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NECESSARY CONDITION OF BASISITY OF POWER SYSTEM IN L_p

Abstract

At the paper the system of power is considered $\{A(t)\varphi^n(t);\, B(t)\overline{\varphi}^k(t)\},\ n,k \ge 0$, *where* $A(t)$, $B(t)$ *and* $\varphi(t)$ *are complex valued functions on* $[a,b]$; $\overline{\varphi}$ *is a complex conjugation. It is proved that at definite conditions on the functions* $A(t)$, $B(t)$ *and* $\varphi(t)$ *necessary conditions of basisity of this system in* L_p, $1 < p < +\infty$ *is* $|\varphi(t)| \equiv const$.

Let's consider the following "double of powers"

$$\{A(t)\varphi^n(t); B(t)\overline{\varphi}^k(t)\},\quad n,k \ge 0\,, \tag{1}$$

$A(t), B(t), \varphi(t)$ are complex-valued functions on $[a,b]$, $\overline{\varphi}$ is a complex conjugation. The necessary and sufficient condition of completeness and minimality of the system (1) in L_p was obtained in paper [1]. The basic properties of the system (1) in L_p was completely investigated in paper [2], in case when $\varphi(t) \equiv e^{it}$. It appears, this case for basisity is exclusive.

So, let in some Banach space B "double" system

$$\{x_n^+, x_k^-\},\ n,k \ge 0 \tag{2}$$

be given.

Definition. *The system (2) makes up a basis in* B, *if for* $\forall x \in B$ *there exists a unique sequence of numbers* $\{a_n^+, a_k^-\}$, $n,k \ge 0$, *such that*

$$\lim_{N^+,N^- \to \infty} \left\| \sum_0^{N^+} a_n^+ x_n^+ + \sum_0^{N^-} a_n^- x_n^- - x \right\| = 0\,,$$

where $\|\cdot\|$ *is a norm in* B.

So, let the functions $A(t)$, $B(t)$ and $\varphi(t)$ satisfy the following conditions:

1) $A(t)$, $B(t)$ and $\varphi'(t)$ are measurable on (a,b), moreover
$$\sup_{[a,b]} \mathrm{rvai} \{|A(t)|^{\pm 1}; |B(t)|^{\pm 1}; |\varphi'(t)|^{\pm}\} \le M < +\infty\,;$$
2) $\Gamma = \varphi\{[a,b]\}$ is a closed $(\varphi(a) = \varphi(b))$ piecewise smooth simple $[0 \in \operatorname{int}\Gamma]$ Radon's curve or simple Lyapunov contour;
3) $\alpha(t) \equiv \arg A(t)$, $\beta(t) \equiv \arg B(t)$ are continuous functions with bounded variation on $[a,b]$.

For definiteness we'll assume that when the point $\varphi = \varphi(t)$ by increasing t runs along the curve Γ, the internal domain $\operatorname{int}\Gamma$ remains at the left.

Theorem. *Let the functions* $A(t)$, $B(t)$ *and* $\varphi(t)$ *satisfy the conditions 1)-3). If the system (1) makes up a basis in* L_p, $1 < p < \infty$, *then* $|\varphi(t)| \equiv const$.

Proof. Let's suppose the contrary. Let

$$R = \max_{[a,b]} |\varphi(t)| > \min_{[a,b]} |\varphi(t)| = r\,.$$

Since the system (1) make up a basis in $L_p(a,b)$, then $\forall f \in L_p$ it holds the following biorthogonal expansion:

$$f(t) = A(t)\sum_{n=0}^{\infty} a_n \varphi^n(t) + B(t)\sum_{n=0}^{\infty} b_n \overline{\varphi}^n(t).$$

Let's denote

$$f^+(t) = \sum_{n=0}^{\infty} a_n \varphi^n(t). \qquad (3)$$

From the basisity and from condition 1) it follows that $f^+ \in L_p(a,b)$. Let's consider the following power series:

$$F(z) \equiv \sum_{n=0}^{\infty} a_n z^n.$$

The radius of convergence of this series we'll denote by R_0. Let's show that $R_0 \geq R$. Let $R_0 < R$. Since, $|\varphi(t)|$ is continuous on $[a,b]$, then it is evident that $\exists t_0 \in [a,b]$, for which $R = |\varphi(t_0)|$. Consequently, $\exists \delta$ is the neighborhood $\overline{G}_\delta(t_0) = [t_0 - \delta\ t_0 + \delta]$ (at $t_0 = a$ or $t_0 = b$ one-sided neighborhood) of point t_0, such that for $\forall t \in \overline{G}_\delta(t_0)$ we have: $R_0 < |\varphi(t)| \leq R$, i.e. $\min\limits_{G_\delta(t_0)} |\varphi(t)| = r_\delta > R_0$. From the convergence of the series (3) in L_p it follows that

$$\left\| a_n \varphi^n(t) \right\| L_p \to 0, \text{ for } n \to \infty, \text{ where } \|f\|_{L_p}^p = \int_a^b |f(t)|^p dt.$$

As a result $\left\| a_n \varphi^n(t) \right\|_{L_p(G_\delta(t_0))} \to 0,\ n \to \infty$.

As known, $r_0 = \dfrac{1}{\overline{\lim\limits_n} \sqrt[n]{|a_n|}}$. So, there exists the sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, $n_k \to \infty$ for which $R_0 = \left[\lim\limits_{k\to\infty} \sqrt[n_k]{|a_{n_k}|}\right]^{-1}$. Since $r_\delta > R_0$ then for sufficiently large $k: r_\delta > \dfrac{1}{\sqrt[n_k]{|a_{n_k}|}}$ as a result $\left|\varphi(t)\sqrt[n_k]{|a_{n_k}|}\right| > 1$, $\forall t \in G_\delta(t_0)$, i.e. $\left|a_{n_k} \cdot \varphi^{n_k}(t)\right| > 1$, $\forall t \in G_\delta(t_0)$.

Thus $\left\| a_{n_k} \cdot \varphi^{n_k}(t) \right\|_{L_p(G_\delta(t_0))} \geq 2\delta > 0$, for sufficiently large k. We get the contradiction. Then $R_0 \geq R$.

Whence it follows that for $\forall f \in L_p(a,b)$ the function $F(z)$ is analytic in the circle $C_R(0) = \{z \in C \mid |z| < R\}$. Since $r < R$, then $\exists \delta_0$ is a neighbourhood of some point $\tau \in (a,b)$, such that $|\varphi(t)| < R$ for $\forall t \in \overline{D}_{\delta_0}(\tau) \equiv [\tau - \delta_0, \tau + \delta_0]$. It is evident that $F^+[\varphi(t)] = f^+(t)$ almost everywhere on $D_{\delta_0}(\tau)$.

From the conditions 1), 2) it follows that the series $\sum\limits_{n=0}^{\infty} a_n \xi^n$ converges in $L_p(\Gamma)$.

Consequently

$$\int_\Gamma \sum_{n=0}^{\infty} a_n \xi^n \cdot \xi^k d\xi = \sum_{n=0}^{\infty} a_n \int_\Gamma \xi^{n+k} d\xi = 0, \quad \forall k \geq 0.$$

From this equation and Smirnov's [3, s.424] theorem it follows that there exists the function $F_1(z)$ from the class of Smirnov $E_1(\operatorname{int}\Gamma)$, for which $F_1^+(\xi)=\sum_{n=0}^{\infty}a_n\xi^n$, almost everywhere on Γ. By the condition 2) $\operatorname{int}\Gamma$ belongs to the class C of Smirnov's [4, s.90] domain. Since $F_1^+(\xi)\in L_p(\Gamma)$ then again by the Smirnov's theorem [4, 92] the function $F_1(z)$ belongs to the class $E_p(\operatorname{int}\Gamma)$. From the previous consideration it follows that $F_1^+[\varphi(t)]=F^+[\varphi(t)]$ almost everywhere on $D_{\delta_0}(\tau)$. Then from the uniqueness theorem of Privalov [3, s.413] it follows that $F_1(z)\equiv F(z)$ in $\operatorname{int}\Gamma$. Consequently $F(z)\in E_p(\operatorname{int}\Gamma)$ and $F^+[\varphi(t)]=f^+(t)$ almost everywhere on $[a,b]$. It is proved analogously that the function $\Phi(z)\equiv\sum_{n=0}^{\infty}\overline{b}_n z^n$ belongs to the class of Smirnov $E_p(\operatorname{int}\Gamma)$ and $\overline{\Phi}[\varphi(t)]=\sum_{n=0}^{\infty}b_n\overline{\varphi}^n(t)$ almost everywhere on $[a,b]$.

As a result we get that the function $F(z)$ and $\Phi(z)$ are the solutions of the following conjugate problem in Smirnov class $E_p(\operatorname{int}\Gamma)$:

$$A(t)F^+[\varphi(t)]+B(t)\overline{\Phi}^+[\varphi(t)]=f(t) \text{ almost everywhere on } [a,b]. \tag{4}$$

Let's consider the system:

$$\left\{\widetilde{A}(t)\varphi^n(t);\ \widetilde{B}(t)\overline{\varphi}^n(t)\right\}_{n\geq 0}. \tag{5}$$

where $\widetilde{A}(t)\equiv A(t)\cdot[\overline{\varphi}'(t)]^{-1}$; $\widetilde{B}(t)\equiv B(t)\cdot[\varphi'(t)]^{-1}$, so,

$$\widetilde{\alpha}(t)\equiv\arg\widetilde{A}(t)=\alpha(t)+\arg\varphi'(t) \text{ and } \widetilde{\beta}(t)\equiv\arg\widetilde{B}(t)=\beta(t)-\arg\varphi'(t).$$

It is easy to show that the system (5) is complete in $L_q(a,b)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ iff the conjugate problem

$$\widetilde{B}(t)\cdot\varphi'(t)\cdot\Phi_1^+[\varphi(t)]+\widetilde{A}(t)\cdot\overline{\varphi}'(t)\overline{\Phi}_2^+[\varphi(t)]=0 \text{ almost everywhere on } [a,b]$$

has only a trivial solution in $E_p(\operatorname{int}\Gamma)$. So we get that if the system (5) is complete in $L_q(a,b)$ then homogeneous problem of conjugation (4) has only a trivial solution in the class $E_p(\operatorname{int}\Gamma)$.

According to paper [1], let's find the value $\widetilde{\omega}$ for the system (5):

$$\widetilde{\omega}=\frac{1}{2\pi}\left[\widetilde{\beta}(a)-\widetilde{\beta}(b)+\widetilde{\alpha}(b)-\widetilde{\alpha}(a)+\frac{2}{p}(\arg\varphi'(a)-\arg\varphi'(b))\right]+\frac{2}{p}-1.$$

Since $\arg\varphi'(b)-\arg\varphi'(a)=2\pi$ we get:

$$\widetilde{\omega}=-\gamma+1\ ,$$

where $\gamma=\frac{1}{2\pi}[\alpha(b)-\alpha(a)-\beta(b)+\beta(a)]$.

By the results of [1] the system (5) is complete in L_q iff $\widetilde{\omega}\leq\frac{1}{q}$. Again by the results of paper [1] the system (1) is minimal in $L_q(a,b)$ if $\omega=\gamma-1>-\frac{1}{q}$ i.e. $\widetilde{\omega}<\frac{1}{q}$. As a result the system (5) is complete in $L_q(a,b)$, so, the homogeneous conjugation problem

From this equation and Smirnov's [3, s.424] theorem it follows that there exists the function $F_1(z)$ from the class of Smirnov $E_1(\operatorname{int}\Gamma)$, for which $F_1^+(\xi)=\sum_{n=0}^{\infty}a_n\xi^n$, almost everywhere on Γ. By the condition 2) $\operatorname{int}\Gamma$ belongs to the class C of Smirnov's [4, s.90] domain. Since $F_1^+(\xi)\in L_p(\Gamma)$ then again by the Smirnov's theorem [4, 92] the function $F_1(z)$ belongs to the class $E_p(\operatorname{int}\Gamma)$. From the previous consideration it follows that $F_1^+[\varphi(t)]=F^+[\varphi(t)]$ almost everywhere on $D_{\delta_0}(\tau)$. Then from the uniqueness theorem of Privalov [3, s.413] it follows that $F_1(z)\equiv F(z)$ in $\operatorname{int}\Gamma$. Consequently $F(z)\in E_p(\operatorname{int}\Gamma)$ and $F^+[\varphi(t)]=f^+(t)$ almost everywhere on $[a,b]$. It is proved analogously that the function $\Phi(z)\equiv\sum_{n=0}^{\infty}\overline{b}_n z^n$ belongs to the class of Smirnov $E_p(\operatorname{int}\Gamma)$ and $\overline{\Phi}[\varphi(t)]=\sum_{n=0}^{\infty}b_n\overline{\varphi}^n(t)$ almost everywhere on $[a,b]$.

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has only a trivial solution in $E_p(\operatorname{int}\Gamma)$. So we get that if the system (5) is complete in $L_q(a,b)$ then homogeneous problem of conjugation (4) has only a trivial solution in the class $E_p(\operatorname{int}\Gamma)$.

According to paper [1], let's find the value $\widetilde{\omega}$ for the system (5):

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Since $\arg\varphi'(b)-\arg\varphi'(a)=2\pi$ we get:

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where $\gamma=\frac{1}{2\pi}[\alpha(b)-\alpha(a)-\beta(b)+\beta(a)]$.

By the results of [1] the system (5) is complete in L_q iff $\widetilde{\omega}\leq\frac{1}{q}$. Again by the results of paper [1] the system (1) is minimal in $L_q(a,b)$ if $\omega=\gamma-1>-\frac{1}{q}$ i.e. $\widetilde{\omega}<\frac{1}{q}$. As a result the system (5) is complete in $L_q(a,b)$, so, the homogeneous conjugation problem

$$A(t)F^+[\varphi(t)]+B(t)\overline{\Phi}^+[\varphi(t)]=0, \text{ almost everywhere on } [a,b]$$

has only a trivial solution in $E_p(\mathrm{int}\Gamma)$. Thus, the problem (4) is uniquely solvable in $E_p(\mathrm{int}\Gamma)$.

Since $r<R$ then it's evident that $\exists z_0 \in ext\Gamma$ such that $|z_0|<R$. Let's consider the function

$$f(t)=\frac{A(t)}{\varphi(t)-z_0}.$$

It is evident that the function $f_0(z)\equiv\dfrac{1}{z-z_0}$ belongs to the class $E_p(\mathrm{int}\Gamma)$ and moreover $A(t)f_0^+[\varphi(t)]=f(t)$ on $[a,b]$.

Comparing this problem with (4) from the uniqueness we receive that $F(z)\equiv f_0(z)$, $\Phi(z)\equiv 0$ in $\mathrm{int}\Gamma$. So $F(z)$ is an analytic continuation of the function $f_0(z)$ from the domain $\mathrm{int}\Gamma$ in $C_R(0)\setminus\mathrm{int}\Gamma$. But from the uniqueness of analytic continuation it follows that it's impossible, because $z_0\in C_R$ is a pole of the function $f_0(z)$ in $C_R(0)$. The theorem is proved.

Let's formulate the following easily provable lemma.

Lemma: *The system*

$$\{\mathrm{Re}[A(t)\varphi^n(t)];\ \mathrm{Im}[A(t)\varphi^n(t)]\}_{n\geq 0} \qquad (6)$$

makes up a basis in $L_p^R(a,b)$, $p\geq 1$ *iff the system*

$$\{A(t)\varphi^n(t);\ \overline{A}(t)\overline{\varphi}^n(t)\}_{n\geq 0}$$

makes up a basis in $L_p(a,b)$.

The corollary follows from this lemma.

Corollary: *Let the function* $A(t)$ *and* $\varphi(t)$ *satisfy the conditions 1)-3). If the system (6) makes up a basis in* $L_p^R(a,b)$, $1<p<+\infty$ *then* $|\varphi(t)|\equiv const$.

Note. The theorem can be proved in common assumptions with respect to the function $A(t)$, $B(t)$ and $\varphi(t)$.

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DZABRAILOV A.D., MAYOROVA I.V.

THE PROPERTIES OF TRACES OF FUNCTIONS ON THE BOUNDARY OF A SURFACE

Abstract

At the paper the properties of L_p - traces of functions of many variables defined on domain $G \subset E_n$ satisfying "the condition of σ-semi-horn" are considered. The estimations of norms of traces of functions by norm of known spaces $B^{<r>}_{p,\theta}(G;s)$ of the generalizing spaces S.M.Nikolsky-O.V.Besov $(s=1)$ and the B-spaces functions with dominate mixed derivatives are proved.

Introduction. At the paper the definitions of the known spaces

$$B^{<r>}_{p,\theta}(G;s), \tag{1}$$

of the functions $f = f(x)$ of the points $x = (x_1;...;x_s) \in E_n = E_{n_1} \times ... E_{n_s}$ of many groups of variables $x_k = (x_{k,1};...;x_{k,n_k}) \in E_{n_k}$ $(k = 1,2,...,s)$, $(n_1 + n_2 + . + n_s = n)$ defined on domain $G \subset E_n$, which satisfies "the condition σ-semi-horn" ([1]) are cited.

A class of surfaces

$$\Gamma_m \in \Pi^1 \tag{2}$$

of dimension $m = m_1 + m_2 + ... + m_\alpha < n = n_1 + n_2 + ... + n_s$ $(1 \le \alpha \le s \le n)$, where $1 \le m_k \le n_k$ $(k = 1,2,...,\alpha)$ is introduced the notation of L_p traces on the surface Γ_m is given, in case when this surface Γ_m is on the boundary ∂G of domain

$$G \in C(\sigma; H) \tag{3}$$

and on the base of new integral representations given in monography [1], the estimations of L_p-traces of functions and their corresponding derivatives on the surface Γ_m by norms of given spaces (1) is proved. Thus it is necessary to note that these spaces in case $s = 1$ are the generalizations of corresponding spaces

$$B^{<r>}_{p,\theta}(G;s),$$

of S.M.Nikolsky-O,V.Besov $(1 < p \le \theta < \infty,\ \theta = \infty)$ and in case $s = n$ generalizations of the known spaces $S^r_{p,\theta}B(G)$ - S.M.Nikolsky $(\theta = \infty)$ - A.D.Dzabrailov $(p = \theta)$ - I.T.Amanov $(1 < p \le \theta < \infty)$.

1.Main definitions and notations.

1.1. Spaces. Let

$$r = (r_1;...;r_s) \tag{1.1}$$

be a "positive vector" with coordinate vectors $r_k = (r_{k,1};...;r_{k,n_k})$ $(k = 1,2,..,s)$.

Let

$$\bar{r} = (\bar{r}_1;...;\bar{r}_s) \tag{1.2}$$

be an "integer non-negative vector" such that $\bar{r}_{k,j}$ is the greatest integer number smaller than $r_{k,j}$ at $j = 1,2,...,n_k$ for all $k = 1,2,...,s$.