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SOBOLEV TYPE THEOREMS FOR $B_{k,n}$-RIESZ POTENTIALS

Abstract

In this paper, we investigate the $B_{k,n}$-Riesz potentials (Riesz-Fourier-Bessel potentials) $I^{\alpha}_{B_{k,n}}$ on the $L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$ and $BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)$.

For Riesz potentials known imbedding $L_p \to L_q$, $q=\dfrac{np}{n-\alpha p}$ for $p=\dfrac{n}{\alpha}$ stopping are valid. In the limit case $p=\dfrac{n}{\alpha}$ Riesz potentials don't defined on all space L_p. How show E.Stein and A.Zigmund for critical exponent $p=\dfrac{n}{\alpha}$ under the condition existence almost everywhere Riesz potentials it boundedly acting from space L_p to bounded mean oscillations space BMO.

In the work this questions was investigated for $B_{k,n}$-Riesz potentials. For $B_{k,n}$-Riesz potentials Sobolev's analog theorem in the space $L_{p,\gamma_{k,n}}(R^n_{k,+})$ is proved. Was show, that for $B_{k,n}$-Riesz potentials for $p=\dfrac{n+|\gamma_{k,n}|}{\alpha}$ imbedding $L_{p,\gamma_{k,n}}(R^n_{k,+}) \to L_{q,\gamma_{k,n}}(R^n_{k,+})$, $q=\dfrac{(n+|\gamma_{k,n}|)p}{n+|\gamma_{k,n}|-\alpha p}$ also stopping is valid. In the limit case $p=\dfrac{n+|\gamma_{k,n}|}{\alpha}$ $B_{k,n}$-Riesz potentials don't defined on all space $L_{p,\gamma_{k,n}}(R^n_{k,+})$.

Proved that for critical exponent $p=\dfrac{n+|\gamma_{k,n}|}{\alpha}$, under the condition existence almost everywhere $B_{k,n}$-Riesz potentials, $B_{k,n}$-Riesz potentials boundedly acting from space $L_{p,\gamma_{k,n}}(R^n_{k,+})$ to $BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)$.

Let R^n n-dimensional Euclidian space of points $x=(x_1,\cdots,x_n)$, $|x|=\left(\sum\limits_{i=1}^{n}x_i^2\right)^{1/2}$, $1\le k\le n-1, x'=x_{1,k}=(x_1,\cdots,x_k)\in R^k, x''=x_{k,n}=(x_{k+1},\cdots,x_n)\in R^{n-k}$
$x=(x',x'')=(x_{1,k},x_{k,n})\in R^n, R^n_{k,+}=\{x=(x_{1,k},x_{k,n})\in R^n; x_{k+1}>0,\ldots,x_n>0\}, B_{k,+}(x,r)=$
$=\{y-x\in R^n_{k,+}; |x-y|<r\}$, $\gamma_{k,n}=(\gamma_{k+1},\ldots,\gamma_n)$, $\gamma_{k+1}>0,\ldots,\gamma_n>0$, $x_{k,n}^{\gamma_{k,n}}=x_{k+1}^{\gamma_{k+1}}\cdot\ldots\cdot x_n^{\gamma_n}$,
$|\gamma_{k,n}|=\gamma_{k+1}+\ldots+\gamma_n$, $S_{k,+}=\{x\in R^n_{k,+}: |x|=1\}$, $(x',y')=x_1y_1+\cdots+x_ny_n$.

In the case $k=0$ $x=x''=x_{0,n}\in R^n$, $R^n_+\equiv R^n_{0,+}=\{x\in R^n; x_1>0,\ldots,x_n>0\}$ and $\gamma=\gamma_{0,n}=(\gamma_1,\ldots,\gamma_n)$.

By $L_{p,\gamma_{k,n}} = L_{p,\gamma_{k,n}}\left(R_{k,+}^n\right)$ we denote the spaces of measurable functions $f(x), x \in R_{k,+}^n$ with finite norm

$$\|f\|_{L_{p,\gamma_{k,n}}\left(R_{k,+}^n\right)} = \|f\|_{p,\gamma_{k,n}} = \left(\int\limits_{R_{k,+}^n} | f(x)|^p x_{k,n}^{\gamma_{k,n}} dx \right)^{1/p}, \quad 1 \le p < \infty .$$

Suppose $L_{\infty,\gamma_{k,n}}\left(R_{k,+}^n\right) = L_\infty\left(R_{k,+}^n\right)$ where $L_\infty\left(R_{k,+}^n\right)$ the space of all essential bounded functions f with finite norm

$$\|f\|_{L_{\infty,\gamma_{k,n}}\left(R_{k,+}^n\right)} = \|f\|_{L_\infty\left(R_{k,+}^n\right)} = \operatorname*{ess\,sup}_{x \in R_{k,+}^n} |f(x)|.$$

The operator of generalized shift ($B_{k,n}$-shift operator) is defined by the following way (see [1], [2]):

$$T^y f(x) = \frac{\prod\limits_{i=k+1}^{n} \Gamma\left(\frac{\gamma_{i+1}}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-k} \prod\limits_{i=k+1}^{n} \Gamma\left(\frac{\gamma_i}{2}\right)} \int\limits_0^\pi \cdots \int\limits_0^\pi f\left(x'-y', \sqrt{x_{k+1}^2 - 2x_{k+1}y_{k+1}\cos\alpha_{k+1} + y_{k+1}^2}, \ldots, \right.$$

$$\left. \ldots, \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2} \right) \cdot \prod_{i=k+1}^{n} \sin^{\gamma_i - 1}\alpha_i d\alpha_{k+1} \cdots d\alpha_n .$$

We will denote by B_j Bessel's singular differential operator $B_j = \frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$, $\gamma_j > 0$ $(j = k+1, \ldots, n)$, $B_{k,n} = \left(B_{k+1}, \ldots, B_n\right)$ and by $\Delta_{B_{k,n}}$-Laplace-Bessel type operator, which is determined by the following way

$$\Delta_{B_{k,n}} = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \sum_{i=k+1}^{n} B_i .$$

We define the isotropic $B_{k,n}$-Morrey spaces $L_{p,\lambda,\gamma_{k,n}}\left(R_{k,+}^n\right), \widetilde{L}_{p,\lambda,\gamma_{k,n}}\left(R_{k,+}^n\right)$ and $B_{k,n}$ – BMO spaces $BMO_{\gamma_{k,n}}\left(R_{k,+}^n\right)$ introduced by V.S.Guliev [5].

Definition 1. *Let $1 \le p < \infty$, $0 \le \lambda \le n + \left|\gamma_{k,n}\right|$. We will be said, the measurable on $R_{k,+}^n$ function f belong to the class $L_{p,\lambda,\gamma_{k,n}}\left(R_{k,+}^n\right)$, if exist constant C_f such, that*

$$\int\limits_{B_{k,+}(0,t)} T^y |f(x)|^p y_{k,n}^{\gamma_{k,n}} dy \le C_f t^\lambda ,$$

for any $(x,t) \in R_{k,+}^n \times (0,\infty)$.

We introduce the norm in the set $L_{p,\lambda,\gamma_{k,n}}\left(R_{k,+}^n\right)$ by the following way

$$\|f\|_{L_{p,\lambda,\gamma_{k,n}}\left(R_{k,+}^n\right)} = \|f\|_{p,\lambda,\gamma_{k,n}} = \sup_{x,t} \left(t^{-\lambda} \int\limits_{B_{k,+}(0,t)} T^y |f(x)|^p y_{k,n}^{\gamma_{k,n}} dy \right)^{1/p} .$$

Definition 2. *Let* $1 \le p < \infty, 0 \le \lambda \le n + |\gamma_{k,n}|$. *We will be said , the measurable on* $R^n_{k,+}$ *function* f *belong to the class* $\widetilde{L}_{p,\lambda,\gamma_{k,n}}(R^n_{k,+})$ *if exist constant* C_f *such, that*

$$\int_{B_{k,+}(0,t)} T^y |f(x)|^p y_{k,n}^{\gamma_{k,n}} dy \le C_f [t]_1^\lambda$$

for any $(x,t) \in R^n_{k,+} \times (0,\infty)$, $[t]_1 = \min\{1,t\}$.

We introduce the norm in the set $\widetilde{L}_{p,\lambda,\gamma_{k,n}}(R^n_{k,+})$ *by the following way*

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma_{k,n}}(R^n_{k,+})} = \sup_{x,t}\left([t]_1^{-\lambda} \int_{B_{k,+}(0,t)} T^y |f(x)|^p y_{k,n}^{\gamma_{k,n}} dy \right)^{1/p}.$$

Obviously,

$$\widetilde{L}_{p,0,\gamma_{k,n}}(R^n_{k,+}) = L_{p,0,\gamma_{k,n}}(R^n_{k,+}) = L_{p,\gamma_{k,n}}(R^n_{k,+})$$

and

$$L_{p,n+|\gamma_{k,n}|,\gamma_{k,n}}(R^n_{k,+}) = L_{\infty,\gamma_{k,n}}(R^n_{k,+}) = L_\infty(R^n_{k,+})$$

$$\widetilde{L}_{p,\lambda,\gamma_{k,n}}(R^n_{k,+}) \subset L_{p,\gamma_{k,n}}(R^n_{k,+}) \text{ and } \|f\|_{L_{p,\gamma_{k,n}}(R^n_{k,+})} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma_{k,n}}(R^n_{k,+})}.$$

Definition 3. *We will be said, the function* $f \in L^{loc}_{1,\gamma_{k,n}}(R^n_{k,+})$ *belong to the space* $BMO_{\gamma_{k,n}}(R^n_{k,+})$ *(Fourier-Bessel-BMO space), if*

$$\|f\|_{BMO_{\gamma_{k,n}}(R^n_{k,+})} = \sup_{x,r} |B_{k,+}(0,r)|^{-1}_{\gamma_{k,n}} \int_{B_{k,+}(0,r)} T^y |f(x) - f_{B_{k,+}(x,r)}| y_{k,n}^{\gamma_{k,n}} dy < \infty,$$

where $f_{B_{k,+}(x,r)} = |B_{k,+}(0,r)|^{-1}_{\gamma_{k,n}} \int_{B_{k,+}(0,r)} T^y f(x) y_{k,n}^{\gamma_{k,n}} dy, |E|_{\gamma_{k,n}} = \int_E x_{k,n}^{\gamma_{k,n}} dx,\ E \subset R^n_{k,+}$.

For $B_{k,n}$-maximal functions

$$M_{B_{k,n}} f(x) = \sup_{\varepsilon>0} |B_{k,+}(0,\varepsilon)|^{-1}_{\gamma_{k,n}} \int_{B_{k,+}(0,\varepsilon)} T^y |f(x)| y_{k,n}^{\gamma_{k,n}} dy.$$

we proved the following theorem.

Theorem 1. [7] **1)** *If* $f \in L_{1,\gamma_{k,n}}(R^n_{k,+})$, *then for any* $\alpha > 0$

$$\left| \{ x \in R^n_{k,+} : M_{B_{k,n}} f(x) > \alpha \} \right|_{\gamma_{k,n}} \le \frac{C_1}{\alpha} \int_{R^n_{k,+}} |f(x)| x_{k,n}^{\gamma_{k,n}} dx,$$

where C_1 *does depend on* f.

2) *If* $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, $1 < p \le \infty$, *then* $M_{B_{k,n}} f(x) \in L_{p,\gamma_{k,n}}(R^n_{k,+})$ *and*

$$\|M_{B_{k,n}} f\|_{p,\gamma_{k,n}} \le C_{p,\gamma_{k,n}} \|f\|_{p,\gamma_{k,n}},$$

where $C_{p,\gamma_{k,n}}$ *-depend only on* $p, \gamma_{k,n}$ *and dimension* n.

Remark 1. In the case $k = 0$ the analog of theorem 1 is proved in [5].

We consider the $B_{k,n}$-Riesz potentials (Riesz-Fourier-Bessel potentials)

$$I^\alpha_{B_{k,n}} f(x) = \int_{R^n_{k,+}} T^y |x|^{\alpha - n - |\gamma_{k,n}|} f(y) y_{k,n}^{\gamma_{k,n}} dy,\quad 0 < \alpha < n + |\gamma_{k,n}| \tag{1}$$

and integral operator type of $B_{k,n}$-potential (Riesz-Fourier-Bessel type's potential)

$$\left[\widetilde{I}^{\alpha}_{B_{k,n}} f\right](x) = \int_{R^n_{k,+}} \left(T^y |x|^{\alpha-n-|\gamma_{k,n}|} - |y|^{\alpha-n-|\gamma_{k,n}|} \chi_{B^*_{k,+}(0,1)}(y) \right) f(y) y_{k,n}^{\gamma_{k,n}} dy, \qquad (2)$$

where $B^*_{k,+}(0,1) = R^n_{k,+} \setminus B_{k,+}(0,1)$.

The operator $\widetilde{I}^{\alpha}_{B_{k,n}}$ is some modification $B_{k,n}$-potentials $I^{\alpha}_{B_{k,n}}$.

In the work show, that if $1 \le p < \frac{n+|\gamma_{k,n}|}{\alpha}$, then $B_{k,n}$-potentials $I^{\alpha}_{B_{k,n}} f$ defined for all function $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$. And also reduced the examples, which shown, that if $p \ge \frac{n+|\gamma_{k,n}|}{\alpha}$, then $B_{k,n}$-potentials $I^{\alpha}_{B_{k,n}}$ don't defined for all function $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$.

For $B_{k,n}$-Riesz potentials the following Sobolev's generalized theorem is valid.

Theorem 2. *Let* $0 < \alpha < n+|\gamma_{k,n}|, 1 \le p < \frac{n+|\gamma_{k,n}|}{\alpha}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma_{k,n}|}$.

a) If $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, *then integral* $I^{\alpha}_{B_{k,n}} f$ *converges absolutely for almost all* $x \in R^n_{k,+}$.

b) If $1 < p < \frac{n+|\gamma_{k,n}|}{\alpha}$, $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, *then* $I^{\alpha}_{B_{k,n}} f \in L_{q,\gamma_{k,n}}(R^n_{k,+})$ *and*

$$\left\| I^{\alpha}_{B_{k,n}} f \right\|_{q,\gamma_{k,n}} \le C_p \|f\|_{p,\gamma_{k,n}} \qquad (3)$$

c) If $f \in L_{1,\gamma_{k,n}}(R^n_{k,+})$, $\frac{1}{q} = 1 - \frac{\alpha}{n+|\gamma_{k,n}|}$, *then*

$$\left|\left\{x \in R^n_{k,+} : I^{\alpha}_{B_{k,n}} f(x) > \beta\right\}\right|^{1/q}_{\gamma_{k,n}} \le \frac{C_1}{\beta} \|f\|_{1,\gamma_{k,n}}. \qquad (4)$$

Remark 2. As is shown the theorem 2 the operator $I^{\alpha}_{B_{k,n}}$ is operator $(p,q)_{\gamma_{k,n}}$-strong type, for $1 < p < \frac{n+|\gamma_{k,n}|}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma_{k,n}|}$, and $(1,q)_{\gamma_{k,n}}$-weak type, for $\frac{1}{q} = 1 - \frac{\alpha}{n+|\gamma_{k,n}|}$.

Remark 3. In the case $k = n-1$ parts a) and b) of theorem 2 are proved in [3] (also see [4]). In the case $k = 0$ the anisotropic analog of theorem 2 is proved in [5], and in the case $k = n-1$ is proved in [6].

Proof of theorem 2. First we prove the part a) of theorem 2. Let $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, $1 \le p < \frac{n+|\gamma_{k,n}|}{\alpha}$. We introduce the following denotes:

$$f_1(x) = f(x)\chi_{B_{k,+}(0,1)}(x),\ f_2(x) = f(x) - f_1(x).$$

Then

$$I^{\alpha}_{B_{k,n}} f(x) = I^{\alpha}_{B_{k,n}} f_1(x) + I^{\alpha}_{B_{k,n}} f_2(x) = J_1(x) + J_2(x).$$

Applying to $J_1(x)$ the Young's inequality for $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, $1 \le p \le \infty$ we have

$$\|J_1(\cdot)\|_{p,\gamma_{k,n}} \le C \left\| |\cdot|^{\alpha-n-|\gamma_{k,n}|} \chi_{B_{k,+}(0,1)} \right\|_{1,\gamma_{k,n}} \left\| T^y f \right\|_{p,\gamma_{k,n}}.$$

By virtue of the next inequality

$$\left\|T^y f(\cdot)\right\|_{p,\gamma_{k,n}} \le \|f\|_{p,\gamma_{k,n}}, \quad \forall y \in R^n_{k,+}, \tag{5}$$

and also, taking into account, that $\left\| |\cdot|^{\alpha-n-|\gamma_{k,n}|} \chi_{B_{k,+}(0,1)} \right\|_{1,\gamma_{k,n}} < \infty$, for $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, $1 \le p \le \infty$ we obtain

$$\|J_1(\cdot)\|_{p,\gamma_{k,n}} \le C\|f\|_{p,\gamma_{k,n}}.$$

Thus, for any $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$, $1 \le p \le \infty$ $J_1(x)$ is finite for almost all $x \in R^n_{k,+}$ i.e. $J_1(x)$ converges absolutely for almost all $x \in R^n_{k,+}$.

We estimate $J_2(x)$. Using the inequality (5) we have

$$|J_2(x)| \le C\|f\|_{p,\gamma_{k,n}} \left(\int_{R^n_{k,+} \setminus B_{k,+}(0,1)} |y|^{(\alpha-n-|\gamma_{k,n}|)p'} y_{k,n}^{\gamma_{k,n}} dy \right)^{1/p'} \le C_1 \|f\|_{p,\gamma_{k,n}}$$

From stated above it is follow, that for $f \in L_{p,\gamma_{k,n}}(R^n_{k,+})$ and $1 \le p < \frac{n+|\gamma_{k,n}|}{\alpha}$ the $B_{k,n}$-Riesz potentials $I^\alpha_{B_{k,n}} f(x)$ converges absolutely for almost all $x \in R^n_{k,+}$.

Now we proved the part b). We have

$$I^\alpha_{B_{k,n}} f(x) = \left(\int_{B_{k,+}(0,t)} + \int_{R^n_{k,+} \setminus B_{k,+}(0,t)} \right) T^y f(x) |y|^{\alpha-n-|\gamma_{k,n}|} y_{k,n}^{\gamma_{k,n}} dy = A(x,t) + C(x,t).$$

Let k -for all integer number. Summable by all $j < 0$ we obtain

$$A(x,t) \le \sum_{j=-\infty}^{-1} \int_{\{2^j t \le |y| < 2^{j+1} t,\, y \in R^n_{k,+}\}} T^y |f(x)| |y|^{\alpha-n-|\gamma_{k,n}|} y_{k,n}^{\gamma_{k,n}} dy \le$$

$$\le C \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-n-|\gamma_{k,n}|} \int_{\{|y| < 2^{j+1} t,\, y \in R^n_{k,+}\}} T^y |f(x)| y_{k,n}^{\gamma_{k,n}} dy \le C t^\alpha M_{B_{k,n}} f(x).$$

Thus, the following estimate

$$|A(x,t)| \le C_\alpha t^\alpha M_{B_{k,n}} f(x), \tag{6}$$

holds, where C_α doesn't depend on f, x and t.

For $C(x,t)$ applying the Holder inequality and inequality (5) we have

$$|C(x,t)| \le C\|f\|_{p,\gamma_{k,n}} t^{-(n+|\gamma_{k,n}|)/q}. \tag{7}$$

Thus, from (6) and (7), we have

$$\left|I^\alpha_{B_{k,n}} f(x)\right| \le C\left(t^\alpha M_{B_{k,n}} f(x) + t^{-(n+|\gamma_{k,n}|)/q} \|f\|_{p,\gamma_{k,n}} \right).$$

Minimizing by t for $t = \left[\left(M_{B_{k,n}} f(x)\right)^{-1} \|f\|_{p,\gamma_{k,n}} \right]^{p/(n+|\gamma_{k,n}|)}$ we obtain

$$\left|I^\alpha_{B_{k,n}} f(x)\right| \le C\left(M_{B_{k,n}} f(x)\right)^{p/q} \|f\|^{1-p/q}_{p,\gamma_{k,n}}.$$

Consequently, by virtue of theorem 1 we have

$$\int\limits_{B_{k,+}(0,t)}\left|I^{\alpha}_{B_{k,n}}f(y)\right|^{q}y_{k,n}^{\gamma_{k,n}}dy\le C\|f\|^{q-p}_{p,\gamma_{k,n}}\int\limits_{R^n_{k,+}}\left(M_{B_{k,n}}f(y)\right)^{p}y_{k,n}^{\gamma_{k,n}}dy\le C\|f\|^{q}_{p,\gamma_{k,n}}.$$

Thus, the part b) of theorem 2 are proved .

Now we prove the part c).

Let $f\in L_{1,\gamma_{k,n}}(R^n_{k,+})$. Sufficiently is proved inequality (4) with 2β instead of β in left side this inequality. Further,

$$\left|\left\{x:\left|I^{\alpha}_{B_{k,n}}f(x)\right|>2\beta\right\}\right|_{\gamma_{k,n}}\le\left|\left\{x:\left|A(x,t)\right|>\beta\right\}\right|_{\gamma_{k,n}}+\left|\left\{x:\left|C(x,t)\right|>\beta\right\}\right|_{\gamma_{k,n}}.$$

By virtue of inequality (6) and theorem 1 we have

$$\beta\left|\left\{x\in R^n_{k,+}:\left|A(x,t)\right|>\beta\right\}\right|_{\gamma_{k,n}}=\beta\int\limits_{\{x\in R^n_{k,+}:|A(x,t)|>\beta\}}x_{k,n}^{\gamma_{k,n}}dx\le\beta\int\limits_{\{x\in R^n_{k,+}:Ct^{\alpha}M_{B_{k,n}}f(x)>\beta\}}x_{k,n}^{\gamma_{k,n}}dx=$$

$$=\beta\left|\left\{x\in R^n_{k,+}:M_{B_{k,n}}f(x)>\frac{\beta}{Ct^{\alpha}}\right\}\right|_{\gamma_{k,n}}\le C_1t^{\alpha}\|f\|_{1,\gamma_{k,n}}.$$

And also

$$\left|C(x,t)\right|\le t^{-\frac{n+|\gamma_{k,n}|}{q}}\int\limits_{R^n_{k,+}}T^{y}\left|f(x)\right|y_{k,n}^{\gamma_{k,n}}dy=t^{-\frac{n+|\gamma_{k,n}|}{q}}\|f\|_{1,\gamma_{k,n}}.$$

Thus, if $t^{-\frac{n+|\gamma_{k,n}|}{q}}\|f\|_{1,\gamma_{k,n}}=\beta$ and $\left|C(x,t)\right|\le\beta$ and consequently $\left|\left\{x:\left|C(x,t)\right|>\beta\right\}\right|_{\gamma_{k,n}}=0$.

Finally,

$$\left|\left\{x:\left|I^{\alpha}_{B_{k,n}}f(x)\right|>2\beta\right\}\right|_{\gamma_{k,n}}\le C_1t^{\alpha+\frac{n+|\gamma_{k,n}|}{q}}=C_1t^{n+|\gamma_{k,n}|}=C_1\beta^{-q}\|f\|^{q}_{1,\gamma_{k,n}}.$$

This we get the inequality (4) of $(1,q)_{\gamma_{k,n}}$-weak type. Thus, the map $f\to I^{\alpha}_{B_{k,n}}f$ is $(1,q)_{\gamma_{k,n}}$-weak type .

The theorem 2 is proved .

Theorem 3. *Let* $0<\alpha<n+|\gamma_{k,n}|$, $p=\dfrac{n+|\gamma_{k,n}|}{\alpha}$ *and* $f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. *Then* $\widetilde{I}^{\alpha}_{B_{k,n}}f\in BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)$ *and*

$$\left\|\widetilde{I}^{\alpha}_{B_{k,n}}f\right\|_{BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)}\le C_p\|f\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}. \tag{8}$$

Corollary. *Let* $0<\alpha<n+|\gamma_{k,n}|$, $p=\dfrac{n+|\gamma_{k,n}|}{\alpha}$ *and* $f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. *If the integral* $I^{\alpha}_{B_{k,n}}f$ *exist almost every, then* $I^{\alpha}_{B_{k,n}}f\in BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)$ *and the estimate*

$$\left\|I^{\alpha}_{B_{k,n}}f\right\|_{BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)}\le C_p\|f\|_{L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)}$$

holds.

Proof of theorem 3. Let $f\in L_{p,\gamma_{k,n}}\left(R^n_{k,+}\right)$. For any is fixed $t>0$ suppose

$$f_1(x)=f(x)\chi_{B_{k,+}(0,2t)}(x),\ f_2(x)=f(x)-f_1(x),$$

where χ_A -the characteristic function of the set A.

Then

$$\widetilde{I}^{\alpha}_{B_{k,n}}f(x)=\widetilde{I}^{\alpha}_{B_{k,n}}f_1(x)+\widetilde{I}^{\alpha}_{B_{k,n}}f_2(x)=F_1(x)+F_2(x),$$

where $F_1(x)=\int\limits_{B_{k,+}(0,2t)}\left(T^y|x|^{\alpha-n-|\gamma_{k,n}|}-|y|^{\alpha-n-|\gamma_{k,n}|}\chi_{B^+_{k,+}(0,1)}(y)\right)f(y)y^{\gamma_{k,n}}_{k,n}dy,$

$$F_2(x)=\int\limits_{R^n_{k,+}\setminus B_{k,+}(0,2t)}\left(T^y|x|^{\alpha-n-|\gamma_{k,n}|}-|y|^{\alpha-n-|\gamma_{k,n}|}\chi_{B^+_{k,+}(0,1)}(y)\right)f(y)y^{\gamma_{k,n}}_{k,n}dy.$$

Note, that f_1 has the compact support. Therefore,

$$\alpha_1=-\int\limits_{B_{k,+}(0,2t)\setminus B_{k,+}(0,\min\{1,2t\})}|y|^{\alpha-n-|\gamma_{k,n}|}f(y)y^{\gamma_{k,n}}_{k,n}dy<\infty$$

Further, we estimate difference $|F_1(x)-a_1|$. We have

$$|F_1(x)-a_1|=\left|\int\limits_{R^n_{k,+}}T^y|x|^{\alpha-n-|\gamma_{k,n}|}f_1(y)y^{\gamma_{k,n}}_{k,n}dy\right|\le$$

$$\le\int\limits_{R^n_{k,+}}|y|^{\alpha-n-|\gamma_{k,n}|}T^y|f_1(x)|y^{\gamma_{k,n}}_{k,n}dy=\int\limits_{\{y\in R^n_{k,+}:T^y|x|<2t\}}|y|^{\alpha-n-|\gamma_{k,n}|}T^y|f(x)|y^{\gamma_{k,n}}_{k,n}dy.$$

For $|x|<t, T^y|x|<2t$ we have $|y|\le|x|+|x-y|\le|x|+T^y|x|<3t$.

Consequently for $x\in B_{k,+}(0,t)$

$$|F_1(x)-a_1|\le\int\limits_{B_{k,+}(0,3t)}|y|^{\alpha-n-|\gamma_{k,n}|}T^y|f(x)|y^{\gamma_{k,n}}_{k,n}dy. \tag{9}$$

For $\alpha p=n+|\gamma_{k,n}|$ we have

$$|B_{k,+}(0,t)|^{-1}_{\gamma_{k,n}}\int\limits_{B_{k,+}(0,t)}T^z|F_1(x)-a_1|z^{\gamma_{k,n}}_{k,n}dz\le$$

$$\le|B_{k,+}(0,t)|^{-1}_{\gamma_{k,n}}\int\limits_{B_{k,+}(0,t)}\left(\int\limits_{B_{k,+}(0,3t)}|y|^{\alpha-n-|\gamma_{k,n}|}T^yT^z|f(x)|y^{\gamma_{k,n}}_{k,n}dy\right)z^{\gamma_{k,n}}_{k,n}dz\le$$

$$\le|B_{k,+}(0,t)|^{-1}_{\gamma_{k,n}}\int\limits_{B_{k,+}(0,t)}t^{\alpha}M_{B_{k,n}}\left(T^z|f(x)|\right)z^{\gamma_{k,n}}_{k,n}dz\le$$

$$\le Ct^{-n-|\gamma_{k,n}|}\cdot t^{\alpha}\cdot t^{(n+|\gamma_{k,n}|)/p'}\left(\int\limits_{B_{k,+}(0,t)}\left(M_{B_{k,n}}\left(T^zf(x)\right)\right)^pz^{\gamma_{k,n}}_{k,n}dz\right)^{1/p}\le C_p\left\|T^zf\right\|_{p,\gamma_{k,n}}\le C_p\|f\|_{p,\gamma_{k,n}}.$$

Thus, for $\alpha p=n+|\gamma_{k,n}|$ we have

$$|B_{k,+}(0,t)|^{-1}_{\gamma_{k,n}}\int\limits_{B_{k,+}(0,t)}\mathrm{T}^z|(F_1(x)-a_1)|z^{\gamma_{k,n}}_{k,n}dz\le C_p\|f\|_{p,\gamma_{k,n}}. \tag{10}$$

We denote $a_2=\int\limits_{B_{k,+}(0,\max\{1,2t\})\setminus B_{k,+}(0,2t)}|y|^{\alpha-n-|\gamma_{k,n}|}f(y)y^{\gamma_{k,n}}_{k,n}dy.$

We estimate $|F_2(x)-a_2|$.

$$|F_2(x)-a_2|\le\int\limits_{R^n_{k,+}\setminus B_{k,+}(0,2t)}|f(y)|T^y\left||x|^{\alpha-n-|\gamma_{k,n}|}-|y|^{\alpha-n-|\gamma_{k,n}|}\right|y^{\gamma_{k,n}}_{k,n}dy.$$

The following lemma is valid.

Lemma 4. *Let* $0<\alpha<n+|\gamma_{k,n}|$. *Then exist constant* $C>0$ *such, that for* $2|x|\le|y|$

$$T^y\left||x|^{\alpha-n-|\gamma_{k,n}|}-|y|^{\alpha-n-|\gamma_{k,n}|}\right|\le C|y|^{\alpha-n-|\gamma_{k,n}|-1}|x| \tag{11}$$

Using inequality (11) and Holder's inequality for $|F_2(x)-a_2|$ we obtain

$$|F_2(x)-a_2|\le C|x|\int\limits_{R^n_{k,+}\setminus B_{k,+}(0,2t)}|f(y)||y|^{\alpha-n-|\gamma_{k,n}|-1}y_{k,n}^{\gamma_{k,n}}dy\le C|x|t^{\alpha-1-\frac{n+|\gamma_{k,n}|}{p}}\|f\|_{p,\gamma_{k,n}}\le$$

$$\le C|x|t^{-1}\|f\|_{p,\gamma_{k,n}}.$$

Note, that $|x|\le t,|z|\le 2t$, the inequality $T^z|x|\le|x|+|z|\le 3t$ holds.

For $\alpha p=n+|\gamma_{k,n}|$ we obtain

$$\left|T^zF_2(x)-a_2\right|\le T^z|F_2(x)-a_2|\le CT^z|x|t^{-1}\|f\|_{p,\gamma_{k,n}}\le C\|f\|_{p,\gamma_{k,n}}.$$

Thus, for $\alpha p=n+|\gamma_{k,n}|$ we have

$$\left|T^zF_2(x)-a_2\right|\le C\|f\|_{p,\gamma_{k,n}}. \tag{12}$$

We denote $a_f=a_1+a_2=\int\limits_{B_{k,+}(0,\max\{1,2t\})}|y|^{\alpha-n-|\gamma_{k,n}|}f(y)y_{k,n}^{\gamma_{k,n}}dy$.

Using inequality (10) and inequality (12) we get

$$\sup_{x,t}\frac{1}{|B_{k,+}(0,t)|_{\gamma_{k,n}}}=\int\limits_{B_{k,+}(0,t)}T^y\left|\tilde{I}^\alpha_{B_{k,n}}f(x)-a_f\right|y_{k,n}^{\gamma_{k,n}}dy\le C\|f\|_{p,\gamma_{k,n}}.$$

From here

$$\left\|\tilde{I}^\alpha_{B_{k,n}}f\right\|_{BMO_{\gamma_{k,n}}(R^n_{k,+})}\le 2\sup_{x,t}\frac{1}{|B_{k,+}(0,t)|_{\gamma_{k,n}}}\int\limits_{B_{k,+}(0,t)}T^y\left|\tilde{I}^\alpha_{B_{k,n}}f(x)-a_f\right|y_{k,n}^{\gamma_{k,n}}dy\le C\|f\|_{p,\gamma_{k,n}}.$$

The theorem 3 is proved.

Theorem 4. *Let* $0<\alpha<n+|\gamma_{k,n}|$, $f\in L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}\left(R^n_{k,+}\right)$. *Then* $\tilde{I}^\alpha_{B_{k,n}}f\in BMO_{\gamma_{k,n}}\left(R^n_{k,+}\right)$ *and*

$$\left\|\tilde{I}^\alpha_{B_{k,n}}f\right\|_{BMO_{\gamma_{k,n}}(R^n_{k,+})}\le C_p\|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}. \tag{13}$$

Remark 4. In theorem 3 the operator $\tilde{I}^\alpha_{B_{k,n}}$ don't replace by $I^\alpha_{B_{k,n}}$. Indeed, taking $f(x)=|x|^{-\alpha}\in L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}\left(R^n_{k,+}\right),0<\alpha<n+|\gamma_{k,n}|$ it is possible verify, that $I^\alpha_{B_{k,n}}f\notin BMO_{\gamma_{k,n}}(R^n_{k,+})$.

Proof. Let $f\in L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}\left(R^n_{k,+}\right)$. Suppose for any be fixed $t>0$, $f_1(x)=f(x)\chi_{B_{k,+}(0,2t)}(x), f_2(x)=f(x)-f_1(x)$ where $\chi_{B_{k,+}}$ -the characteristic function of the set $B_{k,+}$.

Then $\tilde{I}^\alpha_{B_{k,n}}f(x)=\tilde{I}^\alpha_{B_{k,n}}f_1(x)+\tilde{I}^\alpha_{B_{k,n}}f_2(x)=F_1(x)+F_2(x)$.

By virtue of definition $I^{\alpha}_{B_{k,n}} f(x)$ -Riesz potentials and using discussions, which applied for the getting inequality(10) we have

$$\int_{B_{k,+}(0,t)} T^z |F_1(x)-a_1| z_{k,n}^{\gamma_{k,n}} dz \le \int_{B_{k,+}(0,t)} T^z \left(\int_{R^n_{k,+}} |y|^{\alpha-n-|\gamma_{k,n}|} T^y |f_1(x)| y_{k,n}^{\gamma_{k,n}} dy \right) z_{k,n}^{\gamma_{k,n}} dz \le$$

$$\le \int_{B_{k,+}(0,3t)} \left(\int_{B_{k,+}(0,t)} T^y T^z |f(x)| z_{k,n}^{\gamma_{k,n}} dz \right) |y|^{\alpha-n-|\gamma_{k,n}|} y_{k,n}^{\gamma_{k,n}} dy \le$$

$$\le t^{n+|\gamma_{k,n}|-\alpha} \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}} \int_{B_{k,+}(0,3t)} |y|^{\alpha-n-|\gamma_{k,n}|} y_{k,n}^{\gamma_{k,n}} dy \le C_\alpha t^{n+|\gamma_{k,n}|} \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}.$$

Next to the last inequality we taking into account, that $f \in L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}(R^n_{k,+})$ and in last inequality, in the fact that $\int_{B_{k,+}(0,3t)} |y|^{\alpha-n-|\gamma_{k,n}|} y_{k,n}^{\gamma_{k,n}} dy \le C_\alpha t^\alpha$.

Consequently,

$$\frac{1}{t^{n+|\gamma_{k,n}|}} \int_{B_{k,+}(0,t)} T^z |F_1(x)-a_1| z_{k,n}^{\gamma_{k,n}} dz \le C_\alpha \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}. \tag{14}$$

For every $x \in B_{k,+}(0,t)$ by virtue of lemma 1 we have

$$|F_2(x)-a_2| = \left| \int_{R^n_{k,+}\setminus B_{k,+}(0,2t)} \left(T^y |y|^{\alpha-n-|\gamma_{k,n}|} - |y|^{\alpha-n-|\gamma_{k,n}|}\right) f(y) y_{k,n}^{\gamma_{k,n}} dy \right| \le$$

$$\le C_\alpha |x| \int_{R^n_{k,+}\setminus B_{k,+}(0,2t)} |f(y)| |y|^{\alpha-n-|\gamma_{k,n}|-1} y_{k,n}^{\gamma_{k,n}} dy =$$

$$= C_\alpha |x| \sum_{k=0}^{\infty} \int_{2^k t \le |y| < 2^{k+1} t,\, y_{k,n}>0} |f(y)| |y|^{\alpha-n-|\gamma_{k,n}|-1} y_{k,n}^{\gamma_{k,n}} dy \le$$

$$\le C_\alpha |x| \sum_{k=0}^{\infty} (2^k t)^{\alpha-n-|\gamma_{k,n}|-1} \int_{2^k t \le |y| < 2^{k+1} t,\, y_{k,n}>0} |f(y)| y_{k,n}^{\gamma_{k,n}} dy \le$$

$$\le C_\alpha |x| \|f\|_{L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}(R^n_{k,+})} \sum_{k=0}^{\infty} (2^k t)^{\alpha-n-|\gamma_{k,n}|-1} (2^{k+1} t)^{n+|\gamma_{k,n}|-\alpha} \le$$

$$\le C_\alpha t \cdot t^{-1} \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}} \le C_\alpha \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}.$$

We get

$$|F_2(x)-a_2| \le C_\alpha \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}, \quad \forall x \in B_{k,+(0,t)}. \tag{15}$$

Then

$$\left|T^z F_2(x) - a_2\right| \le T^z |F_2(x)-a_2| \le C_\alpha \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}. \tag{16}$$

Finally, from (14) and (16) for $f \in L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}(R^n_{k,+})$ we get

$$\sup_{x,t} \frac{1}{|B_{k,+}(0,t)|_{\gamma_{k,n}}} \int_{B_{k,+}(0,t)} T^z \left|\tilde{I}^{\alpha}_{B_{k,n}} f(x) - a_f\right| z_{k,n}^{\gamma_{k,n}} dz \le C_\alpha \|f\|_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}.$$

Thus, $\tilde{I}^{\alpha}_{B_{k,n}} : L_{1,n+|\gamma_{k,n}|-\alpha,\gamma_{k,n}}(R^n_{k,+}) \to BMO_{\gamma_{k,n}}(R^n_{k,+})$.

The theorem 4 is proved.

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