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THE PROPERTIES OF TRACES OF FUNCTIONS ON THE BOUNDARY OF A SURFACE

Abstract

At the paper the properties of L_p - traces of functions of many variables defined on domain $G \subset E_n$ satisfying "the condition of σ-semi-horn" are considered. The estimations of norms of traces of functions by norm of known spaces $B^{<r>}_{p,\theta}(G;s)$ of the generalizing spaces S.M.Nikolsky-O.V.Besov $(s=1)$ and the B-spaces functions with dominate mixed derivatives are proved.

Introduction. At the paper the definitions of the known spaces

$$B^{<r>}_{p,\theta}(G;s), \tag{1}$$

of the functions $f = f(x)$ of the points $x = (x_1;...;x_s) \in E_n = E_{n_1} \times ... E_{n_s}$ of many groups of variables $x_k = (x_{k,1};...;x_{k,n_k}) \in E_{n_k}$ $(k = 1,2,...,s)$, $(n_1 + n_2 + .+ n_s = n)$ defined on domain $G \subset E_n$, which satisfies "the condition σ-semi-horn" ([1]) are cited.

A class of surfaces

$$\Gamma_m \in \Pi^1 \tag{2}$$

of dimension $m = m_1 + m_2 + ... + m_\alpha < n = n_1 + n_2 + ... + n_s$ $(1 \le \alpha \le s \le n)$, where $1 \le m_k \le n_k$ $(k = 1,2,...,\alpha)$ is introduced the notation of L_p traces on the surface Γ_m is given, in case when this surface Γ_m is on the boundary ∂G of domain

$$G \in C(\sigma; H) \tag{3}$$

and on the base of new integral representations given in monography [1], the estimations of L_p-traces of functions and their corresponding derivatives on the surface Γ_m by norms of given spaces (1) is proved. Thus it is necessary to note that these spaces in case $s = 1$ are the generalizations of corresponding spaces

$$B^{<r>}_{p,\theta}(G;s),$$

of S.M.Nikolsky-O,V.Besov $(1 < p \le \theta < \infty,\ \theta = \infty)$ and in case $s = n$ generalizations of the known spaces $S^r_{p,\theta}B(G)$ - S.M.Nikolsky $(\theta = \infty)$ - A.D.Dzabrailov $(p = \theta)$ - I.T.Amanov $(1 < p \le \theta < \infty)$.

1.Main definitions and notations.

1.1. Spaces. Let

$$r = (r_1;...;r_s) \tag{1.1}$$

be a "positive vector" with coordinate vectors $r_k = (r_{k,1};...;r_{k,n_k})$ $(k = 1,2,..,s)$.

Let

$$\bar{r} = (\bar{r}_1;...;\bar{r}_s) \tag{1.2}$$

be an "integer non-negative vector" such that $\bar{r}_{k,j}$ is the greatest integer number smaller than $r_{k,j}$ at $j = 1,2,...,n_k$ for all $k = 1,2,...,s$.

Then

$$0 < r_{k,j} - \bar{r}_{k,j} \le 1 \quad (j = 1,2,...,n_k) \tag{1.3}$$

for all $k = 1,2,...,s$.

Let now Q be a set of all possible vectors $i = (i_1,...,i_s)$ with coordinates $i_k \in \{0,1,2,...,n_k\}$ $(k = 1,2,...,s)$. The amount of elements of the set Q equals

$$|Q| = \prod_{k=1}^{s}(1 + n_k), \tag{1.4}$$

whence it follows that

$$|Q| = n + 1 \ \text{ for } \ s = 1, \tag{1.5}$$

$$|Q| = 2^n \ \text{ for } \ s = n, \tag{1.6}$$

and in general

$$(n+1) \le |Q| \le 2^n. \tag{1.7}$$

For every $i = (i_1,...,i_s) \in Q$ corresponds the vector $r^i = (r_1^{i_1},...,r_s^{i_s})$ with the coordinate-vectors $r_k^{i_k} = (0,...,0,r_{k,j_k},0,...,0)$ if $i_k \ne 0$; $r_k^0 = (0,...,0,0,0,...,0)$ if $i_k = 0$. Analogously for every vector $i = (i_1,...,i_s) \in Q$ we put the vector $\bar{r}^i = (\bar{r}_1^{i_1},...,\bar{r}_s^{i_s})$ with the coordinate-vectors $\bar{r}_k^{i_k} = (0,...,0,\bar{r}_{k,j_k},0,...,0)$ if $i_k \ne 0$; $\bar{r}_k^0 = (0,...,0,0,0,...,0)$ if $i_k = 0$.

Thus the set of vectors $i \in Q$ determines the set of vectors

$$r^i \ (i \in Q), \ \ \bar{r}^i \ (i \in Q). \tag{1.8}$$

Let's introduce the polynorm by the equality (see [1-3])

$$\|f\|_{L_{p,\theta}^{\langle \omega^i \rangle}(G;s)} = \left(\int_{E_{|\omega^i|}} \left\| \frac{\Delta^{2\omega^i}(t;G) D^{\bar{r}^i} f}{\psi_i(t)} \right\|_{p,G}^{\theta} \frac{dt}{t} \right)^{\frac{1}{\theta}} \tag{1.9}$$

at $1 \le \theta < \infty$, and in case $\theta = \infty$

$$\|f\|_{L_{p,\infty}^{\langle \omega^i \rangle}(G;s)} = \operatorname*{ess\,sup}_{t \in E_{|\omega^i|}} \left\| \frac{\Delta^{2\omega^i}(t;G) D^{\bar{r}^i} f}{\psi_i(t)} \right\|_{p,G}, \tag{1.10}$$

where $1 \le p \le \infty$, $e^i = \operatorname{supp} i$ is a set of indices different from zero coordinates of vector $i = (i_1,...,i_s)$,

$$\frac{dt}{t} = \prod_{k \in e^i} \frac{dt_{k,j_k}}{t_{k,j_k}}, \tag{1.11}$$

$$\psi_i(t) = \prod_{k \in e^i} |t_{k,j_k}|^{(r_{k,j_k} - \bar{r}_{k,j_k})}, \tag{1.12}$$

$$E_{|\omega^i|} = \prod_{k \in e^i} \{t_{k,j_k} \in E_1\}, \tag{1.13}$$

and $\omega^i = (\omega_1^{i_1},...,\omega_s^{i_s})$ is a vector, whose coordinates vectors $\omega_k^{i_k}$ $(k = 1,2,..,s)$ are determined by the equality $\omega_k^{i_k}$ $(0,...,0,1,0,...,0)$ at $i_k \ne 0$; $w_k^0 = (0,...,0,0,0,...,0)$ at $i_k = 0$ for all $k = 1,2,...,s$.

Let's denote that

$$\Delta^{2\omega^i}(t;G) D^{\bar{r}^i} f = \Delta^{2\omega^i}(t) D^{\bar{r}^i} f, \tag{1.14}$$

if the mixed difference of the derivative $D^{r^j} f$ is constructed by the vertex of polyhedron wholly lying on the domain G and in opposite case

$$\Delta^{2\omega^j}(t;G)D^{r^j} f = 0, \tag{1.15}$$

is supposed.

Definition 1.1. *The closure of a set of sufficiently smooth finite in E_n functions by the norm*

$$\|f\|_{B^{<r>}_{p,\beta}(G;s)} = \sum_{i=(i_1,\dots,i_s)\in Q} \|f\|_{L^{<r^i>}_{p,\beta}(G;s)}. \tag{1.16}$$

is called a space $B^{<r>}_{p\beta}(G;s)$.

1.2. The class of domains satisfying "the condition σ -semi horn".

Definition 1.2. *Subdomain*

$$\Omega \subset G \tag{1.17}$$

is called the subdomain satisfying "the condition of σ semi-horn" if

$$x + R_\delta(\sigma;H) \subset G \tag{1.18}$$

for all $x \in \Omega$.

Here $\sigma = (\sigma_1;\dots;\sigma_s)$ is a positive vector with coordinate vectors $\sigma_k = (\sigma_{k,1},\dots,\sigma_{k,n_k})$ $(k=1,2,\dots,s)$, i.e. $\sigma_{k,j} > 0$ $(j=1,2,\dots,n_k)$ at all $k=1,2,\dots,s$; $\delta = (\delta_1,\dots,\delta_s)$ is some vector with coordinate vectors $\delta_k = (\delta_{k,1},\dots,\delta_{k,n_k})(k=1,2,\dots,s)$ and $\delta_{k,j} = +1$ or $\delta_{k,j} = -1$ $(j=1,2,\dots,n_k)$ for all $k=1,2,\dots,s$. The quantity of all possible such vectors δ equals 2^n. In inclusion (1.18) it is sufficient the existence of some vector δ for which true the (1.18) at all $x \in \Omega$.

By $R_\delta = R_\delta(\sigma;H)$, $(H = (H_1,\dots,H_s),\ H_k > 0)$ $(k=1,2,\dots,s)$ we denote a set of points $y \in E_n$ for which are true the inequalities

$$C_j \le \frac{y_{k,j}\delta_{k,j}}{v_k^{\sigma_{k,j}}} \le C_j^* \quad (j=1,2,\dots,n_k) \tag{1.19}$$

for $0 < v_k \le H_k$ at all $k=1,2,\dots,s$ i.e.

$$R_\delta = \bigcup_{0<v_k\le h_k} \left\{ \begin{matrix} y \in E_n; C_j \le \dfrac{y_{k,j}\delta_{k,j}}{v_k^{k,j}} \le C_j^* \\ (j=1,2,\dots,n_k) \end{matrix} \right\} \quad (k=1,2,\dots,s).$$

The set $x + R_\delta(\sigma;H)$ is called "σ -semi-horn" with the vertex at the point $x \in E_n$.

Definition 1.3. *The domain $G \subset E_n$ is called a domain which satisfies "the condition of σ -semi-horn" if there exists a finite system of subdomains*

$$G_1, G_2, \dots, G_M \subset G, \tag{1.20}$$

satisfying "the condition of σ -semi-horn" and covering the domain G i.e.

$$G = \bigcup_{\mu=1}^{M} G_\mu. \tag{1.21}$$

A class of domains $G \subset E_n$ satisfying "the condition σ -semi-horn" we'll denote by

$$C(\sigma;H). \tag{1.22}$$

1.3. A class of m -dimensional surfaces.

Let $m = m_1 + m_2 + ... + m_\alpha < n = n_1 + n_2 + ... + n_s$ $(1 \le \alpha \le s)$,

$$\begin{cases} x_{k,1} = x_{k,1} \\ .... \\ x_{k,m_k} = x_{k,m_k} \\ x_{k,m_k+1} = \varphi_{k,m_k+1}(x_{k,1},...,x_{k,m_k}) \\ .... \\ x_{k,n_k} = \varphi_{k,n_k}(x_{k,1},...,x_{k,m_k}) \end{cases} \quad (1.23)$$

at every $k = 1,2...,\alpha$ where the functions $\varphi_{k,j} = \varphi_{k,j}(x_{k,1},...,x_{k,m_k})(j = m_k + 1,...,n_k)$ are continuous and has the continuous bounded partial derivatives

$$\left| \frac{\partial}{\partial x_{k,\mu}} \varphi_{k,j}(x_{k,1},...,x_{k,m_k}) \right| \le M_{k,j} \quad (1.24)$$

$(j = m_k + 1,...,n_k;\ \mu = 1,2,...,m_k)$ in some m_k -dimensional domain $\Omega_{m_k} \subset E_{m_k}$ $(k = 1,2,...,\alpha)$.

Let's suppose

$$x_k^* = (x_{k,1},...,x_{k,m_k}, \varphi_{k,m_k+1}(x_{k,1},...,x_{k,m_k}),...,\varphi_{k,n_k}(x_{k,1},...,x_{k,m_k})) \quad (1.25)$$
$$(k = 1,2,...,\alpha)$$

A set of points

$$x^* = (x_1^*,...,x_s^*) \in E_n = E_{n_1} \times ... \times E_{n_s}$$

whose coordinate-vectors satisfy the equations

$$\begin{cases} x_1^* = x_1^* \\ .... \\ x_\alpha^* = x_\alpha^* \\ x_{\alpha+1}^* = \psi_{\alpha+1}(x_1^*,...,x_\alpha^*) \\ .... \\ x_s^* = \psi_s(x_1^*,...,x_\alpha^*) \end{cases} \quad (1.26)$$

we'll denote by Γ_m, at this the vector functions

$$\psi_k = \psi_k(x_1^*,...,x_\alpha^*)$$

have the coordinate-functions

$$\psi_{k,j} = \psi_{k,j}(x_1^*,...,x_2^*)(j = 1,2,...,n_k)(k = \alpha + 1,...,s),$$

which are continuous and have continuous bounded derivatives.

$$\left| \frac{\partial}{\partial x_{\gamma,\mu}} \psi_{k,j}(x_1^*,...,x_\alpha^*) \right| \le K_{k,j} \quad (1.27)$$

$(\gamma = 1,2,..,\alpha;\ \mu = 1,2,..,m_\gamma;\ j = 1,2,...,n_k)$ in some domain

$$\Omega_m = \Omega_{m_1} \times ... \times \Omega_{m_\alpha} \subset E_m.$$

A set of points

$$x^* = (x_1^*,...,x_\alpha^*, \psi_{\alpha+1}(x_1^*,...,x_\alpha^*);...;\psi_s(x_1^*,...,x_\alpha^*)) = R(x_1^*,...,x_\alpha^*) = T(x') \quad (1.28)$$

at $x' = (x_{1,1},...,x_{1,m_1},...,x_{\alpha,1},...,x_{\alpha,m_\alpha}) \in \Omega_m$ determines the surface Γ_m.

A class of surfaces Γ_m, satisfying the conditions (1.24), (1.27) is denoted by Π^1. The set of points $x^* + h$ at all $x^* \in \Gamma_m$ is denoted by $\Gamma_m + h$. Let Γ_m be situated on the boundary ∂G of domain G, then vector $h = (h_1;...;h_s) \in E_n$ we can choose such that

$$\Gamma_m + h \subset G. \tag{1.29}$$

1.4. The trace of functions on the boundary of surface.

Let's the surface

$$\Gamma_m \in \Pi^1, \tag{1.30}$$

at this we suppose that

$$\Gamma_m \subset \overline{G}, \tag{1.31}$$

i.e. the surface Γ_m or its part is situated on the boundary ∂G of domain G. In this case "the positive vector" $h = (h_1;...;h_s)$ with coordinate-vectors $h_k = (h_{k,1};...;h_{k,n_k})$ $(k = 1,2,...,s)$ is choosen such that

$$\Gamma_m + h \subset G. \tag{1.32}$$

The contraction of the function $f = f(x)$ on the surface $\Gamma_m + h$ is denoted as

$$f\big|_{\Gamma_m + h}. \tag{1.33}$$

If it is has the meaning in the case when the boundary

$$\left\| f\big|_{\Gamma_m} \right\|_{p,\Gamma_m} \overset{def}{=} \lim_{|h| \to 0} \left\| f\big|_{\Gamma_m + h} \right\|_{p,\Gamma_m + h} \tag{1.34}$$

exists, they say on the existence of L_p - trace of functions on the boundary surface Γ_m and denote this fact as

$$f\big|_{\Gamma_m} \in L_p(\Gamma_m). \tag{1.35}$$

2. Main results.

Theorems 2.1. *1) Let*

$$f \in B_{p,\theta}^{<r>}(G;s), \tag{2.1}$$

where $1 \le p \le \theta < \infty$, $r = (r_1,...,r_s)$ *is a positive vector with coordinate-vectors* $r_k = (r_{k,1},...,r_{k,n_k})$, $(k = 1,2,...,s)$, *i.e.* $r_{k,j} > 0$ $(j = 1,2,...,n_k)$ *at all* $k = 1,2,...,s$.

2) Let the domain

$$G \in C(\sigma; H^0), \tag{2.2}$$

where $H^0 = (H_1^0,...,H_s^0)$, $H_k^0 > 0$ $(k = 1,2,...,s)$ *vector*

$$\sigma = \frac{1}{r} = \left(\frac{1}{r_1};...;\frac{1}{r_s}\right) \tag{2.3}$$

with coordinate-vectors $\frac{1}{r_k} = \left(\frac{1}{r_{k,1}};...;\frac{1}{r_{k,n_k}}\right)$ $(k = 1,2,...,s)$.

3) Let $m = m_1 + m_2 + ... + m_\alpha < n = n_1 + n_2 + ... + n_s$, $(1 \le \alpha \le s \le n)$, $1 \le m_k \le n_k$ $(k = 1,2,...,\alpha)$ *and* m *-dimensional surface*

$$\Gamma_m \in \Pi^1. \tag{2.4}$$

4) Let $\nu = (\nu_1;...;\nu_n)$ *be a "non-negative integer vector" with coordinate-vectors* $\nu_k = (\nu_{k,1};...;\nu_{k,n_k})(k = 1,2,...,s)$ *i.e.* $\nu_{k,j} \geq 0$ $(j = 1,2,...,n_k)$ *are integers for all* $k = 1,...,s$, *thus we suppose validity of inequality*

$$æ_k = 1 - \left(\nu_k, \frac{1}{r_k}\right) - \frac{1}{p}\left|\frac{1}{r_k}\right| + \frac{1}{q}\sum_{j=1}^{m_k}\frac{1}{r_{k,j}} \geq 0 \qquad (2.5)$$

when $k = 1,2,...,\alpha$;

$$æ_k = 1 - \left(\nu_k, \frac{1}{r_k}\right) - \frac{1}{p}\left|\frac{1}{r_k}\right| > 0 \qquad (2.6)$$

when $k = \alpha + 1,...,s$, *where*

$$1 \leq p \leq q \leq \infty. \qquad (2.7)$$

The equality

$$æ_k = 0 \qquad (2.8)$$

at some values $k \in \{1,2,...,\alpha\}$ *is supposed only in case*

$$1 < p < q < \infty \quad \text{and} \quad 1 < p = \theta < \infty. \qquad (2.9)$$

Then there exist L_q*-traces of functions* $D^\nu f(x)$ *on the surface* Γ_m, *i.e.*

$$D^\nu f\big|_{\Gamma_m} \in L_q(\Gamma_m), \qquad (2.10)$$

and the inequalities

$$\left\| D^\nu f\big|_{\Gamma_m} \right\|_{q,\Gamma_m} \leq C \sum_{i=(i_1,...,i_s)\in Q} \left(\prod_{k=1}^{s} H_k^{æ_{k,i_k}}\right) \|f\|_{L_{p,\theta}^{<i>}(G;s)} \qquad (2.11)$$

are true when $0 < H_k \leq H_k^0$ $(k = 1,...,s)$, *where* C *is a constant independent of the function* $f = f(x)$ *and vector* $H = (H_1,...,H_s)$; *the numbers*

$$æ_{k,i_k} = \begin{cases} æ_k & \text{for } k \in e' = \operatorname{supp} i \\ æ_k - 1 & \text{for } k \in e_s \setminus e' \end{cases} \qquad (2.12)$$

$$\left(\nu_k, \frac{1}{r_k}\right) = \sum_{j=1}^{n_k} \frac{\nu_{k,j}}{r_{k,j}} \quad (k = 1,...,s). \qquad (2.13)$$

Remark 2.1. In the conditions of theorem 2.1 the domain $G \subset E_n$ belongs to the class $C(\sigma; H^0)$ which is contracted with the equality

$$\sigma = \frac{1}{r}, \qquad (2.14)$$

i.e. geometry of domain corresponds to the smooth exponent of spaces (2.1).

In case (2.14) we attain the most complete research of differential properties of functions from the spaces (2.1)

Theorem 2.2. *1) Let*

$$f \in B_{p,\theta}^{<r>}(G;s), \qquad (2.15)$$

where $1 \leq p \leq \theta < \infty$, $r = (r_1,...,r_s)$ *is a "positive vector" with coordinate-vectors* $r_k = (r_{k,1},...,r_{k,n_k})(k = 1,...,s)$, *i.e.* $r_{k,j} > 0$ $(j = 1,2,...,n_k)$ *for all* $k = 1,...,s$.

2) Let the domain

$$G \in C(\sigma; H^0), \qquad (2.16)$$

where $H^0=(H_1^0,...,H_s^0)$, $H_k^0>0$ $(k=1,...,s)$; *the vector* $\sigma=(\sigma_1;...;\sigma_s)$ *with coordinate-vectors* $\sigma_k=(\sigma_{k,1},...,\sigma_{k,n_k})(k=1,...,s)$ *is "positive", i.e.* $\sigma_{k,j}>0$ $(j=1,2,...,n_k)$ *for all* $k=1,...,s$.

3) Let $m=m_1+m_2+...+m_\alpha<n=n_1+n_2+...+n_s$, $1\le m_k\le n_k$ $(k=1,...,\alpha;$ $1\le\alpha\le s\le n)$ *and let* m *dimensional surface*

$$\Gamma_m\in\Pi^1 \tag{2.17}$$

4) Let the vector $v=(v_1;...;v_s)$ *with coordinate-vectors* $v_k=(v_{k,1};...;v_{k,n_k})$ $(k=1,...,s)$ *be integer, non-negative, i.e.* $v_{k,j}\ge 0$ $(j=1,2,...,n_k)$ *are integers for all* $k=1,...,s$ *and let's suppose that at every* $i=(i_1,...,i_s)\in Q$ *with the support*

$$e^i=\operatorname{supp} i\ne\emptyset, \tag{2.18}$$

the inequality

$$ж_{k,i_k}=r_{k,i_k}\sigma_{k,i_k}-(v_k,\sigma_k)-\frac{1}{p}|\sigma_k|+\frac{1}{q}\sum_{j=1}^{m_k}\sigma_{k,j}\ge 0 \tag{2.19}$$

for all $k\in e^i\bigcap\{1,2,...,\alpha\}$

$$æ_{k,i_k}=r_{k,i_k}\sigma_{k,i_k}-(v_k,\sigma_k)-\frac{1}{p}|\sigma_k|>0 \tag{2.20}$$

are true for all $k\in e^i\bigcap\{\alpha+1,..,s\}$, *where* $1\le p\le q\le\infty$ *in this the equality*

$$æ_{k,i_k}=0 \tag{2.21}$$

at some $k\in e^i\bigcap\{\alpha+1,..,s\}$ *is supposed only in the case* $1<p<q<\infty$, $1<p=\theta<\infty$.

Then there exist L_q *traces of functions* $D^v f(x)$ *on* m *dimensional surface* Γ_m, *i.e.*

$$D^v f\big|_{\Gamma_m}\in L_q(\Gamma_m), \tag{2.22}$$

and the inequalities

$$\left\|D^v f\big|_{\Gamma_m}\right\|_{q,\Gamma_m}\le C\sum_{i=(i_1,...,i_s)\in Q}\left(\prod_{k=1}^{s}H_k^{æ_{k,i_k}}\right)\|f\|_{L_{p,\theta}^{\langle r\rangle}(G;s)} \tag{2.23}$$

are true when $0<H_k\le H_k^0$ $(k=1,...,s)$, *where* C *is a constant independent of* $f=f(x)$ *and vector* $H=(H_1,...,H_s)$, *and* $æ_{k,0}$ $(k\in e_s\setminus e^i)$ *are determined by the equalities*

$$ж_{k,0}=\begin{cases}-(v_k,\sigma_k)-\dfrac{1}{p}|\sigma_k|+\dfrac{1}{q}\displaystyle\sum_{j=1}^{m_k}\sigma_{k,j} & \text{for } k\in\{1,2,..,\alpha\}\setminus e^i\\[2ex] -(v_k,\sigma_k)-\dfrac{1}{p}|\sigma_k| & \text{for } k\in\{\alpha+1,...,s\}\setminus e^i\end{cases} \tag{2.24}$$

for all $i=(i_1,...,i_s)\in Q$.

Remark 2.2. It is evident that theorem 2.2 is a generalization of theorem 2.1 in case of any domain $G\in C(\sigma;H)$ whence at $\sigma=\dfrac{1}{r}$ follows the case of theorem 2.1.

3. The integral representations of smooth functions.

Let

$$f \in B_{p,\theta}^{<r>}(G;s), \tag{3.1}$$

then taking into account the definition of space not breaking the generality we can suppose this function as quite smooth in domain $G \subset E$, consequently at every point of domain it holds the integral representation (1 [1]):

$$D^{\nu} f(x) = \sum_{i=(i_1,\dots,i_s)\in Q} A_{i,\delta} f(x) \tag{3.2}$$

where integral operators in the right part of equality (3.2) have the form

$$A_{i,\delta} f(x) = c_i \left(\prod_{k \in e_s \setminus e^i} H_k^{-\beta_{k,0}} \right) \int_{\bar{0}}^{\bar{H}} \prod_{k \in e^i} \frac{d\vartheta_k}{\vartheta_k^{1+\beta_{k,i_k}}} \times$$

$$\times \int_{E_{|\omega^i|}} dz^i \int_{E_n} \left\{ \Delta^{2\omega^i} \left(\frac{z^i}{2\omega^i} \right) D^{\bar{r}} f(x+y) \right\} \Phi_{i,\delta}(\dots) dy, \tag{3.3}$$

where $e^i = \operatorname{supp} i$, $e_s = \{1,2,\dots,s\}$ and vector $\nu = (\nu_1;\dots;\nu_s)$ with coordinate-vectors $\nu_k = (\nu_{k,1};\dots;\nu_{k,n_k})$ $(k=1,\dots,s)$ is "non-negative, integer", i.e. $\nu_{k,j} \ge 0$ $(j=1,2,\dots,n_k)$ are integer for all $k=1,\dots,s$ and

$$0 \le \nu_{k,j} \le \bar{r}_{k,j} + 1 \quad (j=1,2,\dots,n_k) \tag{3.4}$$

for all $k=1,\dots,s$.

The numbers $\beta_{k,0}, \beta_{k,i_k}$ $(j=1,2,\dots,n_k)$ $(k=1,\dots,s)$ are determined by the equality

$$\beta_{k,0} = |\sigma_k| + (\nu_k, \sigma_k) = \sum_{j=1}^{n_k} (1+\nu_{k,j}) \sigma_{k,j} \tag{3.5}$$

for all $k \in e_s \setminus e^i$

$$\beta_{k,i_k} = |\sigma_k| + (\nu_k, \sigma_k) - \bar{r}_{k,i_k} \sigma_{k,j_k} + \sigma_{k,i_k} \tag{3.6}$$

for all $k \in e^i$.

The kernels in the right part of identity (3.2) are determined by the equalities

$$\Phi_{i,\delta}(\dots) = \prod_{k \in e_s \setminus e^i} \Phi_{k,\delta_k,0} \left(\frac{y_k}{H_k^{\sigma_k}} \right) \prod_{k \in e^i} \Phi_{k,\delta_k,i_k} \left(\frac{y_k}{\vartheta_k^{\sigma_k}}; \frac{z_{k,i_k}}{\vartheta_k^{\sigma_{k,i_k}}} \right), \tag{3.7}$$

thus the vector

$$\delta = (\delta_1;\dots;\delta_s) \tag{3.8}$$

with such coordinate-vectors $\delta_k = (\delta_{k,1};\dots;\delta_{k,n_k})$ $(k=1,\dots,s)$ that $\delta_{k,j} = +1$ or $\delta_{k,j} = -1$ $(j=1,2,\dots,n_k)$ for all $k=1,\dots,s$.

Let's denote that the notations

$$\frac{y_k}{H_k^{\sigma_k}} = \left(\frac{y_{k,1}}{H_k^{\sigma_{k,1}}}, \dots, \frac{y_{k,n_k}}{H_k^{\sigma_{k,n_k}}} \right), \tag{3.9}$$

$$\frac{y_k}{\vartheta_k^{\sigma_k}} = \left(\frac{y_{k,1}}{\vartheta_k^{\sigma_{k,1}}}, \dots, \frac{y_{k,n_k}}{\vartheta_k^{\sigma_{k,n_k}}} \right); \quad (k=1,\dots,s) \tag{3.10}$$

are saved.

Moreover the kernels are quite smooth finite in $E_n \times E_{|\omega'|}$ with the supports

$$\operatorname{supp} \Phi_{i,\delta}\left(y, z^i\right) \tag{3.11}$$

from the sets

$$\left\{\left(y, z^i\right): \begin{matrix} 0<\delta_{k,j} y_{k,j} \leq 1 \quad (j=1,2,\ldots,n_k) \quad (k=1,\ldots,s), \\ 0<z_{k,i_k} \delta_{k,i_k} \leq 1 \quad \left(k \in e^i\right) \end{matrix}\right\} \tag{3.12}$$

at every $i=\left(i_1,\ldots,i_s\right) \in Q$.

4. The construction of auxiliary functions.

Let

$$G \in C\left(\sigma ; H^0\right). \tag{4.1}$$

From this it follows the existence of system of subdomains

$$G_1, G_2, \ldots, G_M \subset G \tag{4.2}$$

satisfying "the condition of σ-semi-horn" and covering the domain G, consequently at every $\mu=\{1,2,\ldots,M\}$ there exists a vector $\delta=\delta^\mu$ with coordinate-vectors $\delta_k^\mu=\left(\delta_{k,1}^\mu,\ldots,\delta_{k,n_k}^\mu\right)(k=1,\ldots,s)$ such that $\delta_{k,j}^\mu=1$ or $\delta_{k,j}^\mu=-1\left(j=1,2,\ldots,n_k\right)$ at all $k=1,\ldots,s$ and

$$G_\mu+R_{\delta^\mu}\left(\sigma ; H^0\right) \subset G. \tag{4.3}$$

The auxiliary function $f_{\nu; G_\mu+R_{\delta^\mu}}(x)$ determined in E_n coinciding in $G_\mu+R_{\delta^\mu}\left(\sigma ; H^0\right)$ with the function $D^\nu f(x)$ is introduced with the help of integral identity (3.2), (3.3) and it is quite smooth finite in E_n function.

These auxiliary functions are determined by the equalities

$$f_{\nu; G_\mu+R_{\delta^\mu}}(x)=\sum_{i=\left(i_1,\ldots,i_s\right) \in Q} A_{i,\delta^\mu}^* f(x) \tag{4.4}$$

for all $\mu=\{1,2,\ldots,M\}$.

Here

$$\begin{gathered} A_{i,\delta^\mu}^* f(x)=c_i\left(\prod_{k \in e_s \setminus e^i} H_k^{-\beta_{k,0}}\right) \int_{\vec{0}}^{\vec{H}} \prod_{k \in e^i} \frac{d \vartheta_k}{\vartheta_k^{1+\beta_{k,i_k}}} \times \\ \times \int_{E_{|\omega'|}} d z^i \int_{E_n}\left\{\Delta^{2 \omega^i}\left(\frac{z^i}{2 \omega^i}, G_\mu+R_{\delta^\mu}\right) D^{\vec{r}^i} f(x+y)\right\} \Phi_{i,\delta}(\ldots) d y, \end{gathered} \tag{4.5}$$

where all notations given in integral identity (3.2), (3.3) are saved, in addition in the right part (4.5) in case $\vec{\omega}^i=\vec{0}$ instead of $D^{\vec{r}} f(x+y)$ is put $\chi\left(G_\mu+R_{\delta^\mu}\right) D^{\vec{r}} f(x+y)$ ($\chi(\Omega)$ the characteristic function of the set Ω).

As $\Gamma_m \in \Pi^1$, it follows the inequality

$$\left\|f\right|_{\Gamma_m+h}\|_{q, \Gamma_m+h} \leq C\left(\int_{\Omega_m}\left|f\left(T\left(x'\right)+h\right)\right|^q d x'\right)^{\frac{1}{q}}, \tag{4.6}$$

whence we have

$$\left\| f\big|_{\Gamma_m+h} \right\|_{q,\Gamma_m+h} \le C\sum_{\mu=1}^{M}\left\| f_{v,(G_\mu+R_{\delta^\mu})}(T(x')+h) \right\|_{q,E_m}, \tag{4.7}$$

where $E_m = E_{m1}\times\ldots\times E_{m_\alpha}$ $(m = m_1+\ldots+m_\alpha)$

5. The estimations of integral operators.

The integral operator at the point $T(x'+h)$ of surface Γ_m+h are considered in the next form

$$A^*_{i,\delta^\mu} f(T(x')+h) = c_i\left(\prod_{k\in c_s\setminus e^i} H_k^{-\beta_{k,0}}\right)\int_{\bar 0}^{\bar H}\ldots\int\prod_{k\in e^i}\frac{d\vartheta_k}{\vartheta_k^{1+\beta_{k,i_k}}}\times$$

$$\times \int_{E_{|\omega^i|}} dz^i \int_{E_n}\left\{\Delta^{2\omega^i}\left(\frac{z^i}{2\omega^i}, G_\mu + R_{\delta^\mu}\right) D^{\bar r^i} f(x+y)\right\}\Phi_{i,\delta^\mu}(\ldots)dy, \tag{5.1}$$

where the kernels are quite smooth finite functions.

Applying different variants of Hölder's inequalities the Minkovski's generalized inequalities after some calculations we change the order of integrating and we apply in consecutive order one dimensional variants of Hardy-Littlewood inequality (after some transformations) in conditions of theorem 2.2 the inequality

$$\left\| A^*_{i,\delta^\mu} f(T(\cdot)+h) \right\|_{q,E_m} = c\left(\prod_{k=1} H_k^{\text{æ}_{k,i_k}}\right)\|f\|_{L^{<r^i>}_{p,\theta}(G_\mu+R_{\delta^{\mu;s}})}, \tag{5.2}$$

is proved when $0 < H_k \le H_k^0$ $(k=1,\ldots,s)$ uniformly relative to vector $h=(h_1;\ldots;h_s)$ for all $\mu = \{1,2,\ldots,M\}$, $i=(i_1,\ldots,i_s)\in Q^*$.

6. The sketch of the proof of theorem 2.2.

The proof of this theorem is conducted by the method of integral representations on the base of integral representations on the base of integral representations of functions of many bundle of variables given by the equalities (3.2), (3.3). The essential role at proving the theorem 2.2 plays the auxiliary functions determined by the equalities (4.4), (4.5) and by inequality

$$\left\| D^v f\big|_{\Gamma_m+h} \right\|_{q,\Gamma_m+h} \le C\left(\int_{\Omega_m}|f(T(x')+h)|^q dx'\right)^{\frac{1}{q}} \tag{6.1}$$

where $x'=(x_1';\ldots;x_\alpha')\in\Omega_m\subset E_m$, $x_k'=(x_{k,1};\ldots;x_{k,m_k})\in E_{m_k}$ $(k=1,2,\ldots,\alpha)$.

From the inequality (6.1) and the identity (3.2), (3.3) taking into account the auxiliary functions (4.4), (4.5) in conditions of the theorem 2.2. we've the inequality

$$\left\| D^v f\big|_{\Gamma_m+h} \right\|_{q,\Gamma_m+h} \le C\sum_{\mu=1}^{M}\left\| f_{v,(G_\mu+R_{\delta^\mu})}(T(\cdot)+h) \right\|_{q,E_m} \le$$

$$\le C\sum_{\mu=1}^{M}\sum_{i\in Q}\left\| A^*_{i,\delta} f(T(\cdot)+h) \right\|_{q,E_m}, \tag{6.2}$$

whence applying the integral estimations (5.2) for integral operator (5.1) we'll get

$$\left\| D^v f\big|_{\Gamma_m+h} \right\|_{q,\Gamma_m+h} \le C\sum_{\mu=1}^{M}\sum_{i=(i_1,\ldots,i_s)\in Q}\left(\prod_{k=1}^{s} H_k^{\text{æ}_{k,i_k}}\right)\|f\|_{L^{<r^i>}_{p,\theta}(G_\mu+R_{\delta^{\mu;s}})} \tag{6.3}$$

at $0 < H_k \le H_k^0 \quad (k = 1,...,s)$ uniformly relative to vector $h = (h_1,...,h_s)$ which leads to the validity of the inequality

$$\left\| D^\nu f \Big|_{\Gamma_m + h} \right\|_{q, \Gamma_m + h} \le C \sum_{i=(i_1,...,i_s) \in Q} \left(\prod_{k=1}^{s} H_k^{\text{æ}_{k, i_k}} \right) \| f \|_{L_{p,\theta}^{<r^i>}\left(G_\mu + R_{\delta^{\mu;s}}\right)} \qquad (6.4)$$

uniformly relative to vector $h = (h_1;...;h_s)$.

From the last inequality (6.4) when $|h| \to 0$ we'll get the necessary desired estimation

$$\left\| D^\nu f \Big|_{\Gamma_m} \right\|_{q, \Gamma_m} \le C \sum_{i \in Q} \left(\prod_{k=1}^{s} H_k^{\text{æ}_{k, i_k}} \right) \| f \|_{L_{p,\theta}^{<r^i>}\left(G_\mu + R_{\delta^{\mu;s}}\right)}$$

that completes the proof of theorem 2.2.

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