \nu(E) = \int\limits_E \nu(x)dx$ will denote the ν weight measure of the measurable set E. For $\nu(x)=1, x \in E$ we obtain the Lebesque measure of the set E, which we'll denote by $|E|$.

For $\nu \in W(\Omega)$ and $p \geq 1$ by $L_p(\Omega;\nu)$ denote a space of the measurable on Ω functions $u:\Omega \to \mathbf{R}$, for which the norm

$$\left|u:L_p(\Omega;\nu)\right|=\begin{cases}\left|\nu^{\frac{1}{p}}u:L_p(\Omega)\right| & \text{for } 1\le p<+\infty\\ \operatorname*{ess\,sup}_{x\in\Omega}\nu(x)\left|u(x)\right| & \text{for } p=+\infty\end{cases}$$

is finite.

Let $1\le p_i<+\infty$, $\nu_i\in W(\Omega)$, $i=\overline{1,n}$. For the continuous differentiable on Ω functions $u(x)$ we introduce the designation

$$\left|u:W^1_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)\right|=\left|u:L_{p_0}(\Omega;\nu_0)\right|+\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}:L_{p_i}(\Omega;\nu_i)\right|. \tag{1}$$

The closure of a set of all continuous differentiable functions for which the expression (1) is finite, relatively to this norm we'll denote by $W^1_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)$ and call a weight Sobolev space.

The introduced space is a Banach space and for $1<p_i<+\infty$ is a reflexive space ([5]). We also note that the introduced space in general, doesn't coincide with weight Sobolev space introduced by a classical way proceeding from the finiteness of these norms ([7]).

Beforehand we consider the auxiliary statements used for obtaining the imbedding theorem.

Lemma 1. *Let $\Omega\subset\mathbf{R}^n$ be a bounded open set, $\beta>1$ be some number. Then from the family*

$$\left\{\overline{B(x,\eta(x))}:\beta\eta(x)=dist(x,\partial\Omega);x\in\Omega\right\}$$

of closed balls we can choose at most countable family $\left\{\overline{B}_j^k\right\}$ such that

(i) $\Omega=\bigcup_{k=1}^{\xi_n}\bigcup_{j\ge1}B_j^k$ *when ξ_n is a number depending only on the dimension of n;*

(ii) $\overline{B}_j^k\cap\overline{B}_i^k=\varnothing$, $i\ne j$, $k=\overline{1,\xi_n}$.

Lemma 1 is a corallary of Bezikovich theorem on covering (see, for ex. [5]).

Lemma 2. *Let $1\le q<+\infty$, $B(y,\eta)$ be an arbitrary ball in $\mathbf{R}^n$ and $\omega\in W(B(y,\eta))$. Then for any $u\in C^1(B(y,\eta))$ such $\int_{B(y,\eta)}u(x)dx=0$ the inequality*

$$\left|u:L_q(B(y,\eta);\omega)\right|\le C\sum_{k=1}^{n}\left|\int_{B(y,\eta)}\frac{\left|D_k u(z)\right|}{\left|x-z\right|^{n-1}}dz:L(B(y,\eta);\omega)\right| \tag{2}$$

is valid where the constant C depends only on dimension of n.

For proof of lemma 2 it's sufficient to use an integral representation from book [7] (Lemma 1, p.436 and lead simple estimates).

Now we reduce a main result on imbedding of the considered spaces.

Theorem 1. *Let $\Omega\subset\mathbf{R}^n$ be an arbitrary bounded open set with non-empty interior, $1\le p_0\le q<+\infty$, $1<p_i<q<+\infty$ and the following conditions hold*

1) $$\widetilde{C}_0(\Omega)=\sup_{\substack{B(y,\eta)\\ y\in\Omega,\eta=\frac{dist(y,\partial\Omega)}{12\sqrt{n}}}}\frac{1}{|B(y,\eta)|}\left|B(y,\eta)\right|_{\omega}^{\frac{1}{q}}\left|B(y,\eta)\right|_{\nu_0^{1-p_0}}^{1/p_0'}<+\infty;$$

2) $$\widetilde{C}_l(\Omega)=\sup_{\substack{B(y,\eta)\\ y\in\Omega,\eta=\frac{dist(y,\partial\Omega)}{12\sqrt{n}}}}\sup_{x\in B(y,\eta)}\sup_{0<h\le 2\eta}\left|B(x,h)\right|_{\omega}^{\frac{1}{q}}\times$$

$$\times\left(\int_{B(x,2\eta)\setminus B(x,h)}|x-z|^{(1-n)p_l'}\nu_l^{1-p_l'}(z)dz\right)^{1/p_l'}<+\infty;$$

$$\widetilde{C}_l(\Omega)=\sup_{\substack{B(y,\eta)\\ y\in\Omega,\eta=\frac{dist(y,\partial\Omega)}{12\sqrt{n}}}}\sup_{x\in B(y,\eta)}\sup_{0<h\le 2\eta}\left|B(x,h)\right|_{\nu_l^{1-p_l'}}^{\frac{1}{p_l'}}\times$$

$$\times\left(\int_{B(x,2\eta)\setminus B(x,h)}|x-z|^{(1-n)q}\omega(z)dz\right)^{1/q}<+\infty,$$

where $l=\overline{1,n}$.

Then the imbedding

$$W^1_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)\subset L_q(\Omega;\omega) \qquad (3)$$

is valid.

The proof of the theorem is led by the following scheme: the domain Ω is divided into at most countable family of closed balls with the help of lemma 1, then using lemma 2 and theorem 1 from [3] the estimates of corresponding norms obtained on every ball, and then the obtained local estimate are collected to global one using the property of a family of the covering Ω.

Now we pass to the statement of the question on the compactness of imbedding. However before to formulate the main result we cite some results which have also own independent interest.

Let $\Omega\subset\mathbf{R}^n$ be an arbitrary bounded open set with non-empty interior. For $\varepsilon>0$ assume $\Omega_\varepsilon=\{y:dist(y,\Omega)<\varepsilon\}$. It's clear that if $x\in\Omega$, then the open ball $B(x,\varepsilon)\subset\Omega_\varepsilon$.

For $u\in L_1^{loc}(\Omega_\varepsilon)$ we assume

$$u_h(x)=\frac{1}{|B(x,h)|}\int_{B(x,h)}u(y)dy \qquad x\in\overline{\Omega},\quad 0<h<\varepsilon.$$

The following holds

Theorem 2. *Let* $1\le p<+\infty$ *and* $\omega(\Omega)\cap L_1(\Omega)$. *Then if set* $K\subset L_p(\Omega;\omega)$ *satisfies the following conditions:*

1) $K\subset L_l^{loc}(\Omega_\varepsilon)$ *for some* $\varepsilon>0$;

2) K *is uniformly bounded in* $L_p(\Omega;\omega)$;

3) $K_h=\{u_h:u\in K\}$ *is uniformly bounded on* $\overline{\Omega}$ *for every* $h:0<h<\varepsilon$;

4) the functions of the set K_h *are equicontinuous on* $\overline{\Omega}$ *;*

5) $u_h \xrightarrow[h\to 0]{L_p(\Omega;\omega)} u$,

then it's relatively compact in $L_p(\Omega;\omega)$.

The proof of theorem 2 is given in [5].

The reduced theorem is nothing, but axiomatizable variant of a sufficient part of Kolmogorov compactness criterion formulated for the weight space $L_p(\Omega;\omega)$.

Theorem 3. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded measurable set and the countable systems* $\{G^m\}_{m=1}^{\infty}$ *of measurable sets such that* $G^m \subset G^{m+1} \subseteq \Omega$ *and* $\Omega = \bigcup_m G^m$ *. Then the conditions*

1) for each $m \in N$ *the compact imbedding*

$$W^1_{p_0,\dots,p_n}(\Omega;v_0,\dots,v_n) \subset\subset L_q(G^m;\omega);$$

is valid;

$$2)\ \lim_{m\to+\infty}\left(\sup_{\left\|u;W^1_{p_0,\dots,p_n}(\Omega;v_0,\dots,v_n)\right\|} \left\|u : L_q(\Omega \setminus G^m;\omega)\right\|\right) = 0$$

are necessary and sufficient for the compact imbedding

$$W^1_{p_0,\dots,p_n}(\Omega;v_0,\dots,v_n) \subset\subset L_q(\Omega;\omega)$$

to be valid.

The proof of theorem 3 is given in [5].

The sufficiency of theorem 3 generalizes the case $p_0 \neq p_1 \neq \dots \neq p_n$ and intensifies the corresponding results of P.Gurka and B. Opic's articles (see for ex. [2]). And what is more this theorem gives a compactness criterion. Assume

$$A_{q,p}(G;\omega,v) = \lim_{\eta\to 0}\sup_{x\in G}\sup_{0<h\le\eta} \left|B(x,h)\right|_{\omega}^{1/q}\left(\int_{B(x,\eta)\setminus B(x,h)} |x-y|^{(1-n)p'} v^{1-p'}(y)dy\right)^{1/p'},$$

$$A^*_{q,p}(G;\omega,v) = \lim_{\eta\to 0}\sup_{x\in G}\sup_{0<h\le\eta} \left|B(x,h)\right|_{v^{1-p'}}^{1/p'}\left(\int_{B(x,\eta)\setminus B(x,h)} |x-y|^{(1-n)q} \omega(y)dy\right)^{1/q}.$$

One of important steps for obtaining result on the compactness is the following.

Theorem 4. *(Local compactness) Let* $\Omega \subset \mathbf{R}^n$ *be a bounded open set and* $G \subset \Omega$ *such that for sufficiently small* $\varepsilon > 0$, $G_\varepsilon \subset \Omega$, $\omega, v_0, v_l \in W(\Omega)$, $1 < p_0 \le q < \infty$, $1 < p_l < q < \infty$ *and*

1) $\omega, v_0^{1-p_0'} \in L_1^{loc}(\Omega)$;

2) the conditions of theorem 1 are satisfied;

3) the relations

$$A_{q,p_l}(G_\varepsilon;\omega,v_l) = 0\,,\quad A^*_{q,p_l}(G_\varepsilon;\omega,v_l) = 0\,, \tag{4}$$

where $l = \overline{1,n}$ *are satisfied.*

Then the compact imbedding

$$W^{1}_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)\subset\subset L_q(G;\omega)$$

holds.

For proof of theorem 4 allowing for Hausdorff theorem and using theorem 2 for arbitrary $\lambda>0$ and bounded in $W^{1}_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)$ set the finite λ-net is constructed in $L_q(G;\omega)$.

The main result of the article on compactness is the following

Theorem 5. *Let* $\Omega\subset\mathbf{R}^n$, $\omega,\nu_0,\nu_l\in W(\Omega)$, $1<p_0\le q<+\infty$, $1<p_l<q<+\infty$ *and the following conditions be satisfied*

1) Ω *is bounded and* $\Omega=\bigcup\limits_{m\ge1}G^m$, *where* $G^m\subset G^{m+1}\subset\Omega$ *are open and for each* $m\in N$ *there exists a number* $\varepsilon_m>0$ *such that* $G^m_{\varepsilon_m}\subset\Omega$;

2) $\omega,\nu_0^{1-p_0'}\in L_1^{loc}(\Omega)$;

3) $\widetilde{C}(\Omega)=\max\left\{\widetilde{C}_0(\Omega);\widetilde{C}_l(\Omega);\widetilde{C}_l'(\Omega):l=\overline{1,n}\right\}<+\infty$, *where* $\widetilde{C}_0(\Omega),\widetilde{C}_l(\Omega),\widetilde{C}'(\Omega)$ *are determined in theorem 1.*

4) $\lim\limits_{m\to\infty}\widetilde{C}\left(\Omega\setminus G^m\right)=0$;

5) *the relations*

$$A_{q,p_l}\left(G^m_{\varepsilon_m};\omega,\nu_l\right)=0,\quad A^*_{q,p_l}\left(G^m_{\varepsilon_m};\omega,\nu_l\right)=0\quad\forall m\ge1,$$

where $l=\overline{1,n}$ *are satisfied.*

Then the compact imbedding

$$W^{1}_{p_0,\dots,p_n}(\Omega;\nu_0,\dots,\nu_n)\subset\subset L_q(\Omega;\omega)$$

holds.

The proof of theorem 5 is based on theorem 3 applying theorems 1 and 4.

Remark 1. As is known [8] the imbedding and compactness theorem in general doesn't hold without additional conditions on geometry of domain. Therefore the natural question arises:

Do theorems 1 and 5 contradict the known results in this field?

First of all we note that the answer to the formulated question is negative. The matter is that theorems 1 and 5 don't cover "classical" non-weight case and the case of weakly degenerate weights. The considered spaces in conditions imposed on weighted functions in large degree correspond to Sobolev spaces with zero boundary conditions. The last statement is verified by the following example ([9], remark 3.2.6, p.327), let $1<p<+\infty$ and $\Omega\subset\mathbf{R}^n$ be a bounded domain of the class C^∞, then

$$W^1_p\left(\Omega;\rho^{-p}(x),1\right)=\overset{\circ}{W}{}^1_p(\Omega).$$

On the other hand as is known by proving the imbedding theorem for Sobolev spaces with zero boundary values additional conditions on the domain Ω are not necessary. And it in turn confirms the answer of formulated question. We can see the analogy between the Sobolev spaces with strong degenerate weights and non-weight Sobolev spaces with zero boundary values by studying the question on weight boundary values of functions from Sobolev spaces with weights strong degenerating on the boundary of domain [5, 10].

Remark 2. The imbedding and compactness theorems given in paper cover only the case of bounded domains. Note that the analogous results in the case of unbounded domains which will be stated in detail in the posterior publications, are also obtained.

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ON SOLVABILITY OF A FIRST BOUNDARY VALUE PROBLEM FOR NON-UNIFORMLY DEGENERATED SECOND ORDER ELLIPTIC EQUATIONS OF NON-DIVERGENT STRUCTURE

Abstract

At the paper a first boundary value problem is considered for a class of second order elliptic equations of non-divergent structure with non-uniform degeneration. The unique simple strong (almost everywhere) solvability of this problem in corresponding Sobolev weights spaces is proved.

Introduction. Let's D be a bounded domain of n-dimensional euclidean space $\mathbf{R}_n$, $(n\geq 3)$, ∂D be its boundary where $\partial D\in \mathbf{C}^2$. Let's denote by $W^p_{2,\alpha}(D)$ a Banach space of functions $u(x)$ given on D with the norm

$$\|u\|_{W^p_{2,\alpha}(D)}=\left[\int_D\left(|u|^p+\sum_{i=1}^n(\lambda_i(x))^{p/2}|u_i|^p+\sum_{i,j=1}^n\left(\lambda_i(x)\lambda_j(x)\right)^{p/2}|u_{ij}|^p\right)dx\right]^{1/p},$$

and by $\overset{\bullet}{W}{}^p_{2,\alpha}(D)$ - the subspace of $W^p_{2,\alpha}(D)$, in which a set of all infinitely differentiable on $\overline{D}$ functions, vanishing to zero on ∂D is a dense set. Here $p\in(1,\infty)$, $\lambda_i(x)=\left(|x|_\alpha\right)^{\alpha_i}$, $|x|_\alpha=\sum_{i=1}^n|x_i|^{2/(2+\alpha_i)}$. In further we'll use everywhere the following notations:

$$u_i=\frac{\partial u(x)}{\partial x_i},\quad u_{ij}=\frac{\partial^2u(x)}{\partial x_i\partial x_j}.$$

Let $\alpha=(\alpha_1,\ldots,\alpha_n)$ be a vector, $|\alpha|=\alpha_1+\ldots+\alpha_n$ and

$$\alpha^-=\min_i\{\alpha_i\},\quad \alpha^+=\min_i\{\alpha_i\}.$$

For $R\leq 1$, $k>0$, $x^0\in\mathbf{R}_n$ we'll denote by $E_R^{x^0}(k)$ the ellipsoid

$$\left\{x:\sum_{i=1}^n\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}<(kR)^2\right\}.$$

Let's consider in D the Dirichlet problem

$$Lu=\sum_{i,j=1}^n a_{ij}(x)u_{ij}+\sum_{i=1}^n b_i(x)u_i+c(x)u(x)=f(x), \tag{1}$$

$$u|_{\partial D}=0, \tag{2}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix, $c(x)\leq 0$.

Let's suppose that $0\in D$ and for all $x\in D$ and $\xi\in\mathbf{R}_n$ the condition

$$\mu\sum_{i=1}^{n}\lambda_i(x)\xi_i^2 \le \sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j \le \mu^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2, \tag{3}$$

is fulfilled where $\mu \in (0,1]$ is constant.

We'll suppose also that the following conditions:

$$\max\left\{-\frac{2}{n-2};-2+\frac{4p}{n}\right\} < \alpha_i < 2,\ (i=1,...,n), \tag{4}$$

$$f(x)\in L_q(D), \tag{5}$$

$$c(x)\in L_{\frac{pq_1}{(q_1-p)}}(D), \tag{6}$$

$$\frac{b_i(x)}{\sqrt{\lambda_i(x)}}\in L_{q_2^+}(D),\ (i=1,...,n), \tag{7}$$

$$\widetilde{a}_{ij}(x)=\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}\in C(\overline{D}),\ (i,j=1,...,n), \tag{8}$$

are fulfilled, where

$$\frac{n(2+\alpha^+)}{2-\alpha^+}<q<\infty,\ \ 1<p<q_1<q^*=\frac{nm_1}{n-m_1},$$

$$q_2^+=\max\left\{\frac{pq_2}{q_2-p};\frac{n(6+\alpha^+)}{2-\alpha^+}\right\},\ 1<p<q_2<q^{**}=\frac{nm_2}{n-2m_2},$$

$$m_1=\begin{cases} p-\varepsilon^*, \ for\ \alpha^+\le 0;\ \varepsilon^*>0 \\ \dfrac{p(2n+|\alpha|)}{2n+|\alpha|+p\alpha^+},\ \ for\ \alpha^+>0, \end{cases} \tag{9}$$

$$m_2=\begin{cases} p-\varepsilon^{**}, \ for\ \alpha^+\le 0;\ \varepsilon^{**}>0 \\ \dfrac{p(2n+|\alpha|)}{2n+|\alpha|+p(2\alpha^+-\alpha^-)},\ \ for\ \alpha^+>0, \end{cases} \tag{10}$$

$$1<p<n/2. \tag{11}$$

Let $x'\in\partial E_R^0(1+r/2)$. Let's introduce the following notations: $E=E_R^{x'}(r/2),\ E_1=E_R^{x'}(r/4),\ A_R=E_R^0(1+r)\setminus\overline{E_R^0(1)}$. We'll assume that $r\in(0,\frac{1}{n-1}]$. By $mesD$ we'll denote Lebesgue measure of set D.

From the condition (8) it follows that there exists a non-negative, continuous and non-decreasing on $[0,diam\,D]$ function $h(t)$ such that $h(0)=0$ and

$$\left|\widetilde{a}_{ij}(x)-\widetilde{a}_{ij}(y)\right|\le h(|x-y|),\ x,y\in\overline{D};\ i,j=1,...,n,$$

and from the condition (3) it follows that the matrix $\left\|\widetilde{a}_{ij}(x)\right\|$ is uniformly positive defined and there exists the positive constant a_0 such that

$$\left|\widetilde{a}_{ij}(x)\right|\le a_0;\ i,j=1,...,n. \tag{8'}$$

Let D_ρ be a set of points $x\in D$, for which the distance from boundary of domain D is greater than $\rho>0$:

$$D_\rho = \{x : x \in D,\ dist(x, \partial D) > \rho\}.$$

As was shown in [8] if $x \in A_R$ then

$$C_1(n,\alpha)R^{\alpha_i} \le \lambda_i(x) \le C_2(n,\alpha)R^{\alpha_i};\ i = 1,...,n. \tag{12}$$

Here and in further by $C(....)$ we'll denote the positive constants only on the content of parentheses.

The aim of present paper is the proof of the unique simple strong (almost everywhere) solvability of the first boundary value problem (1)-(2) in the space $\overset{\bullet}{W}{}^{p}_{2,\alpha}(D)$ for any $f(x) \in L_q(D)$. Let's denote that in case of uniform elliptic equations the analogous result was received in papers [1]-[2], and for equations whose coefficients satisfy Cordes conditions - in [3]-[4]. Relating to non-uniform degenerated elliptic equations not containing minor coefficients and non-negative α_i, we note paper [5].

Let $L_0 = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i dx_j}$. Analogously to [5] lemmas 1,2 are proved.

Lemma 1. *Let the conditions (3), (8) and (11) be fulfilled. Then for any function $u(x) \in C_0^\infty(E)$ the estimation*

$$\int_E \sum_{i,j=1}^{n} (\lambda_i(x)\lambda_j(x))^{p/2} |u_{ij}|^p dx \le C_3(n,p,\alpha,\mu,h,a_0) \int_E |L_0 u|^p dx,$$

is true where $0 < r < r_0(n,p,\alpha,\mu,h,a_0)$ and $r_0 \le \frac{1}{n-1}$.

Not stipulating specially we'll assume $r = r_0$, and $\overline{E_R^0(1+r)} \subset D$.

Lemma 2. *Let the conditions (3), (8) and (11) be fulfilled. Then for any function $u(x) \in C_0^\infty(E)$ the estimation*

$$\|u\|_{W^p_{2,\alpha}(E)} \le C_4(n,p,\alpha,\mu,h,a_0)\|L_0 u\|_{L_p(E)}$$

is true.

1^0. Imbedding theorems.

Theorem 1. *Let the conditions (4) be fulfilled. If $1 < p < q_1 < q^*$ then $W^p_{1,\alpha}(E)$ boundly embedded in $L_{q_1}(E)$ where m_1 is defined by formula (9).*

Proof. According to [7] if $m_1 \le n$ and $1 \le q_1 < q^*$, then $W_1^{m_1}(E)$ is completely continuously embedded in $L_{q_1}(E)$:

$$\|u\|_{L_{q_1}(E)} \le C_5\|u\|_{W_1^{m_1}(E)},$$

where $C_5 = const$.

For simplicity let's restrict by the case $u(x) \in \overset{\bullet}{W}{}^{p}_{1,\alpha}(E)$. According to the Friedrichs inequality [7]:

$$\int_E |u|^{m_1} dx \le C_6 \int_E \sum_{i=1}^{n} |u_i|^{m_1} dx, \text{ where } C_6 = const.$$

Then

$$\|u\|^{m_1}_{W_1^{m_1}(E)} = \int_E \left(|u|^{m_1} + \sum_{i=1}^n |u_i|^{m_1} \right) dx \le C_7 \int_E \sum_{i=1}^n |u_i|^{m_1} dx .$$

Consequently

$$\|u\|_{L_{q_1}(E)} \le C_8 \left(\int_E \sum_{i=1}^n |u_i|^{m_1} dx \right)^{1/m_1} . \tag{13}$$

The right hand side of this inequality we'll estimate above:

$$\left(\sum_{i=1}^n \int_E |u_i|^{m_1} dx \right)^{1/m_1} = \left(\sum_{i=1}^n \int_E \lambda_i^{-p/2s}(x) \lambda_i^{p/2s}(x) |u_i|^{m_1} dx \right)^{1/m_1} \le$$

$$\le \left[\sum_{i=1}^n \left(\left(\int_E \lambda_i^{-ps'/2s}(x) dx \right)^{1/s'} \cdot \left(\int_E \lambda_i^{p/2}(x) |u_i|^p dx \right)^{1/s} \right) \right]^{1/m_1} \le$$

$$\le \left(\sum_{i=1}^n \int_E \lambda_i^{-ps'/2s}(x) dx \right)^{1/m_1 s'} \cdot \left(\sum_{i=1}^n \int_E \lambda_i^{p/2}(x) |u_i|^p dx \right)^{1/p},$$

where $s = p/m_1$, $s' = s/(s-1)$.

Since the condition (4) is satisfied and m_1, is determined by formula (9), then the first integral on the right hand side will be bounded. Indeed according to (12) and

$$mesE \le C_9 \prod_{i=1}^n R^{1+\alpha_i/2} = C_9 R^{n+|\alpha|/2}, \tag{14}$$

we have

$$\int_E \lambda_i^{-pm_1/2(p-m_1)}(x) dx \le C_{10} R^{-\frac{\alpha_i p m_1}{2(p-m_1)}} \int_E dx \le C_{11} R^{-\frac{\alpha_i p m_1}{2(p-m_1)} + \frac{|\alpha|}{2} + n} \tag{15}$$

From this it's clear that for this integral to be bounded the following must be fulfilled:

$$-\frac{\alpha_i p m_1}{2(p-m_1)} + \frac{|\alpha|}{2} + n \ge 0$$

or

$$m_1 (2n + |\alpha| + \alpha_i p) \le p(2n + |\alpha|).$$

According to (4) and (9) the last inequality is always fulfilled.

Taking into account the estimation (15) in (13) we'll get

$$\|u\|_{L_{q_1}(E)} \le C_{12} \|u\|_{W^p_{1,\alpha}(E)} . \tag{16}$$

The theorem is proved.

Theorem 2. *Let the condition (4) be fulfilled. If* $1 < p < q_2 < q^{**}$ *then*

$$\sum_{i=1}^n \left\| \sqrt{\lambda_i} u_i \right\|_{L_{q_2}(E)} \le C_{13} \|u\|_{W^p_{2,\alpha}(E)},$$

where $C_{13} = const$ *and* m_2 *is defined by the formula (10).*

Proof. According to [7], if $m_2 \le n$ and $1 \le q_2 < q^{**}$ then $W_2^{m_2}(E)$ is completely continuously embedded in $W_1^{q_2}(E)$

$$\|u\|_{W_1^{q_2}(E)} \le C_{14}\|u\|_{W_2^{m_2}(E)}.$$

Again for simplicity let's restrict by the case $u(x)\in \overset{\bullet}{W}{}^{p}_{2,\alpha}(E)$. On the right hand side of this inequality we'll twice successively apply Friedrichs inequality [7]. We have:

$$\left(\int_E\sum_{i=1}^n |u_i|^{q_2}dx\right)^{1/q_2} \le \left(\int_E\left(|u|^{q_2}+\sum_{i=1}^n |u_i|^{q_2}\right)dx\right)^{1/q_2} \le C_{15}\left(\int_E\sum_{i,j=1}^n |u_{ij}|^{m_2}dx\right)^{1/m_2}. \quad (17)$$

The left side of this inequality we'll estimate from below and the right hand side – from above:

$$\left(\int_E\sum_{i=1}^n |u_i|^{q_2}dx\right)^{1/q_2} = \left(\int_E\sum_{i=1}^n \lambda_i^{-q_2/2}(x)\lambda_i^{q_2/2}(x)|u_i|^{q_2}dx\right)^{1/q_2} \ge$$

$$\ge C_{16}R^{-\alpha^-/2}\left(\int_E\sum_{i=1}^n \lambda_i^{q_2/2}(x)|u_i|^{q_2}dx\right)^{1/q_2}, \quad (18)$$

$$\left(\sum_{i,j=1}^n\int_E |u_{ij}|^{m_2}dx\right)^{1/m_2} = \left(\sum_{i,j=1}^n\int_E(\lambda_i(x)\lambda_j(x))^{-p/2s}(\lambda_i(x)\lambda_j(x))^{p/2s}|u_{ij}|^{m_2}dx\right)^{1/m_2} \le$$

$$\le\left(\sum_{i,j=1}^n\left(\int_E(\lambda_i(x)\lambda_j(x))^{-ps'/2s}dx\right)^{1/s'}\cdot\left(\int_E(\lambda_i(x)\lambda_j(x))^{p/2}|u_{ij}|^p dx\right)^{1/s}\right)^{1/m_2} \le$$

$$\le\left(\sum_{i,j=1}^n\int_E(\lambda_i(x)\lambda_j(x))^{-ps'/2s}dx\right)^{1/m_2s'}\cdot\left(\sum_{i,j=1}^n\int_E(\lambda_i(x)\lambda_j(x))^{p/2}|u_{ij}|^p dx\right)^{1/p}, \quad (19)$$

where $s=p/m_2$.

Taking into account (18) and (19) in (17) we have

$$\left(\int_E\sum_{i=1}^n \lambda_i^{q_2/2}(x)|u_i|^{q_2}dx\right)^{1/q_2} \le C_{17}R^{\alpha^-/2}\left(\sum_{i,j=1}^n\int_E(\lambda_i(x)\lambda_j(x))^{-ps'/2s}dx\right)^{1/m_2s'} \times$$

$$\times\left(\sum_{i,j=1}^n\int_E(\lambda_i(x)\lambda_j(x))^{p/2}|u_{ij}|^p dx\right)^{1/p} \le$$

$$\le C_{18}R^{\alpha^-/2}\left(\sum_{i,j=1}^n\int_E(\lambda_i(x)\lambda_j(x))^{-ps'/2s}dx\right)^{1/m_2s'}\cdot\|u\|_{W_{2,\alpha}^p(E)}.$$

In order to complete the proof it's necessary to show that the first integral on the right hand side is bounded. According to (12) and (14)

$$R^{m_2s'\alpha^-/2}\int_E(\lambda_i(x)\lambda_j(x))^{-ps'/2s}dx \le C_{19}R^{\frac{pm_2(\alpha^- -\alpha_i-\alpha_j)}{2(p-m_2)}+\frac{|\alpha|}{2}+n}.$$

This integral will be bounded if

$$\frac{pm_2\left(\alpha^- - \alpha_i - \alpha_j\right)}{2(p-m_2)} + \frac{|\alpha|}{2} + n \geq 0$$

or

$$m_2\left[2n + |\alpha| + p\left(\alpha_i + \alpha_j - \alpha^-\right)\right] \leq p(2n + |\alpha|).$$

The last inequality is always fulfilled according to (4) and (10). Thus we receive that

$$\left[\int_E \sum_{i=1}^n \lambda_i^{q_2/2}(x)|u_i|^{q_2}\,dx\right]^{1/q_2} \leq C_{20}\|u\|_{W_{2,\alpha}^p(E)}$$

or

$$\sum_{i=1}^n \left\|\sqrt{\lambda_i}\,u_i\right\|_{L_{q_2}(E)}^{q_2} \leq C_{20}^{q_2}\|u\|_{W_{2,\alpha}^p(E)}^{q_2}. \tag{20}$$

It is easy to calculate that

$$\sum_{i=1}^n \left\|\sqrt{\lambda_i}\,u_i\right\|_{L_{q_2}(E)}^{q_2} \geq \left(\sum_{i=1}^n \left\|\sqrt{\lambda_i}\,u_i\right\|_{L_{q_2}(E)}\right)^{q_2}.$$

Take it into account in (20):

$$\sum_{i=1}^n \left\|\sqrt{\lambda_i}\,u_i\right\|_{L_{q_2}(E)} \leq C_{20} n^{(q_2-1)/q_2}\|u\|_{W_{2,\alpha}^p(E)},$$

we'll denote $C_{13} = C_{20} n^{(q_2-1)/q_2}$ and the theorem is proved.

2^0. Integral estimations.

Lemma 3. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function $u(x) \in C_0^\infty(E)$ at every $r \leq r_1(n,p,\alpha,\mu,h,a_0)$ the estimation*

$$\|u\|_{W_{2,\alpha}^p(E)} \leq C_{21}\|Lu\|_{L_p(E)}$$

is true.

Proof. According to lemma 2:

$$\|u\|_{W_{2,\alpha}^p(E)} \leq C_4\|L_0 u\|_{L_p(E)} = C_4\|L_0 u - Lu + Lu\|_{L_p(E)} \leq$$
$$\leq C_4\left(\|L_0 u - Lu\|_{L_p(E)} + \|Lu\|_{L_p(E)}\right). \tag{21}$$

We have

$$\|L_0 u - Lu\|_{L_p(E)} \leq \left\|\sum_{i=1}^n b_i u_i\right\|_{L_p(E)} + \|cu\|_{L_p(E)} \leq \sum_{i=1}^n \|b_i u_i\|_{L_p(E)} + \|cu\|_{L_p(E)}. \tag{22}$$

Further we have

$$\|cu\|_{L_p(E)} = \left(\int_E |c|^p|u|^p\,dx\right)^{1/p} \leq \left(\int_E |u|^{ps}\,dx\right)^{1/ps} \cdot \left(\int_E |c|^{ps'}\,dx\right)^{1/ps'} =$$
$$= \|u\|_{L_{q_1}(E)}\|c\|_{L_{\frac{pq_1}{q_1-p}}(E)},$$

where $q_1 = ps$.

According to (16) we have:

$$\|cu\|_{L_p(E)} \le C_{12}\|u\|_{W^p_{1,\alpha}(E)}\|c\|_{L_{\frac{pq_1}{q_1-p}}(E)}.$$

Analogously we have:

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} = \sum_{i=1}^{n}\left(\int_E |b_i u_i|^p dx\right)^{1/p} = \sum_{i=1}^{n}\left(\int_E \left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p \left|\sqrt{\lambda_i}\,u_i\right|^p dx\right)^{1/p} \le$$

$$\le \sum_{i=1}^{n}\left(\int_E \left|\frac{b_i}{\sqrt{\lambda_i}}\right|^{ps'} dx\right)^{1/ps'} \cdot \left(\int_E \left|\sqrt{\lambda_i}\,u_i\right|^{ps} dx\right)^{1/ps} =$$

$$= \sum_{i=1}^{n}\left(\left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_{\frac{pq_2}{q_2-p}}(E)} \left\|\sqrt{\lambda_i}\,u_i\right\|_{L_{q_2}(E)}\right) \le$$

$$\le \left(\sum_{i=1}^{n}\left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|^{q_2'}_{L_{\frac{pq_2}{q_2-p}}(E)}\right)^{1/q_2'} \left(\sum_{i=1}^{n}\left\|\sqrt{\lambda_i}\,u_i\right\|^{q_2}_{L_{q_2}(E)}\right)^{1/q_2},$$

where $q_2 = ps$.

Here we use (20):

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} \le C_{20}\|u\|_{W^p_{2,\alpha}(E)}\left(\sum_{i=1}^{n}\left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|^{q_2/(q_2-1)}_{L_{\frac{pq_2}{q_2-p}}(E)}\right)^{(q_2-1)/q_2}.$$

Let's choose $r_1 \le r_0$ so small, that the following

$$\|c\|_{L_{\frac{pq_1}{q_1-p}}(E)} \le \delta_1, \quad \left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_{\frac{pq_2}{q_2-p}}(E)} \le \delta_2,$$

are fulfilled where $\delta_1 > 0$ and $\delta_2 > 0$ will be chosen later. Then we have

$$\|cu\|_{L_p(E)} \le \delta_1 C_{12}\|u\|_{W^p_{1,\alpha}(E)} \le \delta_1 C_{12}\|u\|_{W^p_{2,\alpha}(E)},$$

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(E)} \le \delta_2 C_{20} n^{(q_2-1)/q_2}\|u\|_{W^p_{2,\alpha}(E)}.$$

Taking into account these estimations in (22):

$$\|L_0 u - Lu\|_{L_p(E)} \le C_{22}\|u\|_{W^p_{2,\alpha}(E)}.$$

where $C_{22} = \delta_1 C_{12} + \delta_2 C_{20} n^{(q_2-1)/q_2}$.

Finally from (21) we get:

$$\|u\|_{W^p_{2,\alpha}(E)} \le C_4 \|Lu\|_{L_p(E)} + C_4 \|L_0 u - Lu\|_{L_p(E)} \le$$

$$\le C_4 \|Lu\|_{L_p(E)} + C_4 C_{22} \|u\|_{W^p_{2,\alpha}(E)} .$$

Let's choose δ_1 and δ_2 so sufficiently small that $C_{22} \le \frac{1}{2C_4}$. Let's denote $C_{21} = C_4 / (1 - C_4 C_{22})$ and the lemma is proved.

Lemma 4. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function* $u(x) \in C^\infty(\overline{E})$ *the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C_{23} \int_E |Lu|^p dx + \varepsilon \|u\|^p_{W^p_{2,\alpha}(E)} + \frac{C_{24}}{\varepsilon R^{2p}} \int_E |u|^p dx ,$$

is true where ε *is any positive number.*

Proof. We'll introduce the function $\theta(x) \in C_0^\infty(E)$ such that

$$\theta(x) = \begin{cases} 1, & for \quad x \in E_1 \\ 0, & for \quad x \notin E_R^{x'}(3r/8), \end{cases} \qquad 0 \le \theta(x) \le 1$$

and

$$|\theta_i| \le C_{25} R^{-1-\alpha_i/2}, \quad |\theta_{ij}| \le C_{26} R^{-2-(\alpha_i+\alpha_j)/2} . \tag{23}$$

Let $\omega(x) = u(x)\theta(x)$. Then $\omega(x) \in C_0^\infty(E)$ and according to lemma 3:

$$\|u\|_{W^p_{2,\alpha}(E_1)} \le C_{21} \|L\omega\|_{L_p(E)}. \tag{24}$$

It is easy to calculate that

$$L\omega = \theta Lu + u\left(\sum_{i,j=1}^n a_{ij}\theta_{ij} + \sum_{i=1}^n b_i\theta_i\right) + 2\sum_{i,j=1}^n a_{ij}u_i\theta_j .$$

According to (8), (8'), (12) and (23) we have

$$\left|\sum_{i,j=1}^n a_{ij}\theta_{ij}\right| = \left|\sum_{i,j=1}^n \widetilde{a}_{ij}(x)\sqrt{\lambda_i(x)\lambda_j(x)}\theta_{ij}\right| \le \frac{C_{27}}{R^2},$$

$$2\left|\sum_{i,j=1}^n a_{ij}u_i\theta_j\right| \le 2\sqrt{\sum_{i,j=1}^n a_{ij}u_iu_j \sum_{i,j=1}^n a_{ij}\theta_i\theta_j} \le$$

$$\le 2\mu^{-1}\sqrt{\sum_{i=1}^n \lambda_i(x)u_i^2}\sqrt{\sum_{i=1}^n \lambda_i(x)\theta_i^2} \le \frac{C_{28}}{R}\sqrt{\sum_{i=1}^n \lambda_i(x)u_i^2} ,$$

$$\left|\sum_{i=1}^n b_i\theta_i\right| = \left|\sum_{i=1}^n \frac{b_i}{\sqrt{\lambda_i}}\sqrt{\lambda_i}\theta_i\right| \le \sum_{i=1}^n \left|\frac{b_i}{\sqrt{\lambda_i}}\right|\left|\sqrt{\lambda_i}\theta_i\right| \le \frac{C_{29}}{R}\sum_{i=1}^n \left|\frac{b_i}{\sqrt{\lambda_i}}\right| .$$

Consequently

$$|L\omega|^p \le C_{30}|Lu|^p + |u|^p\left(\frac{C_{31}}{R^{2p}} + \frac{C_{32}}{R^p}\sum_{i=1}^n\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p\right) + $$
$$+\frac{C_{33}}{R^p}\sum_{i=1}^n\left(\sqrt{\lambda_i}|u_i|\right)^p$$

or

$$\int_E |L\omega|^p dx \le C_{30}\int_E |Lu|^p dx + \frac{1}{R^{2p}}\left(C_{31}\int_E |u|^p dx + C_{32}R^p\sum_{i=1}^n\int_E |u|^p\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p dx\right) +$$
$$+\frac{C_{33}}{R^p}\sum_{i=1}^n\int_E\left(\sqrt{\lambda_i}|u_i|\right)^p dx.$$

Using interpolating inequality [9] we'll get

$$\int_{\tilde E}\sum_{i=1}^n\left|\frac{\partial\tilde u(y)}{\partial y_i}\right|^p dy \le \varepsilon_1\int_{\tilde E}\sum_{i,j=1}^n\left|\frac{\partial^2\tilde u(y)}{\partial y_i\partial y_j}\right|^p dy + \frac{C_{34}}{\varepsilon_1}\int_{\tilde E}|\tilde u|^p dy,$$

where $y_i = R^{-\alpha_i/2}x_i \ (i=1,\dots,n)$, $\tilde E$ and $\tilde u(y)$ are images of the ellipsoid E and the function $u(x)$ respectively.

Returning to the variables x we'll get:

$$\int_E\sum_{i=1}^n(\lambda_i(x))^{p/2}|u_i|^p dx \le \varepsilon_1 C_{35}\int_E\sum_{i,j=1}^n\left(\lambda_i(x)\lambda_j(x)\right)^{p/2}|u_{ij}|^p dx + \frac{C_{36}}{\varepsilon_1}\int_E|u|^p dx.$$

From here we have

$$\int_E|L\omega|^p dx \le C_{30}\int_E|Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p}\right)\int_E|u|^p dx +$$
$$+\frac{\varepsilon_1 C_{33}C_{35}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} + \frac{C_{32}}{R^p}\sum_{i=1}^n\int_E|u|^p\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^p dx \le C_{30}\int_E|Lu|^p dx +$$
$$+\left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p}\right)\int_E|u|^p dx + \frac{\varepsilon_1 C_{33}C_{35}}{R^p}\|u\|^p_{W^p_{2,\alpha}(E)} +$$
$$+\frac{C_{32}}{R^p}\sum_{i=1}^n\left(\int_E\left|\frac{b_i}{\sqrt{\lambda_i}}\right|^{ps'}dx\right)^{1/s'}\cdot\left(\int_E|u|^{ps}dx\right)^{1/s}.$$

Let $s = n/(n-p)$, then according to (16) and (7):

$$\|u\|_{L_{\frac{np}{n-p}}(E)} \le C_{12}\|u\|_{W^p_{1,\alpha}(E)},\quad \left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_n(E)} \le C_{37}.$$

Consequently,

$$\int_E |L\omega|^p dx \le C_{30} \int_E |Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p} \right) \int_E |u|^p dx +$$
$$+ \frac{\varepsilon_1 C_{33} C_{35}}{R^p} \|u\|^p_{W^p_{2,\alpha}(E)} + \frac{nC_{32}(C_{12}C_{37})^p}{R^p} \|u\|^p_{W^p_{1,\alpha}(E)}.$$

Let's apply again the interpolating inequality [9]:

$$\int_E |L\omega|^p dx \le C_{30} \int_E |Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36}}{\varepsilon_1 R^p} \right) \int_E |u|^p dx +$$
$$+ \frac{\varepsilon_1 C_{33} C_{35}}{R^p} \|u\|^p_{W^p_{2,\alpha}(E)} + \frac{\varepsilon_1 C_{38}}{R^p} \|u\|^p_{W^p_{2,\alpha}(E)} + \frac{C_{39}}{\varepsilon_1 R^p} \int_E |u|^p dx =$$
$$= C_{30} \int_E |Lu|^p dx + \left(\frac{C_{31}}{R^{2p}} + \frac{C_{33}C_{36} + C_{39}}{\varepsilon_1 R^p} \right) \int_E |u|^p dx +$$
$$+ \frac{\varepsilon_1 (C_{33}C_{35} + C_{38})}{R^p} \|u\|^p_{W^p_{2,\alpha}(E)}.$$

Take this into account in (24):

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C^p_{21} C_{30} \int_E |Lu|^p dx +$$
$$+ \left(\frac{C^p_{21} C_{31}}{R^{2p}} + \frac{(C_{33}C_{36} + C_{39}) C^p_{21}}{\varepsilon_1 R^p} \right) \int_E |u|^p dx + \frac{\varepsilon_1 C^p_{21} (C_{33}C_{35} + C_{38})}{R^p} \|u\|^p_{W^p_{2,\alpha}(E)}. \quad (25)$$

Let $\varepsilon > 0$ be an arbitrary number. Without loss of generality we assume that $\varepsilon \le 1$. Let ε_1 be so that $\varepsilon = \varepsilon_1 C^p_{21} (C_{33}C_{35} + C_{38}) / R^p$ then from (25) follows

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C^p_{21} C_{30} \int_E |Lu|^p dx + \varepsilon \|u\|^p_{W^p_{2,\alpha}(E)} +$$
$$+ \frac{C^p_{21} C_{31} + \dfrac{C^{2p}_{21} (C_{33}C_{36} + C_{39})(C_{33}C_{35} + C_{38})}{\varepsilon}}{R^{2p}} \int_E |u|^p dx.$$

Lemma has been proved.

Analogously to [5] lemmas 5-8 are proved.

Lemma 5. *Let the conditions (3), (4), (6)-(11) be satisfied. Then for any functions* $u(x) \in W^p_{2,\alpha}\left(E^0_R(1+r)\right)$ *the estimation*

$$\|u\|^p_{W^p_{2,\alpha}\left(E^0_R\left(1+\frac{r}{2}+\frac{r^2}{64}\right)\right)} \le C_{40} \int_{E^0_R(1+r)} |Lu|^p dx + \varepsilon \|u\|^p_{W^p_{2,\alpha}\left(E^0_R(1+r)\right)} + \frac{C_{41}}{\varepsilon} \left(\operatorname*{ess\,sup}_{E^0_R(1+r)} |u| \right)^p,$$

is true, where ε *is any positive number.*

Remark. Since the operator L is degenerated only at the point 0, then the present lemma is true and in the ellipsoids $E^{x''}_R\left(1+\frac{r}{2}+\frac{r^2}{64}\right)$ and $E^{x''}_R(1+r)$ if $\overline{E^{x''}_R(1+r)} \subset D$ and $E^{x''}_R(1+r) \cap E^0_R(r) = \varnothing$.

3^0. Main coercive inequality.

Lemma 6. *Let the conditions (3), (4), (6)-(8) and (11) be fulfilled. Then for any function $u(x)\in \overset{\bullet}{W}{}^{p}_{2,\alpha}(D)$ at every $\varepsilon>0$ and $\rho>0$ the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(D_\rho)}\le C_{42}\left(\int_D |Lu|^p\,dx+\varepsilon\|u\|^p_{W^p_{2,\alpha}(D)}+\frac{1}{\varepsilon}\left(\operatorname*{ess\,sup}_D|u|\right)^p\right)$$

is true.

Lemma 7. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function $u(x)\in \overset{\bullet}{W}{}^{p}_{2,\alpha}(D)$ at every $\rho>0$ the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(D\setminus D_\rho)}\le C_{43}\left(\int_D |Lu|^p\,dx+\left(\operatorname*{ess\,sup}_D|u|\right)^p\right)$$

is true.

Lemma 8. *Let the conditions (3), (4), (6)-(11) be fulfilled. Then for any function $u(x)\in \overset{\bullet}{W}{}^{p}_{2,\alpha}(D)$ the estimation*

$$\|u\|_{W^p_{2,\alpha}(D)}\le C_{44}\left(\|Lu\|_{L_p(D)}+\left(\operatorname*{ess\,sup}_D|u|\right)^p\right)$$

is true.

Theorem 3. *Let the conditions (3)-(11) be fulfilled. Then for any function $u(x)\in \overset{\bullet}{W}{}^{p}_{2,\alpha}(D)$ the estimation*

$$\|u\|_{W^p_{2,\alpha}(D)}\le C_{45}\|Lu\|_{L_q(D)} \tag{26}$$

is true.

Proof. According to our assumption $c(x)\le 0$ is true, then according to the estimation of A.D.Aleksandrov [6]:

$$\operatorname*{ess\,sup}_D|u|\le C_{46}\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\cdot F_n\left(\left\|\frac{b}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\right). \tag{27}$$

It's evident that

$$\det(a_{ij})\ge C_{47}\prod_{i=1}^n\lambda_i(x)=C_{47}\prod_{i=1}^n\left(|x|_\alpha\right)^{\alpha_i}\ge C_{47}\prod_{i=1}^n|x_i|^{2\alpha_i/(2+\alpha_i)}.$$

From here we have:

$$\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}=\left(\int_D\frac{|f|^n}{\det(a_{ij})}dx\right)^{1/n}\le$$

$$\le\left(\int_D|f|^{ns}dx\right)^{1/ns}\cdot\left(\int_D\frac{|f|^n}{\left(\det(a_{ij})\right)^{s'}}dx\right)^{1/ns'}.$$

Let $q=ns$, then

$$\left\| \frac{f}{\sqrt[n]{\det(a_{ij})}} \right\|_{L_n(D)} \leq \|f\|_{L_q(D)} \cdot \left(\int_D \prod_{i=1}^{n} |x_i|^{-\frac{2\alpha_i}{(2+\alpha_i)} \cdot \frac{q}{(q-n)}} \right)^{\frac{(q-n)}{nq}} .$$

Since $q > n(2+\alpha^+)/(2-\alpha^+)$ then integral in right side of the last inequality is finite.

Now let's prove, that the second multiplier in the right hand side of (27) is finite. It's evident that

$$\left\| \frac{b}{\sqrt[n]{\det(a_{ij})}} \right\|_{L_n(D)} = \left\| \sqrt{\sum_{i=1}^{n} \left(\frac{b_i}{\sqrt[n]{\det(a_{ij})}} \right)^2} \right\|_{L_n(D)} = \left[\int_D \left(\sqrt{\sum_{i=1}^{n} \left(\frac{b_i}{\sqrt[n]{\det(a_{ij})}} \right)^2} \right)^n dx \right]^{1/n} \leq$$

$$\leq C_{48} \left(\sum_{i=1}^{n} \int_D \frac{|b_i|^n}{\det(a_{ij})} dx \right)^{1/n} .$$

From here we have

$$\int_D \frac{|b_i|^n}{\det(a_{ij})} dx = \int_D \left| \frac{b_i}{\sqrt{\lambda_i}} \right|^n \cdot \frac{\left(\sqrt{\lambda_i}\right)^n}{\det(a_{ij})} dx \leq \left(\int_D \left| \frac{b_i}{\sqrt{\lambda_i}} \right|^{ns_0} dx \right)^{1/s_0} \cdot \left(\int_D \frac{\left(\sqrt{\lambda_i}\right)^{ns_0'}}{\left(\det(a_{ij})\right)^{s_0'}} dx \right)^{1/s_0'} \leq$$

$$\leq C_{49} \left\| \frac{b_i}{\sqrt{\lambda_i}} \right\|_{L_{ns_0}(D)}^n \cdot \left(\int_D \frac{\left(|x|_\alpha\right)^{\alpha_i n s_0'/2}}{\prod_{j=1}^{n} \left(|x|_\alpha\right)^{\alpha_j s_0'}} dx \right)^{1/s_0'} .$$

Let $s_0 = (6+\alpha^+)/(2-\alpha^+)$. Then according to (7) the first multiplier and according to (4) the second multiplier in the right hand side of the last inequality will be finite.

Thus,

$$\operatorname*{ess\,sup}_D |u| \leq C_{50} \|Lu\|_{L_q(D)} . \tag{28}$$

On the other hand it's known that

$$\|Lu\|_{L_p(D)} \leq (mesD)^{(q-p)/qp} \|Lu\|_{L_q(D)} .$$

We'll take into account (28) and the last inequality in the lemma 8:

$$\|u\|_{W_{2,\alpha}^p(D)} \leq C_{44} \left[(mesD)^{(q-p)/qp} \|Lu\|_{L_q(D)} + C_{50} \|Lu\|_{L_q(D)} \right] = C_{45} \|Lu\|_{L_q(D)} ,$$

where $C_{45} = C_{44} \left[(mesD)^{(q-p)/qp} + C_{50} \right]$.

The theorem is proved.

4^0. The Strong Solvability of the first boundary value problem.

Theorem. *Let the conditions (3)-(11) be fulfilled. Then the problem (1)-(2) is uniquely strong solvable in* $\overset{\bullet}{W}{}_{2,\alpha}^{p}(D)$. *And for the solution* $u(x)$ *of the problem (1)-(2) the estimation*

$$\|u\|_{W^p_{2,\alpha}(D)} \le C_{45} \|f\|_{L_q(D)}$$

is true.

The proof of the theorem is conducted by the scheme, used at the paper [5], subject to the coercive estimation (26).

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